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Upper bounds on the domination number of a graph in terms of order, diameter and minimum degree

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Abstract

A vertex set D of a graph G is a dominating set if every vertex not in D is adjacent to some vertex in D. The domination number γ of a graph G is the minimum cardinality of a dominating set in G. A cycle of length four is denoted by C_4. This paper is concerned with upper bounds for γ as a function of invariants such as order n, minimum degree δ, and diameter d.

If G is a connected C_4-free graph of minimum degree $\delta \geq 1$, then Brigham and Dutton, *Quart. J. Math. Oxford* 41 (1989), 269–275 proved that

$$\gamma \leq \frac{1}{2}\left(n - \frac{\delta(\delta-1)}{2}\right)$$

and if $\delta \geq 3$, then

$$2\gamma \leq n - 1 - (\delta - 1)(\lfloor d/3 \rfloor - 1 + \delta/2).$$

Recently, Volkmann, *J. Combin. Math. Combin. Comput.* 52 (2005), 131-141, gave the following related bound. Let G be a connected graph of minimum degree $\delta \geq 4$. If G does not contain the 4-cycle and the diamond (a 4-cycle with a chord) as induced subgraphs, then

$$2\gamma \leq n - 1 - (\delta - 3)(1 + \lfloor d/2 \rfloor) - \lfloor d/2 \rfloor/\delta.$$

In this paper we present different improvements of these three bounds.

1 Terminology and introduction

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* $N(v) = N(v, G)$ of the vertex v consists of the vertices adjacent to v, and the *closed neighborhood* of v is $N[v] = N[v, G] = N(v) \cup \{v\}$. For a subset $S \subseteq V(G)$, we define $N(S) = N(S, G) = \bigcup_{v \in S} N(v)$ and $N[S] = N[S, G] = N(S) \cup S$. vertex v is an *isolated vertex* if $d(v, G) = 0$, where $d(v) = d(v, G) = |N(v)|$ is the *degree* of $v \in V(G)$. By $\delta = \delta(G)$ we denote the *minimum degree* of the graph G. Furthermore, the *diameter* $d = d(G)$ of a graph G is the maximum distance between two vertices of G. We write C_n for a cycle of length n and K_n for the complete graph of order n. A cycle with length n is also called an n-cycle. A graph is C_4*-free* if it contains no subgraph isomorphic to the cycle C_4.

A set $D \subseteq V(G)$ is a *dominating set* of G if $N[D, G] = V(G)$. The *domination number* $\gamma = \gamma(G)$ of G is the cardinality of any smallest dominating set of G. A set S of vertices is *independent* if every two vertices of S are not adjacent. The *independence number* $\alpha(G) = \alpha$ of a graph G is the maximum cardinality among the independent sets of vertices of G.

For detailed information on domination and related topics see the comprehensive monograph [3] by Haynes, Hedetniemi, and Slater. The following three results are known.

Theorem 1.1 (Brigham, Dutton [2] 1989) If G is a C_4-free graph of minimum degree $\delta \geq 1$, then

$$\gamma \leq \frac{1}{2}\left(n - \frac{\delta(\delta-1)}{2}\right).$$

Theorem 1.2 (Brigham, Dutton [2] 1989) If G is a connected and C_4-free graph of minimum degree $\delta \geq 3$, then

$$\gamma \leq \frac{n - 1 - (\delta-1)(\lfloor d/3 \rfloor - 1 + \delta/2)}{2}.$$

Theorem 1.3 (Volkmann [9] 2005) Let G be a connected graph of minimum degree $\delta \geq 4$. If G does not contain the 4-cycle and the diamond (a 4-cycle with a chord) as induced subgraphs, then

$$\gamma \leq \frac{n - 1 - (\delta-3)(1 + \lfloor d/2 \rfloor) - \lfloor d/2 \rfloor/\delta}{2}.$$

In this paper we present different improvements of Theorems 1.1, 1.2, and 1.3 and some related bounds.

2 Preliminary results

The following well-known results play an important role in our investigations.

Proposition 2.1 (Ore [6] 1962) If G is a graph without isolated vertices, then $\gamma \leq n/2$.

Theorem 2.2 (McCuaig, Shepherd [5] 1989) Let G be a connected graph of minimum degree $\delta \geq 2$. Then $\gamma \leq 2n/5$, unless $G = C_7$ or G belongs to a family of 6 graphs of order at most 7 which all contain a C_4 as a subgraph.

Corollary 2.3. If G is a C_4-free graph of minimum degree $\delta \geq 2$, then $\gamma \leq 3n/7$.

The following proposition is well-known and can be found, for example, in [4].

Proposition 2.4 If G is a C_4-free graph, then

$$n \geq \delta^2 - \delta + 1.$$

Theorem 2.5 (Reed [8] 1996) If G is a graph of minimum degree $\delta \geq 3$, then $\gamma \leq 3n/8$.

Theorem 2.6 (Arnautov [1] 1974, Payan [7] 1975) If G is a graph without isolated vertices, then

$$\gamma \leq n \cdot \frac{1 + \ln(\delta + 1)}{\delta + 1}.$$

3 Upper bounds in terms of order and minimum degree

Theorem 3.1 If G is a C_4-free graph of minimum degree $\delta \geq 2$, then

$$\begin{aligned} \gamma &\leq \delta - 2 + \frac{3}{7}\Big(n - \frac{(\delta-2)(\delta+5)}{2}\Big) \\ &= \frac{3}{7}\Big(n - \frac{(3\delta+1)(\delta-2)}{6}\Big). \end{aligned} \quad (1)$$

Proof. *Case 1.* Assume that $\delta \geq 3$ and $\gamma \geq \delta - 2$. Because of the well-known fact that $\gamma \leq \alpha$, there exists an independent set of vertices X such that $|X| = \delta - 2$. Define the subgraph H by $H = G - N[X]$.

Subcase 1.1. Assume that $H = \emptyset$. It follows that $\gamma \leq \delta - 2$. Since G is C_4-free, we deduce from Proposition 2.4 that

$$2n \geq 2\delta^2 - 2\delta + 2 \geq \delta^2 + 3\delta - 10$$

and thus the desired inequality

$$\gamma \leq \delta - 2 \leq \delta - 2 + \frac{3}{7}\Big(n - \frac{(\delta-2)(\delta+5)}{2}\Big).$$

Subcase 1.2. Assume that $H \neq \emptyset$. Since G is C_4-free, we observe that

$$d_H(v) \geq d_G(v) - (\delta - 2) \geq \delta - (\delta - 2) = 2$$

for an arbitrary vertex v in H. Applying Corollary 2.3 on the subgraph H, we obtain

$$\gamma \leq \delta - 2 + \frac{3}{7}(n - |N[X]|). \tag{2}$$

Since X is an independent set and G is C_4-free, it follows that

$$\begin{aligned} |N[X]| &= \left|\bigcup_{x\in X} N(x)\right| + \delta - 2 \\ &\geq \sum_{x\in X} |N(x)| - \frac{(\delta-3)(\delta-2)}{2} + \delta - 2 \\ &\geq (\delta-2)\delta - \frac{(\delta-3)(\delta-2)}{2} + \delta - 2 \\ &= \frac{(\delta-2)(\delta+5)}{2}. \end{aligned}$$

This implies together with inequality (2) the desired bound

$$\gamma \leq \delta - 2 + \frac{3}{7}\left(n - \frac{(\delta-2)(\delta+5)}{2}\right).$$

Case 2. Assume that $\delta \geq 3$ and $\gamma \leq \delta - 3$. Analogously to Subcase 1.1, we arrive at the desired inequality (1).

Case 3. Assume that $\delta = 2$. Corollary 2.3 leads to $\gamma \leq \frac{3n}{7}$, and this is exactly the bound (1) for $\delta = 2$. □.

Using Proposition 2.4, it is straightforward to verify that in the case that $\delta \geq 2$, inequality (1) is better than Theorem 1.1 by Brigham and Dutton [2].

4 Upper bounds in terms of order, diameter and minimum degree

Our first result in this section is an improvement of Theorem 1.3.

Theorem 4.1 Let G be a connected graph of minimum degree $\delta \geq 4$. If G does not contain the 4-cycle and the diamond as induced subgraphs, then

$$\begin{aligned} \gamma &\leq 1 + \lfloor d/2 \rfloor + \frac{1}{2}(n - 1 - \delta(1 + \lfloor d/2 \rfloor)) \\ &= \frac{1}{2}(n - 1 - (\delta - 2)(1 + \lfloor d/2 \rfloor)). \end{aligned}$$

Proof. Let $d = 2t + r$ with $0 \leq r \leq 1$ and let $x_0x_1 \ldots x_d$ be a minimum length path between the vertices x_0 and x_d. If $A = \{x_0, x_2, \ldots, x_{2t}\}$, then $|A| = 1 + \lfloor d/2 \rfloor = 1 + t$. Since G does not contain the 4-cycle and the diamond as induced subgraphs, we conclude that $N(A) \cap A = \emptyset$. If we define $H = G - N[A]$ and note that A dominates $N[A]$, then we observe that

$$\gamma = \gamma(G) \leq 1 + \lfloor d/2 \rfloor + \gamma(H). \tag{3}$$

Furthermore, the fact $N(A) \cap A = \emptyset$ implies

$$\begin{aligned} |N[A]| &= \left|\bigcup_{i=0}^{t} N[x_{2i}]\right| = \left|\bigcup_{i=0}^{t} N(x_{2i})\right| + |A| \\ &= \sum_{i=0}^{t} |N(x_{2i})| - t + |A| \geq \delta|A| + 1 \end{aligned}$$

and thus $n(H) \leq n - (\delta|A| + 1)$. Now any vertex of H can have, in G, at most three neighbors in $N[A]$, because otherwise, we would obtain a 4-cycle or a diamond as an induced subgraph or a shorter path between x_0 and x_d. Hence $\delta(H) \geq \delta - 3 \geq 1$. It follows from Proposition 2.1 and (3) that

$$\begin{aligned} \gamma = \gamma(G) &\leq 1 + \lfloor d/2 \rfloor + \gamma(H) \\ &\leq 1 + \lfloor d/2 \rfloor + \frac{n - (\delta|A| + 1)}{2} \\ &= 1 + \lfloor d/2 \rfloor + \frac{1}{2}(n - 1 - \delta(1 + \lfloor d/2 \rfloor)). \quad \square \end{aligned}$$

Applying Corollary 2.3 or Theorem 2.5 instead of Proposition 2.1, we obtain analogously the next two results.

Theorem 4.2 Let G be a connected graph of minimum degree $\delta \geq 5$. If G does not contain the 4-cycle and the diamond as induced subgraphs, then

$$\gamma \leq 1 + \lfloor d/2 \rfloor + \frac{3}{7}(n - 1 - \delta(1 + \lfloor d/2 \rfloor)).$$

Theorem 4.3 Let G be a connected graph of minimum degree $\delta \geq 6$. If G does not contain the 4-cycle and the diamond as induced subgraphs, then

$$\gamma \leq 1 + \lfloor d/2 \rfloor + \frac{3}{8}(n - 1 - \delta(1 + \lfloor d/2 \rfloor)).$$

Theorem 4.2 is better than Theorem 4.1 for $\delta \geq 5$, and Theorem 4.3 is better than Theorem 4.2 for $\delta \geq 6$.

Using Theorem 2.6 instead of Proposition 2.1, we arrive at the next result, which is an improvement of Theorem 4.1 for $\delta \geq 8$.

Theorem 4.4 Let G be a connected graph of minimum degree $\delta \geq 4$. If G does not contain the 4-cycle and the diamond as induced subgraphs, then

$$\gamma \leq 1 + \lfloor d/2 \rfloor + \frac{1 + \ln(\delta - 2)}{\delta - 2}(n - 1 - \delta(1 + \lfloor d/2 \rfloor)).$$

Proof. Analogously to the proof of Theorem 4.1, we deduce that

$$\gamma = \gamma(G) \leq \lfloor d/2 \rfloor + 1 + \gamma(H).$$

Because of $\delta(H)+1 \geq \delta-3+1 \geq 2$, and since $\frac{1+\ln x}{x}$ is a monotone decreasing function for $x \geq 2$, it follows from Theorem 2.6 that

$$\begin{aligned}\gamma(G) &\leq 1+\lfloor d/2\rfloor+\frac{n(H)(1+\ln(\delta(H)+1))}{\delta(H)+1}\\ &\leq 1+\lfloor d/2\rfloor+\frac{1+\ln(\delta(H)+1)}{\delta(H)+1}(n-1-\delta(1+\lfloor d/2\rfloor))\\ &\leq 1+\lfloor d/2\rfloor+\frac{1+\ln(\delta-2)}{\delta-2}(n-1-\delta(1+\lfloor d/2\rfloor)). \quad\square\end{aligned}$$

Theorem 4.4 is of particular interest, because the following example will demonstrate that it is asymptotically best possible for $\delta \longrightarrow \infty$.

Example 4.5 Let $H_1, H_2, \ldots, H_p$ be p copies of the complete graph $K_{\delta+1}$ for an integer $\delta \geq 4$, and let x_i and y_i be two different vertices in H_i for $i=1,2,\ldots,p$. We define the graph G as the disjoint union of $H_1, H_2, \ldots, H_p$ together with the edges $x_1y_2, x_2y_3, \ldots, x_{p-1}y_p$. It is easy to see that G does not contain the 4-cycle and the diamond as induced subgraphs and that $\delta(G)=\delta$, $n(G)=n=p(\delta+1)$, $d(G)=d=2p-1$, and $\gamma(G)=\gamma=p$. Now Theorem 4.4 yields

$$\begin{aligned}\gamma &\leq 1+p-1+\frac{1+\ln(\delta-2)}{\delta-2}(p\delta+p-1-p\delta)\\ &= p+(p-1)\frac{1+\ln(\delta-2)}{\delta-2}\\ &\leq p\Big(1+\frac{1+\ln(\delta-2)}{\delta-2}\Big)\\ &= p(1+o(1))\end{aligned}$$

for $\delta \longrightarrow \infty$. Since $\gamma(G)=p$, this inequality chain shows that Theorem 4.4 is asymptotically best possible.

Theorem 4.6 Let G be a connected graph of minimum degree $\delta \geq 3$. If G does not contain the 4-cycle and the diamond as induced subgraphs, then

$$\begin{aligned}\gamma &\leq 1+\lfloor d/3\rfloor+\frac{1}{2}(n-(\delta+1)(1+\lfloor d/3\rfloor))\\ &= \frac{1}{2}(n-(\delta-1)(1+\lfloor d/3\rfloor)).\end{aligned}$$

Proof. Let $d=3t+r$ with $0 \leq r \leq 2$ and let $x_0x_1\ldots x_d$ be a minimum length path between the vertices x_0 and x_d. If $A=\{x_0, x_3, \ldots, x_{3t}\}$, then $|A|=1+\lfloor d/3\rfloor=1+t$. Since G does not contain the 4-cycle and the diamond as induced subgraphs, we conclude that $N(A)\cap A=\emptyset$. If we define $H=G-N[A]$ and note that A dominates $N[A]$, then we observe that

$$\gamma=\gamma(G)\leq 1+\lfloor d/3\rfloor+\gamma(H). \tag{4}$$

Furthermore, the fact $N(A) \cap A = \emptyset$ implies

$$\begin{aligned} |N[A]| &= \left|\bigcup_{i=0}^{t} N[x_{3i}]\right| = \left|\bigcup_{i=0}^{t} N(x_{3i})\right| + |A| \\ &= \sum_{i=0}^{t} |N(x_{2i})| + |A| \geq (\delta+1)|A| \end{aligned}$$

and thus $n(H) \leq n - (\delta+1)|A|$. Now any vertex of H can have, in G, at most two neighbors in $N[A]$, because otherwise, we would obtain a 4-cycle or a diamond as an induced subgraph or a shorter path between x_0 and x_d. Hence $\delta(H) \geq \delta - 2 \geq 1$. It follows from Proposition 2.1 and (4) that

$$\begin{aligned} \gamma = \gamma(G) &\leq 1 + \lfloor d/3 \rfloor + \gamma(H) \\ &\leq 1 + \lfloor d/3 \rfloor + \frac{1}{2}(n - (\delta+1)(1 + \lfloor d/3 \rfloor)). \quad \square \end{aligned}$$

Note that for $\delta = 3$ the bound in Theorem 4.6 is identical with the bound in Theorem 1.2, however, the hypotheses are weaker.

Using Theorem 2.6 instead of Proposition 2.1, we obtain similarly to the proof of Theorem 4.6 the next result, which is an improvement of Theorem 4.6 for $\delta \geq 7$.

Theorem 4.7 Let G be a connected graph of minimum degree $\delta \geq 3$. If G does not contain the 4-cycle and the diamond as induced subgraphs, then

$$\gamma \leq 1 + \lfloor d/3 \rfloor + \frac{1 + \ln(\delta - 1)}{\delta - 1}(n - (\delta+1)(1 + \lfloor d/3 \rfloor)).$$

Theorem 4.8 If G is a connected and C_4-free graph of minimum degree $\delta \geq 4$, then

$$\begin{aligned} \gamma &\leq 1 + \lfloor d/2 \rfloor + \frac{1}{2}\left(n - 1 - \delta(1 + \lfloor d/2 \rfloor) - \frac{(\delta-3)(\delta-4)}{2}\right) \\ &= \frac{1}{2}\left(n - 1 - (\delta-2)(1 + \lfloor d/2 \rfloor) - \frac{(\delta-3)(\delta-4)}{2}\right). \end{aligned}$$

Proof. Let $d = 2t + r$ with $0 \leq r \leq 1$ and let $x_0 x_1 \ldots x_d$ be a minimum length path between the vertices x_0 and x_d. If $A = \{x_0, x_2, \ldots, x_{2t}\}$, then $|A| = 1 + \lfloor d/2 \rfloor = 1 + t$ and $N(A) \cap A = \emptyset$. If we define $H = G - N[A]$, then we observe that $\gamma = \gamma(G) \leq 1 + \lfloor d/2 \rfloor + \gamma(H)$. Furthermore, the fact $N(A) \cap A = \emptyset$ implies, as in the proof of Theorem 4.1, that $|N[A]| \geq \delta|A| + 1$ and thus $n(H) \leq n - (\delta|A| + 1)$. Now any vertex of H can have, in G, at most three neighbors in $N[A]$. Hence $\delta(H) \geq \delta - 3 \geq 1$. It follows from Theorem 1.1 that

$$\begin{aligned} \gamma &\leq 1 + \lfloor d/2 \rfloor + \gamma(H) \\ &\leq 1 + \lfloor d/2 \rfloor + \frac{1}{2}\left(n - 1 - \delta(1 + \lfloor d/2 \rfloor) - \frac{(\delta-3)(\delta-4)}{2}\right). \quad \square \end{aligned}$$

It is a simple matter to verify that for $d = 6s + 2$, $d = 6s + 4$, or $d = 5s + 5$ and $s \geq 0$, as well as for $d = 6s$, $d = 6s + 1$, or $d = 6s + 3$ and $\delta \geq 5$ and $s \geq \frac{\delta-2}{\delta-4}$, Theorem 4.8 is an improvement of Theorem 1.2 by Brigham and Dutton [2].

Applying Theorem 3.1 instead of Theorem 1.1, we obtain analogously to the proof of Theorem 4.8 the following better bound for $\delta \geq 5$.

Theorem 4.9 If G is a connected and C_4-free graph of minimum degree $\delta \geq 5$, then

$$\gamma \leq 1 + \lfloor d/2 \rfloor + \frac{3}{7}\Big(n - 1 - \delta(1 + \lfloor d/2 \rfloor) - \frac{(3\delta - 8)(\delta - 5)}{6}\Big).$$

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