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Construction for an OGDD of type 24^4

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Abstract

Colbourn and Gibbons showed there exists an OGDD of type g^4 for all positive integers $g \equiv 0 \pmod 4$ if there exists an OGDD of type g^4 for $g = 4, 8, 12$ and 24. OGDDs of type 8^4 and 12^4 were constructed by Dukes, and an OGDD of type 4^4 was constructed by the author. In this article, we will construct an OGDD of type 24^4 to obtain the existence of an OGDD of type g^4 for all positive integers $g \equiv 0 \pmod 4$.

1 Introduction

A *group-divisible design with block size* 3 (briefly, 3-GDD), $(X, \mathcal{G}, \mathcal{A})$, is a set X and a partition $\mathcal{G}$ of X into classes (usually called *groups*), and a set $\mathcal{A}$ of 3-subsets of X, so that each pair $\{x, y\}$ of elements of X appears once in a 3-subset of $\mathcal{A}$ if x and y are from different groups, and does not appear in a 3-subset of $\mathcal{A}$ if x and y are from the same group.

An *orthogonal group-divisible design* (briefly, OGDD), $(X, \mathcal{G}, \mathcal{A}, \mathcal{B})$, is a pair of 3-GDDs $(X, \mathcal{G}, \mathcal{A})$ and $(X, \mathcal{G}, \mathcal{B})$ satisfying two orthogonal conditions:

(i) if $\{x, y, z\} \in \mathcal{A}$ and $\{x, y, w\} \in \mathcal{B}$, then z and w are in different groups; and

(ii) for two distinct intersecting triples $\{x, y, z\}$ and $\{u, v, z\}$ of $\mathcal{A}$, the triples $\{x, y, w\}$ and $\{u, v, t\}$ of $\mathcal{B}$ satisfy $w \neq t$.

For the existence of an *OGDD* of type g^u (that is, the group size is g and the number of groups is u), Colbourn and Gibbons [4] have done excellent work. The following were their concluding remarks:

The main question that remains open is whether there is any value of g for which an OGDD of type g^4 exists. On the basis of the nonexistence when $g = 2$ and $g = 4$, one might be tempted to conjecture that the answer is negative.

The following theorem is Theorem 2.10 in [4] by Colbourn and Gibbons.

Theorem 1.1 *If m is a positive integer and $m \notin \{2, 3, 6, 10, 12, 14, 18, 26, 30, 38, 42\}$, and there is an OGDD of type g^u, then there exists an OGDD of type $(mg)^u$.*

Theorem 1.2 *If there exists an OGDD of type g^4 for $g = 4, 8, 12$ and 24, then there exists an OGDD of type g^4 for all positive integers $g \equiv 0 \pmod 4$.*

Proof. Apply Theorem 1.1 with $u = 4$, $g = 4$ to obtain an OGDD of type $(4m)^4$ for all positive integers $m \notin \{2, 3, 6, 10, 12, 14, 18, 26, 30, 38, 42\}$. Apply Theorem 1.1 with $u = 4$, $g = 8$, $m = 5, 7, 9, 13, 15, 19$ and 21 to obtain an OGDD of type $(4k)^4$ for $k \in \{10, 14, 18, 26, 30, 38, 42\}$. Apply Theorem 1.1 with $u = 4$, $g = 12$, $m = 4$ to obtain an OGDD of type $(4 \cdot 12)^4$. Since there exists an OGDD of type g^4 for $g = 8, 12$ and 24, there exists an OGDD of type $(4k)^4$ for $k = 2, 3$ and 6. Hence there exists an OGDD of type $(4k)^4$ for all positive integers k, that is, there exists an OGDD of type g^4 for all positive integers $g \equiv 0 \pmod 4$. □

OGDDs of type 8^4 and 12^4 were constructed by Dukes in [2] and an OGDD of type 4^4 was constructed by the author in [7].

In this article, we will construct an OGDD of type 24^4 to obtain existence of an OGDD of type g^4 for all positive integers $g \equiv 0 \pmod 4$.

2 The construction of an OGDD of type 24^4

It is natural that we hope to construct an OGDD of type $(2h)^4$ by base blocks under Z_{8h}. Unfortunately, there is no such design from Theorem 3.1 in Appendix A.

In this section we let

$$G_i = \{0, 3, 6, ..., 69\} + i, \quad i = 0, 1, 2;$$

$$H = \{\infty_1, \infty_2, ..., \infty_{24}\}; \quad \mathcal{G} = \{G_0, G_1, G_2, H\}; \quad X = G_0 \cup G_1 \cup G_2 \cup H.$$

Definition 2.1 Let $(X, \mathcal{G}, \mathcal{B})$ be a 3-GDD of type 24^4. For $i = 1, ..., 24$ and $j = 0, 1, 2$, define:

$$\mathcal{B}_g = \{B \in \mathcal{B} : B \cap H = \emptyset\}; \quad \mathcal{B}_h = \{B \in \mathcal{B} : B \cap H \neq \emptyset\};$$

$$\mathcal{P}_{\mathcal{B},i} = \{\{x, y\} : \{\infty_i, x, y\} \in \mathcal{B}\}.$$

From the definition of a 3-GDD, we have

Lemma 2.2 *If $(X, \mathcal{G}, \mathcal{B})$ is a 3-GDD of type 24^4 then*
(i) each $\mathcal{P}_{\mathcal{B},i}$ is a partition of $X \setminus H$;
(ii) each point of $X \setminus H$ appears exactly 12 times in $\mathcal{B}_g$.

From the definition of an OGDD, we have

Lemma 2.3 *If $(X, \mathcal{G}, \mathcal{A}, \mathcal{B})$ is an OGDD of type 24^4 then $\mathcal{B}_g \cup \mathcal{A}_g$ are the blocks of a 3-GDD of type 24^3.*

Lemma 2.4 *If $(X, \mathcal{G}, \mathcal{A}, \mathcal{B})$ is an OGDD of type 24^4 then $\{\mathcal{P}_{\mathcal{B},i} : i = 1, 2, ..., 24\}$ is a partition of $\{\{x, y\}, \{y, z\}, \{z, x\} : \{x, y, z\} \in \mathcal{A}_g\}$.*

First, by Lemma 2.3, we will construct a 3-GDD of type 24^3 for which the three groups are G_0, G_1, G_2.

It is natural that we hope to construct it by base blocks under Z_{72}. Unfortunately, there is no such design from Theorem 4.2 in Appendix B. So we consider constructing it by base blocks under subgroups of Z_{72}.

Let $E = \{0, 2, 4, ..., 70\}, F = \{0, 6, 12, ..., 66\}$ be two subgroups of Z_{72}.

Let $\mathcal{T}_1 = \{$

$$\{1, 9, -55\}, \{1, 57, -31\}, \{1, 33, -7\}, \{0, 8, -62\}, \{0, 14, -50\}, \{0, 26, -8\},$$

$$\{0, 56, -2\}, \{0, 62, -26\}, \{0, 38, -56\}, \{0, 32, -14\}, \{0, 2, -38\}, \{0, 50, -32\}\}$$

be a set of base blocks under F and $\mathcal{T}_2 = \{$

$$\{1, 38, 3\}, \{1, 44, 15\}, \{1, 50, 27\}, \{1, 56, 39\}, \{1, 62, 51\}, \{1, 32, 63\},$$

$$\{0, 1, 68\}, \{0, 19, 44\}, \{0, 7, 20\}, \{1, 2, 69\}, \{1, 20, 45\}, \{1, 8, 21\}\}$$

be a set of base blocks under E.

It is easily checked that $\mathcal{T}_1$ under F and $\mathcal{T}_2$ under E form a 3-GDD of type 24^3.

Second, by Lemma 2.2, we will partition $\mathcal{T}_1$ into $\mathcal{A}_1$ and $\mathcal{B}_1$, and partition $\mathcal{T}_2$ into $\mathcal{A}_2$ and $\mathcal{B}_2$ as follows.

$\mathcal{A}_1 = \{\{1, 9, -55\}, \{1, 57, -31\}, \{1, 33, -7\}, \{0, 8, -62\}, \{0, 14, -50\}, \{0, 26, -8\}\}$;

$\mathcal{A}_2 = \{\{0, 1, 68\}, \{0, 19, 44\}, \{0, 7, 20\}, \{1, 38, 3\}, \{1, 44, 15\}, \{1, 50, 27\}\}$;

$\mathcal{B}_1 = \{\{0, 56, -2\}, \{0, 62, -26\}, \{0, 38, -56\}, \{0, 32, -14\}, \{0, 2, -38\}, \{0, 50, -32\}\}$;

$\mathcal{B}_2 = \{\{1, 2, 69\}, \{1, 20, 45\}, \{1, 8, 21\}, \{1, 56, 39\}, \{1, 62, 51\}, \{1, 32, 63\}\}$.

Finally, by Lemma 2.2 and Lemma 2.4, we will arrange $\mathcal{P}_{\mathcal{A},i}$ and $\mathcal{P}_{\mathcal{B},i}$ to form an OGDD of type 24^4.

The following pairs under F come from $\mathcal{B}_1$ under the subgroup F and $\mathcal{B}_2$ under the subgroup E.

$\{0, 2+6s\} : s = 9, 10, 6, 5, 0, 8$	$\{2, 4+6s\} : s = 2, 9, 8, 4, 5, 10$
$\{4, 0+6s\} : s = 1, 5, 10, 3, 7, 6$	$\{1, 3+6s\} : s = 11, 7, 3, 6, 8, 10$
$\{3, 5+6s\} : s = 11, 7, 3, 6, 8, 10$	$\{5, 1+6s\} : s = 0, 8, 4, 7, 9, 11$
$\{1, 2+6s\} : s = 0, 3, 1, 9, 10, 5$	$\{3, 4+6s\} : s = 0, 3, 1, 9, 10, 5$
$\{5, 0+6s\} : s = 1, 4, 2, 10, 11, 6$	$\{0, 1+6s\} : s = 11, 4, 2, 9, 10, 5$
$\{2, 3+6s\} : s = 11, 4, 2, 9, 10, 5$	$\{4, 5+6s\} : s = 11, 4, 2, 9, 10, 5$

Arrange $\mathcal{P}_{\mathcal{A},i}, i = 1, 2, \ldots, 24$ using the above pairs to obtain the following $\mathcal{A}_3$:

$$\begin{aligned}\mathcal{A}_3 = \ & \{\{\infty_1, 0, 56\}, \{\infty_1, 1, 69\}, \{\infty_1, 4, 71\}, \{\infty_2, 0, 62\}, \{\infty_2, 1, 45\}, \{\infty_2, 4, 29\}, \\ & \{\infty_3, 0, 38\}, \{\infty_3, 1, 21\}, \{\infty_3, 4, 17\}, \{\infty_4, 0, 32\}, \{\infty_4, 5, 1\}, \{\infty_4, 3, 4\}, \\ & \{\infty_5, 0, 2\}, \{\infty_5, 5, 49\}, \{\infty_5, 3, 22\}, \{\infty_6, 0, 50\}, \{\infty_6, 5, 25\}, \{\infty_6, 3, 10\},\end{aligned}$$

$\{\infty_7, 2, 16\}, \{\infty_7, 1, 39\}, \{\infty_7, 5, 6\}, \{\infty_8, 2, 58\}, \{\infty_8, 1, 51\}, \{\infty_8, 5, 24\},$
$\{\infty_9, 2, 52\}, \{\infty_9, 1, 63\}, \{\infty_9, 5, 12\}, \{\infty_{10}, 2, 28\}, \{\infty_{10}, 3, 71\}, \{\infty_{10}, 0, 67\},$
$\{\infty_{11}, 2, 34\}, \{\infty_{11}, 3, 47\}, \{\infty_{11}, 0, 25\}, \{\infty_{12}, 2, 64\}, \{\infty_{12}, 3, 23\}, \{\infty_{12}, 0, 13\},$
$\{\infty_{13}, 4, 6\}, \{\infty_{13}, 3, 41\}, \{\infty_{13}, 1, 2\}, \{\infty_{14}, 4, 30\}, \{\infty_{14}, 3, 53\}, \{\infty_{14}, 1, 20\},$
$\{\infty_{15}, 4, 60\}, \{\infty_{15}, 3, 65\}, \{\infty_{15}, 1, 8\}, \{\infty_{16}, 4, 18\}, \{\infty_{16}, 5, 43\}, \{\infty_{16}, 2, 69\},$
$\{\infty_{17}, 4, 42\}, \{\infty_{17}, 5, 55\}, \{\infty_{17}, 2, 27\}, \{\infty_{18}, 4, 36\}, \{\infty_{18}, 5, 67\}, \{\infty_{18}, 2, 15\},$
$\{\infty_{19}, 1, 56\}, \{\infty_{19}, 3, 58\}, \{\infty_{19}, 5, 60\}, \{\infty_{20}, 1, 62\}, \{\infty_{20}, 3, 64\}, \{\infty_{20}, 5, 66\},$
$\{\infty_{21}, 1, 32\}, \{\infty_{21}, 3, 34\}, \{\infty_{21}, 5, 36\}, \{\infty_{22}, 0, 55\}, \{\infty_{22}, 2, 57\}, \{\infty_{22}, 4, 59\},$
$\{\infty_{23}, 0, 61\}, \{\infty_{23}, 2, 63\}, \{\infty_{23}, 4, 65\}, \{\infty_{24}, 0, 31\}, \{\infty_{24}, 2, 33\}, \{\infty_{24}, 4, 35\}\}.$

The following pairs under F come from $\mathcal{A}_1$ under the subgroup F and $\mathcal{A}_2$ under the subgroup E.

$\{0, 2+6s\} : s = 3, 7, 11, 4, 2, 1$	$\{2, 4+6s\} : s = 0, 1, 6, 11, 7, 3$
$\{4, 0+6s\} : s = 9, 11, 8, 0, 2, 4$	$\{1, 3+6s\} : s = 1, 9, 5, 0, 2, 4$
$\{3, 5+6s\} : s = 4, 5, 1, 0, 9, 2$	$\{5, 1+6s\} : s = 3, 2, 5, 10, 6, 1$
$\{1, 2+6s\} : s = 8, 7, 6, 11, 4, 2$	$\{3, 4+6s\} : s = 7, 6, 8, 11, 4, 2$
$\{5, 0+6s\} : s = 5, 8, 3, 9, 0, 7$	$\{0, 1+6s\} : s = 1, 3, 8, 0, 7, 6$
$\{2, 3+6s\} : s = 3, 1, 6, 7, 0, 8$	$\{4, 5+6s\} : s = 1, 8, 3, 0, 7, 6$

Arrange $\mathcal{P}_{\mathcal{B},i}, i = 1, 2, \ldots, 24$ using the above pairs to obtain the following $\mathcal{B}_3$.

$$\begin{aligned}
\mathcal{B}_3 = \; & \{\{\infty_1, 4, 54\}, \{\infty_1, 3, 5\}, \{\infty_1, 1, 50\}, \{\infty_2, 4, 66\}, \{\infty_2, 3, 59\}, \{\infty_2, 1, 44\}, \\
& \{\infty_3, 4, 48\}, \{\infty_3, 3, 17\}, \{\infty_3, 1, 38\}, \{\infty_4, 2, 70\}, \{\infty_4, 3, 29\}, \{\infty_4, 0, 7\}, \\
& \{\infty_5, 2, 46\}, \{\infty_5, 3, 35\}, \{\infty_5, 0, 19\}, \{\infty_6, 2, 22\}, \{\infty_6, 3, 11\}, \{\infty_6, 0, 49\}, \\
& \{\infty_7, 4, 0\}, \{\infty_7, 5, 61\}, \{\infty_7, 2, 21\}, \{\infty_8, 4, 12\}, \{\infty_8, 5, 37\}, \{\infty_8, 2, 9\}, \\
& \{\infty_9, 4, 24\}, \{\infty_9, 5, 7\}, \{\infty_9, 2, 39\}, \{\infty_{10}, 0, 26\}, \{\infty_{10}, 5, 19\}, \{\infty_{10}, 3, 46\}, \\
& \{\infty_{11}, 0, 14\}, \{\infty_{11}, 5, 13\}, \{\infty_{11}, 3, 40\}, \{\infty_{12}, 0, 8\}, \{\infty_{12}, 5, 31\}, \{\infty_{12}, 3, 52\}, \\
& \{\infty_{13}, 0, 20\}, \{\infty_{13}, 1, 9\}, \{\infty_{13}, 4, 11\}, \{\infty_{14}, 0, 44\}, \{\infty_{14}, 1, 57\}, \{\infty_{14}, 4, 53\}, \\
& \{\infty_{15}, 0, 68\}, \{\infty_{15}, 1, 33\}, \{\infty_{15}, 4, 23\}, \{\infty_{16}, 2, 4\}, \{\infty_{16}, 1, 3\}, \{\infty_{16}, 5, 30\}, \\
& \{\infty_{17}, 2, 10\}, \{\infty_{17}, 1, 15\}, \{\infty_{17}, 5, 48\}, \{\infty_{18}, 2, 40\}, \{\infty_{18}, 1, 27\}, \{\infty_{18}, 5, 18\}, \\
& \{\infty_{19}, 0, 1\}, \{\infty_{19}, 2, 45\}, \{\infty_{19}, 4, 5\}, \{\infty_{20}, 0, 43\}, \{\infty_{20}, 2, 3\}, \{\infty_{20}, 4, 47\}, \\
& \{\infty_{21}, 0, 37\}, \{\infty_{21}, 2, 51\}, \{\infty_{21}, 4, 41\}, \{\infty_{22}, 1, 68\}, \{\infty_{22}, 3, 70\}, \{\infty_{22}, 5, 54\}, \\
& \{\infty_{23}, 1, 26\}, \{\infty_{23}, 3, 28\}, \{\infty_{23}, 5, 0\}, \{\infty_{24}, 1, 14\}, \{\infty_{24}, 3, 16\}, \{\infty_{24}, 5, 42\}\}.
\end{aligned}$$

Theorem 2.5 *There exists an OGDD of type* 24^4 *and furthermore there exists an OGDD of type* g^4 *for all positive integers* $g \equiv 0 \pmod 4$.

Proof. Define a 3-GDD of type 24^4 by developing the three sets of base blocks: $\mathcal{A}_1$ under the subgroup F, $\mathcal{A}_2$ under the subgroup E and $\mathcal{A}_3$ under the subgroup F.

Form a second 3-GDD by developing the three sets of base blocks: $\mathcal{B}_1$ under the subgroup F, $\mathcal{B}_2$ under the subgroup E and $\mathcal{B}_3$ under the subgroup F. It is readily checked that the two 3-GDDs are orthogonal (see Appendix C). □

The question that still remains open is whether there is any value of $g \equiv 2 \pmod 4$ for which an OGDD of type g^4 exists.

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Appendix A

Let $X = Z_{8h}, H = \{0, 4, 8, ..., 8h-4\}$ be a subgroup of Z_{8h}, and $G_i = H + i, i = 0, 1, 2, 3$. In the following we will show

Theorem 3.1 *There is no OGDD of type* $(2h)^4$ *for which all blocks are developed by base blocks under* Z_{8h}.

Assume that $\mathcal{A}$ and $\mathcal{B}$ are two sets of base blocks under Z_{8h} for an OGDD of type $(2h)^4$, for which the four groups are G_0, G_1, G_2 and G_3.

Without loss of generality, we can let

$$\mathcal{A} = \{\{0, a_i, a_i + b_i\} : i = 0, 1, \ldots, h-1\}$$

be the base blocks of the first 3-GDD, and

$$\mathcal{B} = \{\{0, a_i, a_i + d_i\} : i = 0, 1, \ldots, h-1\}$$

be the base blocks of the second 3-GDD, where

$$a_i = 4i + 2,\ b_i, d_i \equiv 1, 3 \pmod 4,\ i = 0, 1, \ldots, h-1.$$

From the orthogonality of an OGDD, it is easy to see that

(i) $\{b_i, a_i + b_i, d_i, a_i + d_i : i = 0, 1, 2, \ldots, h-1\} = \{1, 3, 5, \ldots, 8h-1\}$;

(ii) $d_i - b_i \equiv 2 \pmod 4$.

Without loss of generality, we can let

$$b_i \equiv 1 \pmod 4, i = 0, 1, \ldots, s-1; \qquad b_j \equiv 3 \pmod 4, j = s, s+1, \ldots, h-1.$$

Hence

$$\begin{aligned} a_i + b_i &\equiv 3 \pmod 4, i = 0, 1, \ldots, s-1; \\ a_j + b_j &\equiv 1 \pmod 4, j = s, s+1, \ldots, h-1. \end{aligned}$$

By (ii)

$$d_i \equiv 3 \pmod 4,\ i = 0, 1, \ldots, s-1; \qquad d_j \equiv 1 \pmod 4,\ j = s, s+1, \ldots, h-1.$$

Hence

$$\begin{aligned} a_i + d_i &\equiv 1 \pmod 4,\ i = 0, 1, \ldots, s-1; \\ a_j + d_j &\equiv 3 \pmod 4,\ j = s, s+1, \ldots, h-1. \end{aligned}$$

By (i), the sum of all numbers which is 1 modulo 4 is

$$\Sigma b_i + \Sigma(a_j + b_j) + \Sigma d_j + \Sigma(a_i + d_i) \equiv (8h-2)h \pmod{8h}.$$

By (i), the sum of all numbers which is 3 modulo 4 is

$$\Sigma b_j + \Sigma(a_i + b_i) + \Sigma d_i + \Sigma(a_j + d_j) \equiv (8h-2)h \pmod{8h}.$$

It is clear that the left sides of the above two equalities are the same; this forces $(8h-2)h \equiv (8h+2)h \pmod{8h}$, that is, $4h \equiv 0 \pmod{8h}$, which is impossible.

Appendix B

Let $X = Z_{6h}, H = \{0, 3, 6, ..., 6h-3\}$ be a subgroup of Z_{6h}, and $G_i = H+i, i = 0, 1, 2$. In the following we will show

Theorem 4.1 *There is no 3-GDD of type 3^{2h} for which all blocks are developed by base blocks under Z_{6h}.*

Assume that $\mathcal{A}$ is a set of base blocks under Z_{6h} for a 3-GDD of type 3^{2h}, for which the three groups are G_0, G_1, G_2.

It is easy to see that the number of base blocks is $2h/3$, so $h \equiv 0 \pmod 3$. It is clear that one base block yields four or zero odd differences, so $h \equiv 0 \pmod 2$. Hence $h = 6n$. Without loss of generality, we can let

$$\mathcal{A} = \{\{0, a_i, a_i + b_i\} : i = 0, 1, \ldots, 4n-1\}$$

be the base blocks of the 3-GDD, where

$$\begin{aligned} a_i \equiv b_i &\equiv 1 \pmod 6,\ i = 0, 1, \ldots, 3n-1; \\ a_j \equiv b_j &\equiv 2 \pmod 6,\ j = 3n, 3n+1, \ldots, 4n-1. \end{aligned}$$

Since

$$\begin{aligned} &\{a_i, b_i, a_i + b_i, a_j, b_j, 36n - (a_j + b_j) : i = 0, 1, \ldots, 3n-1,\ j = 3n, 3n+1, \ldots, 4n-1\} \\ &\qquad = \{6k+1, 6k+2 : k = 0, 1, \ldots, 6n-1\}, \end{aligned}$$

we have

$$\Sigma(a_i + b_i) + \Sigma a_j + \Sigma b_j + \Sigma(36n - a_j - b_j) \equiv \Sigma a_i + \Sigma b_i \pmod{36n}.$$

Hence

$$2 + 8 + \ldots + 36n - 4 \equiv 1 + 7 + \ldots + 36n - 5 \pmod{36n}.$$

That is, $6n \equiv 0 \pmod{36n}$, which is impossible.

Appendix C

Let $\mathcal{A} = \{A + g : A \in \mathcal{A}_1 \cup \mathcal{A}_3, g \in F\} \cup \{A + g : A \in \mathcal{A}_2, g \in E\}$,
$\mathcal{B} = \{B + g : B \in \mathcal{B}_1 \cup \mathcal{B}_3, g \in F\} \cup \{B + g : B \in \mathcal{B}_2, g \in E\}$.

The following seven tables show that $\mathcal{A}$ and $\mathcal{B}$ satisfy the two orthogonal conditions. The first table means

$$\{0,4,5\}, \{0,28,47\}, \ldots, \{0,23,49\}, \{0,\infty_1,56\}, \{0,\infty_2,62\}, \ldots, \{0,\infty_{24},31\} \in \mathcal{A}$$

and $\{4,5,\infty_{19}\}$, $\{28,47,\infty_{15}\}$, $\ldots$, $\{23,49,\infty_{12}\}$, $\{\infty_1,56,7\}$, $\{\infty_2,62,19\}$, $\ldots$, $\{\infty_{24},31,44\} \in \mathcal{B}$. Since the 36 points, $\infty_{19}, \infty_{15}, \ldots, \infty_{12}, 7, 19, \ldots, 44$, are distinct and in different groups with 0, we have that condition (i) holds with

$$\{x,y\} \in \{\{4,5\}, \{28,47\}, \ldots, \{23,49\}, \{\infty_1,56\}, \{\infty_2,62\}, \ldots, \{\infty_{24},31\}$$

and the condition (ii) holds with $z = 0$.

The last table means

$$\{\infty_1,0,56\}, \{\infty_1,1,69\}, \{\infty_1,4,71\} \in \mathcal{A}$$

and

$$\{0,56,70\}, \{1,69,2\}, \{4,71,3\} \in \mathcal{B}.$$

Hence

$$\{\infty_1,0,56\} + g, \{\infty_1,1,69\} + g, \{\infty_1,,4,71\} + g \in \mathcal{A}, \ g \in F$$

and

$$\{0,56,70\} + g, \{1,69,2\} + g, \{4,71,3\} + g \in \mathcal{B}, \ g \in F.$$

Since the 36 points, $70 + g, 2 + g, 3 + g : g \in F$, are distinct and in different groups with ∞_1, we have that condition (i) holds with

$$\{x,y\} \in \{\{0,56\} + g, \{1,69\} + g, \{4,71\} + g : g \in F\}$$

and condition (ii) holds with $z = \infty_1$.

0 orthogonality

$4,5$	∞_{19}	$28,47$	∞_{15}	$52,59$	∞_{13}
$7,20$	∞_{24}	$19,44$	∞_{23}	$1,68$	∞_{22}
$8,10$	∞_{16}	$14,22$	∞_{17}	$26,64$	∞_{18}
$35,37$	∞_9	$29,43$	∞_{10}	$23,49$	∞_{12}
$\infty_1,56$	7	$\infty_2,62$	19	$\infty_3,38$	1
$\infty_4,32$	28	$\infty_5,2$	46	$\infty_6,50$	70
$\infty_7,71$	55	$\infty_8,53$	13	$\infty_9,65$	67
$\infty_{10},67$	53	$\infty_{11},25$	17	$\infty_{12},13$	59
$\infty_{13},70$	5	$\infty_{14},46$	23	$\infty_{15},16$	35
$\infty_{16},58$	56	$\infty_{17},34$	26	$\infty_{18},40$	2
$\infty_{19},17$	16	$\infty_{20},11$	40	$\infty_{21},41$	4
$\infty_{22},55$	50	$\infty_{23},61$	14	$\infty_{24},31$	44

1 orthogonality

$9, 17$	∞_6	$57, 41$	∞_2	$33, 65$	∞_5
$0, 68$	∞_{15}	$54, 26$	∞_{14}	$66, 14$	∞_{13}
$38, 3$	∞_9	$71, 36$	∞_{24}	$44, 15$	∞_{19}
$59, 30$	∞_{17}	$50, 27$	∞_{21}	$47, 24$	∞_{22}
$\infty_1, 69$	71	$\infty_2, 45$	29	$\infty_3, 21$	35
$\infty_4, 5$	51	$\infty_5, 29$	69	$\infty_6, 53$	45
$\infty_7, 39$	20	$\infty_8, 51$	44	$\infty_9, 63$	26
$\infty_{10}, 6$	32	$\infty_{11}, 48$	62	$\infty_{12}, 60$	68
$\infty_{13}, 2$	54	$\infty_{14}, 20$	48	$\infty_{15}, 8$	12
$\infty_{16}, 35$	60	$\infty_{17}, 23$	66	$\infty_{18}, 11$	24
$\infty_{19}, 56$	27	$\infty_{20}, 62$	63	$\infty_{21}, 32$	9
$\infty_{22}, 18$	41	$\infty_{23}, 12$	17	$\infty_{24}, 42$	5

2 orthogonality

$66, 4$	∞_2	$60, 10$	∞_1	$48, 40$	∞_8
$3, 70$	∞_{22}	$6, 7$	∞_{19}	$21, 46$	∞_{23}
$30, 49$	∞_5	$9, 22$	∞_{24}	$54, 61$	∞_4
$37, 39$	∞_{16}	$31, 45$	∞_{17}	$25, 51$	∞_{18}
$\infty_1, 18$	40	$\infty_2, 12$	22	$\infty_3, 36$	64
$\infty_4, 42$	49	$\infty_5, 0$	19	$\infty_6, 24$	1
$\infty_7, 16$	12	$\infty_8, 58$	66	$\infty_9, 52$	0
$\infty_{10}, 28$	57	$\infty_{11}, 34$	69	$\infty_{12}, 64$	15
$\infty_{13}, 1$	9	$\infty_{14}, 55$	39	$\infty_{15}, 67$	27
$\infty_{16}, 69$	67	$\infty_{17}, 27$	13	$\infty_{18}, 15$	61
$\infty_{19}, 19$	18	$\infty_{20}, 13$	42	$\infty_{21}, 43$	6
$\infty_{22}, 57$	52	$\infty_{23}, 63$	16	$\infty_{24}, 33$	46

3 orthogonality

$67, 11$	∞_7	$19, 59$	∞_8	$43, 35$	∞_{11}
$2, 70$	∞_4	$56, 28$	∞_5	$68, 16$	∞_6
$40, 5$	∞_{21}	$1, 38$	∞_3	$46, 17$	∞_{20}
$61, 32$	∞_2	$52, 29$	∞_{14}	$49, 26$	∞_1
$\infty_1, 7$	56	$\infty_2, 31$	2	$\infty_3, 55$	20
$\infty_4, 4$	8	$\infty_5, 22$	50	$\infty_6, 10$	62
$\infty_7, 37$	53	$\infty_8, 25$	65	$\infty_9, 13$	11
$\infty_{10}, 71$	13	$\infty_{11}, 47$	55	$\infty_{12}, 23$	49
$\infty_{13}, 41$	34	$\infty_{14}, 53$	4	$\infty_{15}, 65$	46
$\infty_{16}, 8$	10	$\infty_{17}, 50$	58	$\infty_{18}, 62$	28
$\infty_{19}, 58$	59	$\infty_{20}, 64$	35	$\infty_{21}, 34$	71
$\infty_{22}, 20$	25	$\infty_{23}, 14$	61	$\infty_{24}, 44$	31

4 orthogonality

$66, 2$	∞_{12}	$54, 68$	∞_{11}	$12, 38$	∞_{10}
$5, 72$	∞_{23}	$8, 9$	∞_{20}	$23, 48$	∞_{16}
$32, 51$	∞_{7}	$11, 24$	∞_{18}	$56, 63$	∞_{8}
$39, 41$	∞_{1}	$33, 47$	∞_{3}	$27, 53$	∞_{4}
$\infty_1, 71$	69	$\infty_2, 29$	45	$\infty_3, 17$	3
$\infty_4, 3$	29	$\infty_5, 57$	17	$\infty_6, 69$	5
$\infty_7, 62$	9	$\infty_8, 20$	27	$\infty_9, 26$	63
$\infty_{10}, 50$	24	$\infty_{11}, 44$	30	$\infty_{12}, 14$	6
$\infty_{13}, 6$	26	$\infty_{14}, 30$	2	$\infty_{15}, 60$	56
$\infty_{16}, 18$	65	$\infty_{17}, 42$	71	$\infty_{18}, 36$	23
$\infty_{19}, 21$	50	$\infty_{20}, 15$	14	$\infty_{21}, 45$	68
$\infty_{22}, 59$	36	$\infty_{23}, 65$	60	$\infty_{24}, 35$	0

5 orthogonality

$61, 69$	∞_{13}	$37, 21$	∞_{14}	$13, 45$	∞_{15}
$4, 0$	∞_{7}	$58, 30$	∞_{3}	$70, 18$	∞_{9}
$42, 7$	∞_{21}	$3, 40$	∞_{11}	$48, 19$	∞_{20}
$63, 34$	∞_{10}	$54, 31$	∞_{6}	$51, 28$	∞_{12}
$\infty_1, 10$	60	$\infty_2, 52$	42	$\infty_3, 64$	36
$\infty_4, 1$	66	$\infty_5, 49$	30	$\infty_6, 25$	48
$\infty_7, 6$	10	$\infty_8, 24$	16	$\infty_9, 12$	64
$\infty_{10}, 9$	52	$\infty_{11}, 33$	70	$\infty_{12}, 57$	34
$\infty_{13}, 39$	31	$\infty_{14}, 27$	43	$\infty_{15}, 15$	55
$\infty_{16}, 43$	45	$\infty_{17}, 55$	69	$\infty_{18}, 67$	21
$\infty_{19}, 60$	61	$\infty_{20}, 66$	37	$\infty_{21}, 36$	1
$\infty_{22}, 22$	27	$\infty_{23}, 16$	63	$\infty_{24}, 46$	33

∞ **orthogonality**

∞_1	$0,56$	70	∞_1	$1,69$	2	∞_1	$4,71$	3
∞_2	$0,62$	46	∞_2	$1,45$	20	∞_2	$4,29$	57
∞_3	$0,38$	16	∞_3	$1,21$	8	∞_3	$4,17$	69
∞_4	$0,32$	58	∞_4	$5,1$	6	∞_4	$3,4$	71
∞_5	$0,2$	34	∞_5	$5,49$	24	∞_5	$3,22$	47
∞_6	$0,50$	40	∞_6	$5,25$	12	∞_6	$3,10$	23
∞_7	$2,16$	18	∞_7	$1,39$	56	∞_7	$5,6$	1
∞_8	$2,58$	12	∞_8	$1,51$	62	∞_8	$5,24$	49
∞_9	$2,52$	36	∞_9	$1,63$	32	∞_9	$5,12$	25
∞_{10}	$2,28$	42	∞_{10}	$3,71$	4	∞_{10}	$0,67$	71
∞_{11}	$2,34$	0	∞_{11}	$3,47$	22	∞_{11}	$0,25$	53
∞_{12}	$2,64$	24	∞_{12}	$3,23$	10	∞_{12}	$0,13$	65
∞_{13}	$4,6$	62	∞_{13}	$3,41$	58	∞_{13}	$1,2$	69
∞_{14}	$4,30$	20	∞_{14}	$3,53$	64	∞_{14}	$1,20$	45
∞_{15}	$4,60$	26	∞_{15}	$3,65$	34	∞_{15}	$1,8$	21
∞_{16}	$4,18$	50	∞_{16}	$5,43$	60	∞_{16}	$2,69$	1
∞_{17}	$4,42$	44	∞_{17}	$5,55$	66	∞_{17}	$2,27$	55
∞_{18}	$4,36$	14	∞_{18}	$5,67$	36	∞_{18}	$2,15$	67
∞_{19}	$1,56$	39	∞_{19}	$3,58$	41	∞_{19}	$5,60$	43
∞_{20}	$1,62$	51	∞_{20}	$3,64$	53	∞_{20}	$5,66$	55
∞_{21}	$1,32$	63	∞_{21}	$3,34$	65	∞_{21}	$5,36$	67
∞_{22}	$0,55$	17	∞_{22}	$2,57$	19	∞_{22}	$4,59$	21
∞_{23}	$0,61$	11	∞_{23}	$2,63$	13	∞_{23}	$4,65$	15
∞_{24}	$0,31$	41	∞_{24}	$2,33$	43	∞_{24}	$4,35$	45

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