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Decycling connected regular graphs

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Abstract

For a graph G and $S \subseteq V(G)$, if $G - S$ is acyclic, then S is said to be a *decycling set* of G. The cardinality of a smallest decycling set of G is called the *decycling number* of G and is denoted by $\phi(G)$. We prove in this paper that if G runs over the set of connected graphs with a fixed degree sequence $\mathbf{d}$, then the values $\phi(G)$ completely cover a line segment $[A, B]$ of positive integers. Let $\mathcal{CR}(\mathbf{d})$ be the class of all connected graphs having degree sequence $\mathbf{d}$. For an arbitrary graphic degree sequence $\mathbf{d}$, two invariants

$$A := \mathrm{Min}(\phi, \mathbf{d}) = \min\{\phi(G) : G \in \mathcal{CR}(\mathbf{d})\}$$

and

$$B := \mathrm{Max}(\phi, \mathbf{d}) = \max\{\phi(G) : G \in \mathcal{CR}(\mathbf{d})\},$$

arise naturally. For a regular graphic degree sequence $\mathbf{d} = r^n := (r, r, \ldots, r)$, where r is the vertex degree and n is the order of the graph, we obtain some significant results on the values of $\mathrm{Min}(\phi, r^n)$ and $\mathrm{Max}(\phi, r^n)$.

1. Introduction

Let G be a connected graph and $X \subseteq E(G)$. Then the minimum $|X|$ such that $G - X$ is acyclic is known as the dimension of the cycle space of G and it is equal to $|E(G)| - |V(G)| + 1$. It is natural to investigate the corresponding problem in terms of vertices, and this was indeed considered by Kirchhoff [8] in his work on spanning trees.

The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph G is difficult even for some simply defined graphs. For

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a graph G, this minimum number is known as the *decycling number* of G, and is denoted by $\phi(G)$. The class of those graphs G for which $\phi(G) = 0$ consists of all forests, and $\phi(G) = 1$ if and only if G has at least one cycle and a vertex is on all of its cycles. It is also easy to see that $\phi(K_n) = n - 2$ and $K_{p,\,q} = p - 1$ if $p \leq q$, where K_n denotes the complete graph of order n and $K_{p,\,q}$ denotes the complete bipartite graph with partite sets of cardinality p and q. The exact values of decycling numbers for many classes of graphs were obtained and cited in [2].

We proved recently in [10] that if G runs over the set of graphs with a fixed degree sequence $\mathbf{d}$, the values $\phi(G)$ completely cover a line segment $[a, b]$ of nonnegative integers. Let $\mathcal{R}(\mathbf{d})$ be the class of all graphs having degree sequence $\mathbf{d}$. Thus for an arbitrary graphic degree sequence $\mathbf{d}$, two invariants

$$a := \min(\phi, \mathbf{d}) = \min\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}$$

and

$$b := \max(\phi, \mathbf{d}) = \max\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\},$$

arise naturally. For a regular graphic degree sequence $\mathbf{d} = r^n := (r, r, \ldots, r)$ where r is the vertex degree and n is the number of graph vertices, we obtained in [10] the exact values of $\min(\phi, r^n)$ and $\max(\phi, r^n)$ in all situations. It is natural to extend this problem to the class of connected graphs with a degree sequence $\mathbf{d}$. As a direct consequence of Taylor [15] and our result in [10], we have that if G runs over the set of connected graphs with a fixed degree sequence $\mathbf{d}$, the values $\phi(G)$ completely cover a line segment $[A, B]$ of nonnegative integers. Let $\mathcal{CR}(\mathbf{d})$ be the class of all connected graphs having degree sequence $\mathbf{d}$. Thus for an arbitrary graphic degree sequence $\mathbf{d}$, two invariants

$$A := \mathrm{Min}(\phi, \mathbf{d}) = \min\{\phi(G) : G \in \mathcal{CR}(\mathbf{d})\}$$

and

$$B := \mathrm{Max}(\phi, \mathbf{d}) = \max\{\phi(G) : G \in \mathcal{CR}(\mathbf{d})\},$$

arise naturally. We will find the values of $\mathrm{Min}(\phi, r^n)$ and $\mathrm{Max}(\phi, r^n)$.

Only finite simple graphs are considered in this paper. For the most part, our notation and terminology follows that of Bondy and Murty [3]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We will use the following notation and terminology for a typical graph G. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. We use $|S|$ to denote the cardinality of a set S and therefore we define $n = |V|$ to be the *order* of G and $m = |E|$ the *size* of G. To simplify writing, we write $e = uv$ for the edge e that joins the vertex u to the vertex v. A *path* of length k in a graph G, denoted by P_k, is a sequence of distinct vertices $u_1, u_2, \ldots, u_k$ of G such that for all $i = 1, 2, \ldots, k-1$, $u_i u_{i+1}$ are edges of G. A u, v-path is a path which has u as its first vertex and v as its last vertex in the path. The *degree* of a vertex v of a graph G is defined as $d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree of a graph G is usually denoted by $\Delta(G)$. If $S \subseteq V(G)$, the graph $G[S]$ is the subgraph induced by S in G. For a graph G, if $X \subseteq E(G)$, we denote by

$G - X$ the graph obtained from G by removing all edges in X. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph G, if $X \subseteq V(G)$, the graph $G - X$ is the graph obtained from G by removing all vertices in X and all edges incident with vertices in X. For a graph G with $X \subseteq E(\overline{G})$, we denote by $G + X$ the graph obtained from G by adding all edges in X. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. Two graphs G and H are disjoint if $V(G) \cap V(H) = \emptyset$. For any two disjoint graphs G and H, we define $G \cup H$, their union, by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. We can extend this definition to a finite union of pairwise disjoint graphs, since the operation "$\cup$" is associative. For a graph G and $s \in V(G)$, the *neighborhood* of s in G is defined by

$$N(s) = \{v \in V(G) : sv \in E(G)\}.$$

If $S \subseteq V(G)$, then we define

$$N(S) = \bigcup_{s \in S} N(s).$$

If $F \subseteq V(G)$, we write $N_F(S)$ for $N(S) \cap F$. A graph G is said to be *regular* if all of its vertices have the same degree. A 3-regular graph is called a *cubic graph.*

Let G be a graph of order n and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is called a *degree sequence* of G, and we simply write $(d(v_1), d(v_2), \ldots, d(v_n))$ if the underlying graph G is clear from the context. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G. In this case, G is called a *realization* of $\mathbf{d}$.

An algorithm for determining whether or not a given sequence of non-negative integers is graphic was independently obtained by Havel [7] and Hakimi [6]. We state their results in the following theorem.

Theorem 1.1 *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a non-increasing sequence of non-negative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n) = \mathbf{d}'.$$

Then $\mathbf{d}$ *is graphic if and only if* $\mathbf{d}'$ *is graphic.* □

Let G be a graph and $ab, cd \in E(G)$ be independent, where $ac, bd \notin E(G)$. Put

$$G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) + \{ac, bd\}.$$

The operation $\sigma(a, b; c, d)$ is called a *switching operation.* It is easy to see that the graph obtained from G by a switching has the same degree sequence as G. The following theorem has been shown by Havel [7] and Hakimi [6].

Theorem 1.2 *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a graphic degree sequence. If* G_1 *and* G_2 *are any two realizations of* $\mathbf{d}$*, then* G_2 *can be obtained from* G_1 *by a finite sequence of switchings.* □

As a consequence of Theorem 1.2, Eggleton and Holton [4] defined in 1978 *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ whose vertices are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. They obtained the following theorem.

Theorem 1.3 *The graph $\mathcal{R}(\mathbf{d})$ is connected.* □

The following theorem was shown by Taylor [15] in 1980.

Theorem 1.4 *For a graphic degree sequence $\mathbf{d}$, let $\mathcal{CR}(d)$ be the set of all connected realizations of $\mathbf{d}$. Then the induced subgraph $\mathcal{CR}(\mathbf{d})$ of $\mathcal{R}(\mathbf{d})$ is connected.* □

2. Interpolation theorem

Let $\mathbb{G}$ be the class of all simple graphs, a function $f : \mathbb{G} \to \mathbb{Z}$ is called a *graph parameter* if $G \cong H$, then $f(G) = f(H)$. If f is a graph parameter and $\mathbb{J} \subseteq \mathbb{G}$, f is called an *interpolation graph parameter with respect to* $\mathbb{J}$ if there exist integers x and y such that

$$\{f(G) : G \in \mathbb{J}\} = [x, y] = \{k \in \mathbb{Z} : x \leq k \leq y\}.$$

We have shown in [11, 12, 13] that the chromatic number χ, the clique number ω, and the matching number α_1 are interpolation graph parameters with respect to $\mathcal{R}(\mathbf{d})$. If f is an interpolation graph parameter with respect to $\mathbb{J}$, $\{f(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(f, \mathbb{J}) = \min\{f(G) : G \in \mathbb{J}\}$ and $\max(f, \mathbb{J}) = \max\{f(G) : G \in \mathbb{J}\}$. In the case where $\mathbb{J} = \mathcal{R}(\mathbf{d})$ we simply write $\min(f, \mathbf{d})$ and $\max(f, \mathbf{d})$ for $\min(f, \mathcal{R}(\mathbf{d}))$ and $\max(f, \mathcal{R}(\mathbf{d}))$ respectively and in the case where $\mathbb{J} = \mathcal{CR}(\mathbf{d})$ we write $\mathrm{Min}(f, \mathbf{d})$ and $\mathrm{Max}(f, \mathbf{d})$ for $\min(f, \mathcal{CR}(\mathbf{d}))$ and $\max(f, \mathcal{CR}(\mathbf{d}))$ respectively.

We proved in [10] the following results.

Theorem 2.1 *If σ is a switching on G, then $|\phi(G) - \phi(G^\sigma)| \leq 1$.* □

Theorem 2.2 *For a given graphic degree sequence $\mathbf{d}$, there exist integers a and b such that there is a graph G with degree sequence $\mathbf{d}$ and $\phi(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.* □

By Theorem 1.4 and Theorem 2.1, we have the following interpolation theorem with respect to $\mathcal{CR}(\mathbf{d})$.

Theorem 2.3 *For a given graphic degree sequence $\mathbf{d}$, there exist integers A and B such that there is a connected graph G with degree sequence $\mathbf{d}$ and $\phi(G) = c$ if and only if c is an integer satisfying $A \leq c \leq B$.* □

Let G be a graph and D be a minimum decycling set of G. Then $G - D$ is an induced forest of G of maximum order. Erdős et al. [5] first defined a counterpart graph parameter I as follows. Let G be a graph and $F \subseteq V(G)$. F is called an *induced forest* of G if $G[F]$ contains no cycle. An induced forest F of G is *maximal* if for every $v \in G - F$, $F \cup \{v\}$ is not an induced forest of G. Let $I(G)$ be defined as

$$I(G) := \max\{|F| : F \text{ is an induced forest of } G\}.$$

Thus $I(G)$ is the maximum cardinality of induced forests of G. An induced forest F of G with $|F| = I(G)$ is called a *maximum induced forest* of G. It is clear that $\phi(G) + I(G) = |V(G)|$ for any graph G. Consequently, if $\mathbf{d}$ is a graphic degree sequence of length n, then $n = \min(\phi, \mathbf{d}) + \max(I, \mathbf{d}) = \max(\phi, \mathbf{d}) + \min(I, \mathbf{d}) = \mathrm{Min}(\phi, \mathbf{d}) + \mathrm{Max}(I, \mathbf{d}) = \mathrm{Max}(\phi, \mathbf{d}) + \mathrm{Min}(I, \mathbf{d})$. Since ϕ is an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$ and $\mathcal{CR}(\mathbf{d})$, I is an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$ and $\mathcal{CR}(\mathbf{d})$. A *linear forest* is a forest with each component is a path.

3. Cubic graphs

It is easy to observe that the values of $\mathrm{Min}(\phi, r^n)$ and $\mathrm{Max}(\phi, r^n)$ are easily obtained for all $r \in \{0, 1, 2\}$. The problems of finding $\mathrm{Min}(\phi, 3^n)$ and $\mathrm{Min}(\phi, 3^n)$ are more difficult. We will consider such problems in terms of the graph parameter I. A *cubic tree* is a tree in which its vertices consisting of degree 1 or 3. It is easy to see that if T is a cubic tree of order n, then $n = 2k + 2$, where k is the number of vertices of degree 3 of T. Let $\mathbb{T}$ denote the family of cubic graphs obtained by taking cubic trees and replacing each vertex of degree 3 by a triangle and attaching a copy of K_4 with one subdivided edge (the graph K_4' in **Fig. 3.1**) at every vertex of degree 1.

It is easy to see that $\mathrm{Min}(I, 3^4) = 2$, $\mathrm{Min}(I, 3^6) = 4$, $\mathrm{Min}(I, 3^8) = 5$ and $\mathrm{Min}(I, 3^{10}) = 6$.

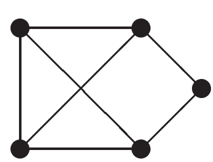
$K_4' =$ 

Fig. 3.1

A lower bound for the order of maximum induced forest in connected cubic graphs has been obtained by Liu and Zhao [9] as stated in the following theorem.

Theorem 3.1 *Let G be a connected cubic graph of order $n \geq 12$. Then $I(G) = \frac{5}{8}n - \frac{1}{4}$ if $G \in \mathbb{T}$ and $I(G) \geq \frac{5}{8}n$ if $G \notin \mathbb{T}$.* □

It is clear that if $G \in \mathbb{T}$, then G has order $8k + 10$, where k is the number of vertices of degree 3 in the corresponding cubic tree. Thus $I(G) = \mathrm{Min}(I, 3^{8k+10}) = 5k + 6$. We now consider a cubic graph of order $8k + 8$. Let C be a cubic graph of order $8k+8$. Then by Theorem 3.1, $I(C) \geq \frac{5}{8}(8k+8) = 5(k+1)$. A cubic graph T obtained by taking cubic tree with k vertices of degree 3, replacing $k - 1$ of the vertices by a triangle and attaching a copy of K at every vertex of degree 1. Thus T has order $8k + 8$ and $I(T) = 5(k+1)$. Thus $\mathrm{Min}(I, 3^{8k+8}) = 5(k+1)$. The value of $\mathrm{Min}(I, 3^n)$, $n = 8k + 4, 8k + 6$, can be obtained in the following argument. Since a switching changes the order of induced forest by at most 1, we have $\mathrm{Min}(I, 3^{p+q}) \leq \mathrm{Min}(I, 3^p) + \mathrm{Min}(I, 3^q) + 1$ for all even integers p and q with $4 \leq p \leq q$. Thus $5k+4 = \lceil \frac{5}{8}(8k+6) \rceil \leq \mathrm{Min}(I, 3^{8k+6}) \leq \mathrm{Min}(I, 3^4) + \mathrm{Min}(I, 3^{8(k-1)+10}) + 1 = 2 + 5(k-1) + 6 + 1 = 5k+4$. Finally $5k+3 = \lceil \frac{5}{8}(8k+4) \rceil \leq \mathrm{Min}(I, 3^{8k+4}) \leq \mathrm{Min}(I, 3^4) + \mathrm{Min}(I, 3^{8(k-1)+8}) + 1 = 2 + 5k + 1 = 5k+3$. Therefore we obtain the following theorem and corollary.

Theorem 3.2 *Let n be an even integer with $n \geq 12$. Then*

$$\mathrm{Min}(I, 3^n) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & if\ n \equiv 2 (\mathrm{mod}\ 8), \\ \lceil \frac{5}{8}n \rceil & otherwise. \end{cases}$$

□

Corollary 3.3 *Let n be an even integer with $n \geq 12$. Then*

$$\mathrm{Max}(\phi, 3^n) = \begin{cases} \frac{3}{8}n + \frac{1}{4} & if\ n \equiv 2 (\mathrm{mod}\ 8), \\ \lfloor \frac{3}{8}n \rfloor & otherwise. \end{cases}$$

□

Let H be a graph. A graph G is called an *H-free graph* if G does not contain H as an induced subgraph. Let X be a set of graphs. Then a graph G is called an *X-free graph* if for every $H \in X$, G is an H-free graph. In [14], there are five connected cubic graphs of order 8, all of which having maximum induced forests of order 5. Alon et al. proved in [1] that if G is a $\{K_4, K_4'\}$-free graph with maximum degree 3. If G is of order n and of size m, then $I(G) \geq n - \frac{m}{4}$. Consequently, if G is a cubic $\{K_4, K_4'\}$-free graph of order $n \geq 10$, then $I(G) \geq \frac{5n}{8}$. Zheng and Lu proved in [16] that $I(G) \geq \frac{2n}{3}$ for any connected cubic graph G of order n without triangles, except for two cubic graphs with $n = 8$. They also pointed out that this lower bound is best possible. It is easy to see that there exists cubic graph G of order n containing triangles and $I(G) \geq \frac{2n}{3}$. We extend their result by proving that $I(G) \geq \frac{2n}{3}$ for any connected cubic K_4'-free graph G of order $n \geq 10$.

Let P be a graph with $V(P) = \{v_0, v_1, \ldots, v_8\}$ and $E(P) = \{v_i v_{i+1} : i = 1, 2, \ldots, 8 \ (\mathrm{mod}\ 9)\} \cup \{v_1 v_4, v_5 v_8, v_2 v_7, v_3 v_6\}$. Thus P is a triangle-free graph of order 9 and of size 13. By Alon et al. [1], $I(P) \geq \lceil 9 - \frac{13}{4} \rceil = 6$. It is easy to find a set of 6 vertices of P, for example $\{v_0, v_1, v_2, v_3, v_5, v_6\}$, which is induced a forest. Therefore $I(P) = 6$.

By applying the result in [1] we find that if G is a connected K_4'-free graph of order 8 and $\Delta(G) = 3$, then $I(G) = 5$ if and only if G is a cubic graph.

Lemma 3.4 *Let G be a connected triangle-free graph of order n and $\Delta(G) = 3$. If G is not a cubic graph, then $I(G) \geq \frac{2n}{3}$.*

Proof. Suppose that G does not contain a vertex of degree 1. Thus G contains at least one vertex of degree 2. Let k be the number of vertices of degree 2 in the graph G and let $v_1, v_2, \ldots, v_k$ be the k vertices of degree 2. Let $P_1, P_2, \ldots, P_k$ be k graphs each of which is isomorphic to the graph P. The graph G^* can be constructed from G by adding an edge from $v_i (1 \leq i \leq k)$ to the vertex of degree 2 of $P_i (1 \leq i \leq k)$.

Therefore G^* is a cubic triangle-free graph of order $n+9k$. Since $\frac{2(n+9k)}{3} \leq I(G^*) = I(G) + kI(P)$ and $I(P) = 6$, $I(G) \geq \frac{2n}{3}$. Suppose that G contains a vertex v of degree 1. Thus, by induction on n, we have $I(G) \geq I(G-v)+1 \geq \frac{2(n-1)}{3} + 1 \geq \frac{2n}{3}$. □

Lemma 3.5 *Let* $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ *and let* G *be an* X*-free graph of order* n *with* $\Delta(G) = 3$. *Then* $I(G) \geq \frac{2n}{3}$.

Proof. Let $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ and let G be an X-free graph of order n. Then, by Alon et al. [1], $I(G) \geq \frac{5n}{8}$. By calculation we found that $\lceil \frac{5n}{8} \rceil = \lceil \frac{2n}{3} \rceil$ for all n with $4 \leq n \leq 10$ and $n \neq 8$. If $n = 8$, then G is not cubic. Thus we also have that $I(G) \geq \frac{2n}{3}$. Thus the lemma is verified for all n with $4 \leq n \leq 10$. Now suppose that $n \geq 11$. By Lemma 3.4, we may assume that G contains a triangle T with $V(T) = \{x, y, z\}$. If there exists a vertex in $V(T)$, say x such that $d_G(x) = 2$, then by induction on n there exists a maximum induced forest F_1 of $G - T$ with $|F_1| \geq \frac{2(n-3)}{3}$. Hence $F = F_1 \cup \{x, y\}$ is an induced forest of G and $|F| \geq \frac{2n}{3}$. Suppose that for all triangles $T = \{x, y, z\}$ of G, $d_G(x) = d_G(y) = d_G(z) = 3$. Since G is a K_4-free graph, $|N(T)| \geq 2$.

Case 1.
Suppose that x and y have a common neighbor u, and let v be the neighbor of z. Since G is a K_4'-free graph, u and v are not adjacent in G. Thus by induction on n, $G - T$ contains an induced forest of order at least $\frac{2(n-3)}{3}$. Since $d_{G-T}(u) \leq 1$, any maximum induced forest of $G - T$ must contain u. If there is a maximum induced forest F_1 of $G - T$ does not contain u, v-path, then $F_1 \cup \{y, z\}$ is an induced forest of G of order at least $\frac{2n}{3}$. Suppose that for any maximum induced forest F_1 of $G - T$, F_1 contains u, v-path. Since $G' = G - T + uv$ satisfies conditions of the lemma, there is a maximum induced forest F' of G' of order at least $\frac{2(n-3)}{3}$. If $uv \notin E(F')$, then $F = F' \cup \{y, z\}$ is a maximum induced forest of G of order at least $\frac{2n}{3}$. If $uv \in E(F')$, then $F = (F' - uv) \cup \{y, z\}$ is a maximum induced forest of G of order at least $\frac{2n}{3}$.

Case 2.
Suppose that x, y, z have different neighbors. Let u, v, w be the neighbors of x, y, z respectively. Put $G_1 = G[\{u, v, w\}]$. If $|E(G_1)| = 3$, then G is not connected and $I(G) \geq 4 + I(G - G_1) \geq 4 + 2(n-6)/3 = 2n/3$. Suppose that G_1 is not a triangle and suppose further that there exist two vertices in $\{u, v, w\}$, say u, v, such that $uv \notin E(G)$ and $G' = G - T + uv$ satisfies conditions of the lemma. By induction on n, there exists a maximum induced forest F_1 of G' of order at least $\frac{2(n-3)}{3}$. If $uv \notin E(F_1)$, then $F = F_1 \cup \{x, y\}$ is an induced forest of G of order at least $\frac{2n}{3}$. If $uv \in E(F)$, then $F = (F_1 - uv) \cup \{x, y\}$ is an induced forest of G of order at least $\frac{2n}{3}$. If $|E(H)| = 2$ and $uv \notin E(H)$, then $G' = G - T + uv$ satisfies conditions of the lemma. If $E(H) = \{vw\}$ and $G' = G - T + uw$ does not satisfy conditions of the lemma, then $G' = G - T + uv$ satisfies conditions of the lemma. Finally if $E(H) = \emptyset$, $G' = G - T + vw$ and $G' = G - T + uw$ do not satisfy conditions of the lemma, then $G' = G - T + uv$ satisfies conditions of the lemma. Thus the proof is complete.

□

We have the following theorem.

Theorem 3.6 *Let G be a connected cubic K_4'-free graph of order n, $n \geq 6$ and $n \neq 8$. Then $I(G) \geq \frac{2n}{3}$.* □

We proved in [10] the following theorem.

Theorem 3.7 *Let $r \geq 3$ and nr be even. Then*

$$\min(\phi, r^n) = \begin{cases} r-1 & if\ r+1 \leq n \leq 2r-1, \\ \lceil \frac{nr-2n+2}{2(r-1)} \rceil & if\ n \geq 2r. \end{cases}$$

□

Let G be a connected r-regular graph and S be a minimum decycling set of G. Since for any $v \in S$ there is a connected component C of $G - S$ such that v is adjacent to at least two vertices of C, there exists $u \in G - S$ such that $vu = e \in E(G)$ and $G - e$ is connected. Thus for two disjoint connected r-regular graphs G and H with minimum decycling set S and T of G and H respectively, there exist $u \in S$, $v \in G - S$, $x \in T$, $y \in H - T$ such that $uv = e \in E(G)$, $xy = f \in E(H)$ and $G - e$, $H - f$ are connected. A connected r-regular graph $K = ((G - e) \cup (H - f)) + \{ux, vy\}$ satisfies

$$\phi(K) \leq \phi(G \cup H) = \phi(G) + \phi(H),$$

and the following corollary holds.

Corollary 3.8 *Let $r \geq 3$ and nr be even. Then*

$$\text{Min}(\phi, r^n) = \begin{cases} r-1 & if\ r+1 \leq n \leq 2r-1, \\ \lceil \frac{nr-2n+2}{2(r-1)} \rceil & if\ n \geq 2r. \end{cases}$$

□

Thus the values of $\text{Min}(\phi, r^n)$ for all r and n are already obtained. In particular the values of $\text{Min}(\phi, 3^{2n})$ and $\text{Max}(\phi, 3^{2n})$ are found for all n.

4. $\text{Max}(\phi, r^n)$

We will discuss the problem of determining the values of $\text{Max}(\phi, r^n)$ for $r \geq 4$ in this section. Note that $\mathcal{R}(r^n) = \mathcal{CR}(r^n)$ if and only if $r + 1 \leq n \leq 2r + 1$. Thus $\text{Max}(\phi, r^n) = \max(\phi, r^n)$ for all $n \in \{r + 1, r + 2, \ldots, 2r + 1\}$. In this case we have already obtained in [10] as stated in the following theorem.

Theorem 4.1 *For $r \geq 4$, and $n = r + j$, $1 \leq j \leq r + 1$, then*
(1) $\max(\phi, r^n) = n - 2$, *if and only if $j = 1$,*
(2) $\max(\phi, r^n) = n - 3$, *if and only if $j = 2$,*
(3) $\max(\phi, r^n) = n - 4$, *for all even integers $n = r + j$, $3 \leq j \leq r + 1$,*
(4) $\max(\phi, r^n) = n - 4$, *for all odd integers $n = r + j$, $3 \leq j \leq r + 1$ and $n \geq f(j)$,*

(5) $\max(\phi, r^n) = n - 5$, *for all odd integers* $n = r + j$, $3 \le j \le r + 1$ *and* $n < f(j)$, *where* $f(j) = \frac{5}{2}(j-1)$ *if* $j \equiv 3 \pmod 4$, *and* $f(j) = 1 + \frac{5}{2}(j-1)$ *if* $j \equiv 1 \pmod 4$. □

Thus the values of $\mathrm{Max}(\phi, r^n)$ are already obtained for all r and n with $n \le 2r + 1$. The problem of determining the decycling number of a graph is equivalent to finding the greatest order of an induced forest and the sum of the two numbers equals the order of the graph. In particular $\mathrm{Max}(\phi, r^n) = n - \mathrm{Min}(I, r^n)$.

Let G be a K_5-free graph of order n, $\Delta(G) = 4$. Let F be a maximal induced forest of G. We denote by $c(F)$ the number of cycles in $G - F$. A pair (X, Y), where $X \subseteq F$ and $Y \subseteq G - F$, is an *interchangeable pair of vertices with respect to* F if $(F - X) \cup Y$ is a forest, $|(F - X) \cup Y| \ge |F|$, and $c((F - X) \cup Y) < c(F)$. In general we can define an interchangeable pair of vertices for a graph G with $\Delta(G) > 4$ as follows. Let G be a $K_{\Delta+1}$-free graph of order n with $\Delta(G) = \Delta > 4$. Let F be a maximal induced forest of G. We denote by $k(F)$ the number of $K_{\Delta-1}$ in $G - F$. A pair (X, Y), where $X \subseteq F$ and $Y \subseteq G - F$, is an *interchangeable pair of vertices with respect to* F if $(F - X) \cup Y$ is a forest, $|(F - X) \cup Y| \ge |F|$, and $k((F - X) \cup Y) < k(F)$.

Let G be a K_5-free graph of order n and $\Delta(G) = 4$. Thus for any maximal induced forest F of G, $G - F$ is a union of cycles and paths. We choose a maximal induced forest F of G with minimum $c(F)$. In other word, the forest F is chosen in such a way that it contains no interchangeable pair of vertices with respect to F. Suppose that $c(F) \ge 1$. Let C be a cycle in $G - F$. Then each vertex of C must be adjacent to exactly two vertices in F. Suppose that there exists a vertex $u \in F$, $d_F(u) \ge 2$, and u is adjacent to a vertex $v \in V(C)$, then $(\{u\}, \{v\})$ is an interchangeable pair of vertices with respect to F. Thus for all cycles C of $G - F$, each vertex $v \in V(C)$, v must be adjacent to exactly two vertices $u_1, u_2 \in F$ with $d_F(u_1) = d_F(u_2) = 1$. By maximality of F, u_1 and u_2 must be in the same connected component of F. Since F is a forest, there exists a unique path in $G[F]$ from u_1 to u_2. If u_1 and u_2 are not adjacent and there is a vertex $u \in F$ in the path such that $d_F(u) \ge 3$, then $(\{u\}, \{v\})$ is an interchangeable pair of vertices with respect to F. Therefore the connected component of F containing u_1 and u_2 must be a path. Suppose that there exist exactly two vertices v, w of C adjacent to a vertex $u \in F$, then $(\{u\}, \{v\})$ is an interchangeable pair of vertices with respect to F. Finally suppose that there are three vertices v, w, z of C adjacent to a vertex $u \in F$, then the path P in $G[F]$ containing u has order at least 3 or the cycle C has order at least 4, since otherwise G would contain K_5. Let u and u' be the end vertices of P in $G[F]$. Then $N_C(\{u, u'\}) = \{v, w, z\} \subseteq V(C)$. If P has order at least 3, then $(\{u, u'\}, \{v, w\})$ is an interchangeable pair of vertices with respect to F. If P has order 2 and C has order at least 4, then $(\{u, u'\}, \{v, w, z\})$ is an interchangeable pair of vertices with respect to F. Thus the corresponding paths in $G[F]$ of vertices in C are pairwise disjoint.

Theorem 4.2 *Let G be a K_5-free graph of order n, $\Delta(G) = 4$. Then $\phi(G) \le \frac{n}{2}$.*

Proof. We may assume that G is connected K_5-free graph of order n and $\Delta(G) = 4$.

If G contains a maximal induced forest F with $G-F$ is a forest, then $\phi(G) \leq \frac{n}{2}$. Suppose for each maximal induced forest F of G, $G-F$ contains at least one cycle. Choose a maximal induced F of G with minimum $c(F)$. Let C be a cycle in $G-F$ with $V(C) = \{v_0, v_1, \ldots, v_{k-1}\}$ and $E(C) = \{v_i v_{i+1} : i = 0, 1, \ldots, k-1 (\text{mod } k)\}$. From above observation, for each $i = 0, 1, \ldots, k-1$, let $P(v_i)$ be the corresponding path in $G[F]$ containing $N_F(v_i) = \{u_{i1}, u_{i2}\}$ as its end vertices. Note that the paths $P(v_0), P(v_1), \ldots, P(v_{k-1})$ are pairwise disjoint. Moreover, since G has no interchangeable pair of vertices with respect to F, for each $u_{ij} (0 \leq i \leq k-1, 1 \leq j \leq 2)$, there is a corresponding path $P(u_{ij})$ in $G-F$ and all the corresponding paths are pairwise disjoint.

Case 1.
If k is even, we can form a new graph G' in which $V(G') = V(G-C)$ and $E(G') = E(G-C) \cup \{u_{i1}u_{(i+1)1}, u_{i2}u_{(i+1)2} : i = 0, 2, 4, \ldots, k-2\}$. Since the corresponding paths $P(u_{ij})$ are pairwise disjoint, the graph G' is a K_5-free graph of order $n-k$. By induction, G' contains an induced forest F' of order at least $\frac{n-k}{2}$. Since $G'[V(P(v_i)) \cup V(P(v_{i+1}))]$ is a cycle for all $i = 0, 2, 4, \ldots, k-2$, there exists $u \in V(P(v_i)) \cup V(P(v_{i+1}))$ such that $u \notin F'$. If $u \in V(P(v_i))$, then $F' \cup \{v_i\}$ is a forest of G. Similarly if $u \in V(P(v_{i+1})$. Thus $\phi(G) \leq \frac{n}{2}$.

Case 2.
If k is odd and there exists i such that $P(v_i)$ has order at least three, then we can analogously form a graph G' in which $V(G') = V(G-C)$, pairing the $k-1$ paths $P(v_j)$ with $j \neq i$, and adding $u_{i1}u_{i2}$ to the edge set of G'. The proof follows by similar argument as in Case 1.

Case 3.
If k is odd and for each $i = 0, 1, 2, \ldots k-1$, $P(v_i)$ has order two, then since G has no interchangeable pair of vertices, for each $i = 0, 1, 2, \ldots, k-1$ and $j = 1, 2$, there is a path $P(u_{ij})$ in $G-F$ and $P(u_{ij})$ has $N_{G-F}(u_{ij})$ as its end vertices. Put $N_{G-F}(u_{ij}) = \{v_{ij(1)}, v_{ij(2)}\}$. Moreover, the paths $P(u_{ij})$ are pairwise disjoint. We can now form a graph G' in which $V(G') = V(G) - (C \cup N_F(V(C)))$ and $E(G') = E(G[V(G')] \cup E_1$, where $E_1 = \{v_{i1(1)}v_{i2(1)}, v_{i1(2)}v_{i2(2)} : i = 0, 1, 2, \ldots, k-1\}$. By induction on n, there exists an induced forest F' of G' with $|F'| \geq \frac{n-3k}{2}$. Since for each $i = 0, 1, 2, \ldots, k-1$, F' can not contain all vertices in $V(P(u_{i1})) \cup V(P(u_{i2}))$, there exists $v \in V(P(u_{i1})) \cup V(P(u_{i2}))$ such that $v \notin F'$. If $v \in V(P(u_{i1})$, then $F' \cup \{u_{i1}, v_i\}$ is an induced forest of G. Similarly if $v \in V(P(u_{i2}))$. Thus there is a set X containing either u_{i1} or u_{i2}, but not both, according to $F' \cup \{u_{i1}, v_i\}$ or $F' \cup \{u_{i2}, v_i\}$ is an induced forest of G. Therefore $F' \cup \{v_0, v_1, \ldots, v_{k-2}\} \cup X$ is an induced forest of G of order at least $\frac{n-3k}{2} + 2k - 1 \geq \frac{n}{2}$. This completes the proof. □

Corollary 4.3 $\text{Max}(\phi, 4^n) \leq \frac{n}{2}$. □

Let G be a $K_{\Delta+1}$-free graph with $\Delta(G) = \Delta \geq 5$ and let F be a maximal induced of G with minimum $k(F)$. Then for each $v \in G-F$, there exists a connected component T of F such that v is adjacent to at least two vertices of T. Thus $\Delta(G-F) \leq \Delta - 2$. Suppose that $k(F) \geq 1$. Let K be a complete subgraph of $G-F$ of order $\Delta - 1$.

Put $V(K) = \{v_1, v_2, \ldots, v_{\Delta-1}\}$. Thus for each v_i there exists a connected component $P(v_i)$ of $G[F]$ such that v_i is adjacent to exactly two vertices of $P(v_i)$. If there exists $u \in V(P(v_i))$ such that $uv_i \in E(G)$ and $d_F(u) \geq 2$, then $(\{u\}, \{v_i\})$ is an interchangeable pair of vertices with respect to F. Thus for each $i = 1, 2, \ldots, \Delta - 1$, v_i must be adjacent to exactly two vertices of degree one in $P(v_i)$. Suppose that there exists $u \in V(P(v_i))$ such that $d_F(u) \geq 3$, $(\{u\}, \{v_i\})$ is an interchangeable pair of vertices with respect to F. Thus the corresponding $P(v_i)$ of v_i in K is a path. Furthermore all such paths are pairwise disjoint.

Lemma 4.4 *Let G be a $K_{\Delta+1}$-free graph of order n with $\Delta(G) = \Delta \geq 5$. Then $I(G) \geq \frac{2n}{\Delta}$ or there exists an induced forest F of G such that $G - F$ is a $K_{\Delta-1}$-free graph.*

Proof. With above observation in mind, suppose that for all maximal induced forests F of G, $k(F) \geq 1$. Let F be a maximal induced forest of G with minimum $k(F)$ and let K be a complete subgraph of $G - F$ of order $\Delta - 1$. Put $V(K) = \{v_1, v_2, \ldots, v_{\Delta-1}\}$. Let $P(v_i)$ be defined as above and let u_{i1}, u_{i2} be the two vertices with degree one of $P(v_i)$. We now form a graph G' with $V(G') = V(G - K)$ and $E(G') = E(G[V(G')]) \cup X$, where $X = \{u_{11}u_{21}, u_{12}u_{22}, u_{31}u_{41}, u_{32}u_{42}\}$. By induction on n, $I(G') \geq \frac{2n-\Delta+1}{\Delta}$ or there exists an induced forest F' of G' such that $G' - F'$ is a $K_{\Delta-1}$-free graph. Since F' does not contain all vertices of $P(v_1) \cup P(v_2)$ and likewise of $P(v_3) \cup P(v_4)$, there exist two vertices $x, y \in \{v_1, v_2, v_3, v_4\}$ such that $F'' = F' \cup \{x, y\}$ is an induced forest of G and $G - F''$ is a $K_{\Delta-1}$-free graph. This means that if $G' - F'$ is a $K_{\Delta-1}$-free graph, then there exists a maximal induced forest F'' of G such that $G - F''$ is a $K_{\Delta-1}$-free graph. Thus we may assume that $|F'| \geq \frac{2n-\Delta+1}{\Delta}$. Hence there exist two vertices $x, y \in \{v_1, v_2, v_3, v_4\}$ such that $F'' = F' \cup \{x, y\}$ is an induced forest of G and $|F''| \geq \frac{2n-\Delta+1}{\Delta} + 2 \geq \frac{2n}{\Delta}$. This completes the proof. □

Lemma 4.5 *Let G be a connected K_5-free graph of order n and $\Delta(G) = 5$. Then $I(G) \geq \frac{2n}{5}$.*

Proof. *Case 1.*

Suppose that for each maximal induced forest F of G, $G - F$ contains K_4 as its component. Choose maximal induced forest F of G with minimum $k(F)$. Let K be a copy K_4 in $G - F$ and $V(K) = \{v_0, v_1, v_2, v_3\}$. Since G is connected and G does not have an interchangeable pair of vertices with respect to F, for each vertex v_i, there is a path $P(v_i)$ of F such that v_i is adjacent to two vertices with degree one of $P(v_i)$ and for any two distinct vertices v_i and v_j, $P(v_i)$ and $P(v_j)$ are disjoint. For each v_i, let u_{i1} and u_{i2} be the end vertices of $P(v_i)$, $i = 0, 1, 2, 3$. We now form a graph G' with $V(G') = V(G - K)$ and $E(G') = E(G - K) \cup E_1$, where $E_1 = \{u_{01}u_{11}, u_{02}u_{12}, u_{21}u_{31}, u_{22}u_{32}\}$. By induction on n, G' contains an induced forest F' of G' such that $|F'| \geq \frac{2(n-4)}{5}$. It is clear by forming the graph G' that there exist two distinct vertices v_i and v_j such that $F' \cup \{v_i, v_j\}$ is an induced forest of G and of order at least $\frac{2n}{5}$.

Case 2.

Suppose that for each induced forest F of G, $G-F$ contains at least one $H \in \mathcal{CR}(3^8)$ as its component. Choose a maximal induced forest F of G in such a way that F contains minimum number of copies of graphs in $H \in \mathcal{CR}(3^8)$. Let K be a copy of graph in $\mathcal{CR}(3^8)$ in $G-F$ and put $V(K) = \{v_0, v_1, \ldots, v_7\}$. By choosing F in this way, we have for each v_i, there is a path $P(v_i)$ of F such that v_i is adjacent to the end vertices of $P(v_i)$. Moreover for any two distinct v_i and v_j, $P(v_i)$ and $P(v_j)$ are disjoint. Let u_{i1}, u_{i2} be the end vertices of $P(v_i)$, $i = 0, 1, 2, \ldots, 7$. We now form a graph G' with $V(G') = V(G-K)$ and $E(G') = E(G-K) \cup E_1$, where $E_1 = \{u_{i1}u_{(i+1)1}, u_{i2}u_{(i+1)2} : i = 0, 2, 4, 6\}$. By induction on n, G' contains an induced forest F' of order at least $\frac{2(n-8)}{5}$. It is clear by forming the graph G' that there are at least four vertices $x_1, x_2, x_3, x_4 \in V(K)$ such that $F' \cup \{x_1, x_2, x_3, x_4\}$ forms an induced forest of G of order at least $\frac{2n}{5}$.

Case 3.
Suppose that for each induced forest F of G, $G-F$ contains at least one copy K_4' as its induced subgraph. Choose a maximum induced forest F of G in such a way that $G-F$ contains minimum number of copies of K_4'. Let K be a copy of K_4' in $G-F$. Put $V(K) = \{v_0, v_1, v_2, v_3, v_4\}$ and $d_K(v_0) = 2$. For each $v_i, i = 1, 2, 3, 4$, there exists a connected component $P(v_i)$ of F such that v_i is adjacent to exactly two vertices of $P(v_i)$. Again let $\{u_{i1}, u_{i2}\}$ be two vertices of $P(v_i)$ that are adjacent by v_i, $i = 1, 2, 3, 4$. Since K is not a cubic graph, $P(v_i)$ and $P(v_j)$ may not be disjoint. It is clear by choosing the minimum number of copies of K_4' in $G-F$ that for each vertex u of $P(v_i)$, $i = 1, 2, 3, 4$, there are at most two vertices of K that are adjacent to u. Let G' be a graph with $V(G') = V(G-K)$ and $E(G') = E(G-K) \cup E_1$. A graph G' will be formed according to the following cases. We then apply induction to each such a forming of G', there exists an induced forest F' of G' such that F' together with two vertices of K forms an induced forest of order at least $\frac{2n}{5}$. Put $N = \cup_{i=1}^{4} N_F(v_i)$. For two disjoint nonempty subsets X, Y of $V(G)$ we denote $e(X, Y)$ the number of edges in G joining between X and Y. Note that F was chosen as a maximal induced forest of G with minimum $k(F)$, we have the following observation.

1. $4 \leq |N| \leq 8$.

2. $d_F(u) \in \{1, 2\}$, for all $u \in N$.

3. For each $u \in N$, there are at most two vertices in $\{v_1, v_2, v_3, v_4\}$ that are adjacent to u.

4. If $u \in N$ and $d_F(u) = 2$, then there is exactly one vertex in $\{v_1, v_2, v_3, v_4\}$ that is adjacent to u.

With the above observation in mind, suppose $|N| = 4$. Thus we may assume that $u_{11} = u_{21}$, $u_{12} = u_{22}$, $u_{31} = u_{41}$ and $u_{32} = u_{42}$. We can choose $E_1 = \{u_{11}u_{31}, u_{11}u_{32}, u_{12}u_{31}, u_{12}u_{32}\}$.

Suppose $|N| = 8$ and suppose further that there exist i and i' with $i \neq i'$ such that $e(\{u_{i1}\}, \{u_{i'1}, u_{i'2}\}) = 2$. Then u_{i1}, u_{i2} and $u_{i'1}, u_{i'2}$ are not adjacent. We can choose $E_1 = \{u_{i1}u_{i2}, u_{i'1}u_{i'2}\}$. Suppose that for i and i', $e(\{u_{i1}\}, \{u_{i'1}, u_{i'2}\}) \leq 1$. We may assume without loss of generality that for pairs i, i' with $1 \leq i < i' \leq 4$, u_{ij} and $u_{i'j}$

are not adjacent. Thus we can choose $E_1 = \{u_{11}u_{21}, u_{12}u_{22}, u_{31}u_{41}, u_{32}u_{42}\}$.

Suppose $4 < |N| < 8$ and suppose further that $u_{11} = u_{21}, u_{12} = u_{22}$. Thus $d_F(u_{11}) = d_F(u_{12}) = 1$. We can choose $E_1 = \{u_{11}u_{31}, u_{11}u_{32}, u_{12}u_{41}, u_{12}u_{42}\}$. Finally suppose that any pair of $1 \le i < i' \le 4$, $|N_F(v_i) \cap N_F(v_{i'})| \le 1$. Since $|N| < 8$, we may assume without loss of generality that $u_{11} = u_{21}$ and $u_{12} \ne u_{22}$. Thus u_{11}, u_{12}, u_{22} lie in the same component of F. Since $d_F(u_{11}) = 1$, $d_F(u_{12}) = 2$ or $d_F(u_{22}) = 2$. Suppose that $d_F(u_{12}) = 2$ and $d_F(u_{22}) = 2$, then $\{u_{11}, u_{12}, u_{22}\} \cap \{u_{31}, u_{32}, u_{41}, u_{42}\} = \emptyset$. We can choose $E_1 = \{u_{11}u_{31}, u_{11}u_{32}, u_{12}u_{41}u_{22}u_{42}\}$ or $E_1 = \{u_{11}u_{31}, u_{11}u_{32}, u_{12}u_{42}u_{22}u_{41}\}$. Suppose that $d_F(u_{12}) = 2$ and $d_F(u_{22}) = 1$. Thus $\{u_{11}, u_{12}\} \cap \{u_{31}, u_{32}, u_{41}, u_{42}\} = \emptyset$. If $\{u_{11}, u_{12}, u_{22}\} \cap \{u_{31}, u_{32}, u_{41}, u_{42}\} = \emptyset$, then we can choose E_1 as in the previous case. If $u_{22} \in \{u_{31}, u_{32}, u_{41}, u_{42}\}$, say $u_{22} = u_{32}$, we can choose $E_1 = \{u_{11}u_{41}, u_{11}u_{42}, u_{12}u_{32}\}$ or $E_1 = \{u_{11}u_{41}, u_{11}u_{42}, u_{12}u_{31}\}$.

Case 4.
Suppose that $G - F$ is an X-free graph, where $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$. If $|F| < \frac{2n}{5}$, then, by Lemma 3.5, $G - F$ contains an induced forest F' of order at least $\frac{2(n-|F|)}{3} \ge \frac{2(n-\frac{2}{5}n)}{3} \ge \frac{2n}{5}$. This completes the proof. □

Lemma 4.6 *Let G be a K_6-free graph of order n with $\Delta(G) = 5$. Then $I(G) \ge \frac{2n}{5}$.*

Proof. If G does not contain K_5 as a subgraph, then the result follows from Lemma 4.5. Suppose that G contains K_5 as a subgraph. Put $V(K) = \{v_1, v_2, \ldots v_5\}$. Since G is a K_6-free graph, $N_{G-K}(K)$ contains at least two vertices. If there exists a maximum induced forest F_1 of $G - K$ such that $N_{G-K}(K) \not\subseteq F_1$ or $G[N_{F_1}(K)]$ is disconnected, then there exist $x, y \in V(K)$ such that $F_1 \cup \{x, y\}$ is an induced forest of G. By induction on n, we have $I(G) \ge |F_1| + 2 \ge \frac{2(n-5)}{5} + 2 = \frac{2n}{5}$. We now suppose that each maximum induced forest F of $G - K$, $G[N_F(K)]$ is a connected component of $G[F]$. Suppose further that $N_F(K)$ contains exactly two elements $x, y \in F$. Put $L = K \cup \{x, y\}$ and $H = G[L]$. Thus $d_H(x) \ge 4$ or $d_H(y) \ge 4$. Thus $I(G) \ge I(L) + I(G-L) \ge 3 + \frac{2(n-7)}{5} \ge \frac{2n}{5}$. If $N_F(K)$ contains more than two elements, we can form a graph G' with $V(G') = V(G) - K$ and $E(G') = E(G-K) \cup \{e\}$, where e is a new edge connecting two vertices in $N_F(K)$. By induction on n, G' contains an induced forest F' of order at least $\frac{2(n-5)}{5}$. Since F' can not contain all vertices of $N_F(K)$, there exist $v_i, v_j \in V(K)$ such that $F'' = F' \cup \{v_i, v_j\}$ is an induced forest of G and $|F''| \ge \frac{2n}{5}$. □

As a direct consequence of Lemma 4.6 we have the following theorem.

Theorem 4.7 *Let G be a connected 5-regular graph of order $n \ge 12$. Then $\phi(G) \le \frac{3n}{5}$.* □

Lemma 4.8 *Let G be a $K_{\Delta+1}$-free graph of order n with $\Delta(G) = \Delta \ge 5$. Then $I(G) \ge \frac{2n}{\Delta}$.*

Proof. The result follows for $\Delta = 4, 5$. Suppose that $\Delta \ge 6$, by Lemma 4.4 we have $I(G) \ge \frac{2n}{\Delta}$ or there exists an induced forest F of G such that $G - F$ is a $K_{\Delta-1}$-free graph. If $|F| < \frac{2n}{\Delta}$, then $|G - F| > \frac{(\Delta-2)n}{\Delta}$. Since $G - F$ is a $K_{\Delta-1}$-free graph, we have $I(G) \geq I(G - F) \geq \frac{\frac{2(\Delta-2)n}{\Delta}}{\Delta-2} = \frac{2n}{\Delta}$. □

We have the following theorem.

Theorem 4.9 *Let G be a connected r-regular graph of order $n \geq 2r + 2$. Then $\phi(G) \leq \frac{n(r-2)}{r}$ for all $r \geq 4$.* □

Let G be a connected r-regular graph of order $n = rq + t$, $0 \leq t \leq r - 1$, $r \geq 4$ and $q \geq 1$. Then by theorem 4.9, we have $I(G) \geq 2q + \lceil \frac{2t}{r} \rceil$. It is easy to construct a connected r-regular graph G of order n with $I(G) = 2q$ if $t = 0$, $IG) = 2q + 1$ if $t = 1, 2$ and $I(G) = 2q+2$ if $3 \leq t \leq r-1$. Consequently, we have $\mathrm{Max}(\phi, r^n) = n-2q$ if $t = 0$, $\mathrm{Max}(\phi, r^n) = n - 2q - 1$ if $t = 1, 2$, $\mathrm{Max}(\phi, r^n) = n - 2q - 2$ if $2t > r$ and $\mathrm{Max}(\phi, r^n) \in \{n - 2q - 2, n - 2q - 1\}$ if $3 \leq t \leq \frac{r}{2}$.

We close this paper with the following conjecture.

Conjecture $\mathrm{Max}(\phi, r^n) = n - 2q - 2$ *if* $3 \leq t \leq \frac{r}{2}$, *for all* $r \geq 6$. □

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