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On the defining sets of some special graphs

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Abstract

In a given graph G, a set S of vertices with an assignment of colors to them is a defining set of vertex coloring for G, if there exists a unique extension of the colors of S to a $\chi(G)$-coloring of the vertices of G. A defining set with minimum cardinality is called a smallest defining set (of vertex coloring) and its cardinality, the defining number, is denoted by $d(G,\chi)$. Mahmoodian et al. (1999) determined the defining number of graph $C_3 \times C_n$. In this paper, we study the defining number of graph $C_m \times C_n$, and show that

$$\max\left\{\left\lceil\frac{m}{2}\right\rceil,\left\lceil\frac{n}{2}\right\rceil\right\} \le d(C_m \times C_n,\chi) \le \left\lfloor\frac{m+n-1}{2}\right\rfloor \quad (n \text{ odd}).$$

Also, we prove a similar result for the defining number of graph $C_m \times P_n$.

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1 Introduction

A k-coloring of a graph G is an assignment of k different colors to the vertices of G such that no two adjacent vertices receive the same color. The (vertex) chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number k, for which there exists a k-coloring for G. A graph G with $\chi(G) = k$ is called a k–chromatic graph. In a given graph G, a set of vertices S with an assignment of colors to them is called a defining set of vertex coloring for G, if there exists a unique extension of the colors of S to a $\chi(G)$-coloring of the vertices of G. A defining set with minimum cardinality is called a smallest defining set (of a vertex coloring) and its cardinality, the defining number, is denoted by $d(G, \chi)$.

Defining sets of vertex coloring are closely related to the list coloring of a graph. In a list coloring for each vertex v there is a given list of colors $\mathcal{L}_v$ allowable on that vertex. Coloring must be done so that each vertex is colored with an allowable color and no two adjacent vertices receive the same color. It should be noted that any defining set S in a graph G naturally induces a list of possible colors for the vertices of the induced subgraph $\langle V(G) \setminus S \rangle$. Furthermore, using this list of colors, $\langle V(G) \setminus S \rangle$ is uniquely list colorable.

A graph G with n vertices, is called uniquely 2-list colorable, if for each vertex $v \in V(G)$, there exists a list of colors $\mathcal{L}_v$ with $|\mathcal{L}_v| = 2$, such that there is a unique list coloring for G using this list. In the following theorem, Mahdian and Mahmoodian characterized uniquely 2-list colorable graphs.

Theorem A ([3]) *A connected graph is uniquely* 2*-list colorable if and only if at least one of its blocks is not one of the following graphs:*

(a) *a cycle;*

(b) *a complete graph; or*

(c) *a complete bipartite graph.*

The following proposition is very useful in our discussion.

Proposition 1. *Suppose there are list of colors $\mathcal{L}_i$, $i = 1, \ldots, n$, on the vertices of a cycle C_n, such that each $\mathcal{L}_i$ is of size* 2 *and $| \cup_{i=1}^{n} \mathcal{L}_i| = 3$. Then there exist two different colorings for C_n from these lists.*

Proof. Since $|\mathcal{L}_i| = 2$ for each i and $|\cup_{i=1}^{n} \mathcal{L}_i| = 3$, there exists some i with $\mathcal{L}_i \neq \mathcal{L}_{i+1}$. We may assume $\mathcal{L}_1 \neq \mathcal{L}_n$. Then we have a proper coloring c such that $c(v_1) = \mathcal{L}_1 \setminus \mathcal{L}_n$ and $c(v_i) = \mathcal{L}_i \setminus c(v_{i-1})$ for $i > 1$. Now since by Theorem A, C_n is not a uniquely 2-list colorable graph, there exists another coloring from the lists $\mathcal{L}_i$ for C_n. □

There are some results on defining numbers in [5] (see also [2], and [4]). Nasesasr et al. in [6] showed that $d(C_3 \times C_n, \chi) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, and in [5] it is shown that $d(P_2 \times C_n, \chi) = \left\lceil \frac{n}{2} \right\rceil$ (n odd).

Note that the graph $C_m \times C_n$ (respectively, $C_m \times P_n$) can be viewed as a graph with vertex set as an $m \times n$ array $[v_{ij}]$, where each induced subgraph on the vertices on row i is a copy of cycle C_n (path P_n), and each induced subgraph on the vertices on column j is a copy of cycle C_m.

Here we find a lower bound and an upper bound for $d(C_m \times C_n, \chi)$ and also for $d(C_m \times P_n, \chi)$. We also propose two conjectures for their exact values.

2 Defining number of $C_m \times C_n$

First we note that when both m and n are even, then the defining number of $C_m \times C_n$ is easy to determine. Indeed, since $\chi(C_m \times C_n) = \max\{\chi(C_m), \chi(C_n)\}$, and when m and n both are even, we have $\chi(C_m \times C_n) = 2$, thus $C_m \times C_n$ is a connected bipartite graph and we have $d(C_m \times C_n, \chi) = 1$.

Next theorem which is about the graph $C_m \times C_n$, when at least one of m or n is odd, is our tool in finding a lower bound for $d(C_m \times C_n, \chi)$. To prove this theorem first we introduce the following lemma. Throughout our discussion about $G = K_2 \times C_n$, we suppose that the vertices of two copies of C_n in G, C_n^1 and C_n^2, are also labeled by $v_{11}, ..., v_{1n}$ and $v_{21}, ..., v_{2n}$.

Lemma 1. *Let $G = K_2 \times C_n$ and let $\mathcal{L}_{ij}$ be a list of colors on v_{ij}, where*

(i) $|\mathcal{L}_{ij}| = 2$, *and* $| \cup_{i=1}^{2} \cup_{j=1}^{n} \mathcal{L}_{ij}| = 3$*;*

(ii) $\mathcal{L}_{ij} \neq \mathcal{L}_{i(j+1)}$ *where $j+1$ is the unique number k in $\{1, 2, \ldots, n\}$ such that* $j + 1 \equiv k \pmod n$*, for $i = 1$ and 2, and $j = 1, 2, ..., n$.*

Then G either has no coloring from $\mathcal{L}_{ij}$, or it has at least two different colorings.

Proof. There are three cases to be considered:

Case 1. There exists $p \in \{1, ..., n\}$ such that $\mathcal{L}_{1p} = \mathcal{L}_{2p}$ and there exists $q \in \{1, ..., n\}$ such that $\mathcal{L}_{1q} \neq \mathcal{L}_{2q}$.

Without loss of generality assume that $\mathcal{L}_{11} = \mathcal{L}_{21}$ and $\mathcal{L}_{12} \neq \mathcal{L}_{22}$. Also, since $\mathcal{L}_{11} \neq \mathcal{L}_{12}$ and $\mathcal{L}_{21} \neq \mathcal{L}_{22}$, we can assume $\mathcal{L}_{11} = \mathcal{L}_{21} = \{\alpha, \beta\}$, $\mathcal{L}_{12} = \{\beta, \gamma\}$, and $\mathcal{L}_{22} = \{\alpha, \gamma\}$. Now if c is a coloring for G, we must have $c(v_{11}) = \alpha$, for if $c(v_{11}) = \beta$ then the color of v_{12} and v_{21} will be forced to γ and α, respectively, which make it impossible to have a color for v_{22}. Therefore, $c(v_{11}) = \alpha$ and $c(v_{21}) = \beta$.

Now for each fixed ordered pair (r, s), $(1 \leq r \leq 2,\ 1 \leq s \leq n)$, and with respect to coloring c, we define an assignment $\mathcal{F}_{rs} : V(G) \longrightarrow \{\alpha, \beta, \gamma\}$ as follows:

$$\mathcal{F}_{rs}(v_{ij}) = \begin{cases} c(v_{ij}), & \text{if } \ (i,j) \neq (r,s), \ \text{ and} \\ \mathcal{L}_{rs} \setminus \{c(v_{rs})\}, & \text{if } \ (i,j) = (r,s). \end{cases}$$

Note that without loss of generality we may assume that $c(v_{12}) = \beta$. For, if $c(v_{12}) = \gamma$ then $c(v_{22}) = \alpha$, which can be changed to the case when $c(v_{12}) = \beta$ by renumbering the vertices of G with the permutation $v_{1j} \leftrightarrow v_{2j}$ and changing the colors by the permutation $\alpha \leftrightarrow \beta$.

Thus we have two cases to consider.

Case 1.1. $c(v_{12}) = \beta$ and $c(v_{22}) = \alpha$.

We claim that either $\mathcal{F}_{12}$ or $\mathcal{F}_{22}$ is a coloring for G which is obviously different from c. To prove our claim, on the contrary assume that neither $\mathcal{F}_{12}$ nor $\mathcal{F}_{22}$ is a coloring for G. Since $\mathcal{F}_{12}(v_{12}) = \gamma$, we must have $c(v_{13}) = \gamma$. Also $\mathcal{F}_{22}(v_{22}) = \gamma$ implies that $c(v_{23}) = \gamma$, which is a contradiction. So either $\mathcal{F}_{12}$ or $\mathcal{F}_{22}$ is a coloring for G.

Case 1.2. $c(v_{12}) = \beta$ and $c(v_{22}) = \gamma$.

We proceed by applying strong mathematical induction on $n \geq 3$. It is obvious that $\mathcal{F}_{12}$ is not a coloring for G. Now if $\mathcal{F}_{22}$ is also not a coloring for G, then we must have $c(v_{23}) = \alpha$. Since $\mathcal{L}_{23} \neq \mathcal{L}_{22}$, $\mathcal{L}_{23} = \{\alpha, \beta\}$. Note that if $n = 3$, then $\mathcal{L}_{23} = \{\alpha, \beta\}$ is impossible because we have $\mathcal{L}_{21} = \{\alpha, \beta\}$. This implies that if $n = 3$, then $\mathcal{F}_{22}$ is a coloring for G.

If $n \geq 4$, then $c(v_{23}) = \alpha$ and $c(v_{12}) = \beta$ force the color of v_{13} to be γ. By a similar argument as above we have $\mathcal{L}_{13} = \{\alpha, \gamma\}$. It is obvious that $\mathcal{F}_{13}$ is not a coloring for G. Moreover, if $\mathcal{F}_{23}$ is not a coloring for G we must have $c(v_{24}) = \beta$ and $\mathcal{L}_{24} = \{\beta, \gamma\}$. $c(v_{24}) = \beta$ and $c(v_{13}) = \gamma$ force that $c(v_{14}) = \alpha$. So $\mathcal{L}_{14} = \{\alpha, \beta\}$. Note that if $n = 4$, $\mathcal{L}_{14} = \{\alpha, \beta\}$ is impossible, because $\mathcal{L}_{11} = \{\alpha, \beta\}$. It means that for $n = 4$, $\mathcal{F}_{23}$ is a coloring for G. Similarly, for $n \geq 5$, $\mathcal{F}_{14}$ is not a coloring. If $\mathcal{F}_{24}$ also is not a coloring for G, we have $c(v_{25}) = \gamma$, $\mathcal{L}_{25} = \{\alpha, \gamma\}$. $c(v_{25}) = \gamma$, $c(v_{14}) = \alpha$ force that $c(v_{15}) = \beta$. So $\mathcal{L}_{15} = \{\gamma, \beta\}$. Note that if $n = 5$, then also in this case, $\mathcal{F}_{25}$ is another coloring, so we may assume that $n \geq 6$.

Now it can be seen that the second column in G has the same list of colors with the fifth column. In graph G we delete the vertices $v_{12}, v_{13}, v_{14}, v_{22}, v_{23}, v_{24}$ and join v_{11} to v_{15} and v_{21} to v_{25} and obtain the graph $H = K_2 \times C_{n-3}$, which by induction hypothesis has two different colorings. Note that in each coloring of H, v_{11} has the color α and v_{21} has the color β. Also independent of the coloring of v_{15} and v_{25} from their lists in H, we can add six deleted vertices to H, with their colors from c and obtain two different colorings for G.

Case 2. $\mathcal{L}_{1j} = \mathcal{L}_{2j}$ for each $j \in \{1, ..., n\}$.

Assume that c is a coloring for G. By interchanging the color of vertex v_{1j} in C_n^1 with the color of vertex v_{2j} in C_n^2 and vice versa, we obtain another list coloring for

G. Thus, there are two different list colorings for the graph G.

Case 3. $\mathcal{L}_{1j} \neq \mathcal{L}_{2j}$ for every $j \in \{1, ..., n\}$.

In this case we show that the graph G, indeed has two different colorings. By assumption (ii), one may check that the assignment $c : V(C_n^1) \longrightarrow \cup_{j=1}^{n} \mathcal{L}_{1j}$, where $c(v_{1j}) = \mathcal{L}_{1j} \setminus \mathcal{L}_{2j}$, is a list coloring for C_n^1. Now given this coloring for C_n^1, each vertex v_{2j} of C_n^2 still has two choices of colors $\mathcal{L}_{2j}$. By Proposition 1, C_n^2 with given list of colors, $\mathcal{L}_{2j}$, has two different colorings c_1 and c_2.

Obviously, $c \cup c_1$ and $c \cup c_2$ are two different list colorings for the graph G.

For example, in Figure 1 we show two different list colorings for the graph $G = K_2 \times C_5$, where its list of colors satisfy Case 3. In each coloring the color of each vertex is shown in bold face characters.

The proof of the lemma is now complete. $\square$

C_5^1 :	$1\mathbf{2}$	$2\mathbf{3}$	$\mathbf{1}3$	$\mathbf{2}3$	$1\mathbf{3}$	C_5^1 :	$1\mathbf{2}$	$2\mathbf{3}$	$\mathbf{1}3$	$\mathbf{2}3$	$1\mathbf{3}$
C_5^2 :	$1\mathbf{3}$	$1\mathbf{2}$	$2\mathbf{3}$	$\mathbf{1}3$	$1\mathbf{2}$	C_5^2 :	$1\mathbf{3}$	$\mathbf{1}2$	$\mathbf{2}3$	$1\mathbf{3}$	$\mathbf{1}2$

Figure 1: $G = K_2 \times C_5$

Theorem 1. *Each defining set for the graph $G = C_m \times C_n$, where at least one of m and n is odd, must contain at least one vertex from every two consecutive rows (columns) of the array of vertices of G.*

Proof. Suppose that S is a defining set for the graph $G = C_m \times C_n$ which contains no vertices from consecutive rows, say r_1 and r_2. Also, assume that all vertices in G are colored by unique extension of the colors of S, except the vertices in rows r_1 and r_2. Thus the colors of other vertices of G induce a list of possible colors for the vertices in rows r_1 and r_2, say $\mathcal{L}_{11}, ..., \mathcal{L}_{1n}$, and $\mathcal{L}_{21}, ..., \mathcal{L}_{2n}$, respectively. Since each two adjacent vertices have different colors, we have $\mathcal{L}_{ij} \neq \mathcal{L}_{i(j+1)}$, for $i = 1, 2$; and $j = 1, 2, ..., n$. Also since $\chi(C_m \times C_n) = 3$, for each $\mathcal{L}_{ij}$ we have $|\mathcal{L}_{ij}| = 2$ and $|\cup_{j=1}^{n} \mathcal{L}_{ij}| = 3$, for $i = 1$ and 2. Now by Lemma 2, this contradicts the fact that S is a defining set for the graph G. Similar argument holds for consecutive columns. $\square$

Corollary 1. *If at least one of the positive integers m and n is odd, then*

$$d(C_m \times C_n, \chi) \geq \max \left\{ \left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil \right\}.$$

Proof. By Theorem 1, we have: $d(C_m \times C_n, \chi) \geq \left\lceil \frac{m}{2} \right\rceil$ and $d(C_m \times C_n, \chi) \geq \left\lceil \frac{n}{2} \right\rceil$. Therefore, $d(C_m \times C_n, \chi) \geq \max\{\left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil\}$. □

Next we find an upper bound for $d(C_m \times C_n, \chi)$.

Theorem 2. *If at least one of the positive integers m and n is odd, then*

$$d(C_m \times C_n, \chi) \leq \left\lfloor \frac{m+n-1}{2} \right\rfloor.$$

Proof. To prove this theorem, we present in Table 1a, diagrams showing a defining set S of size $\left\lfloor \frac{m+n-1}{2} \right\rfloor$ for the graph $C_m \times C_n$. In each diagram, a vertex in S is denoted by the symbol $*$. Also, the symbol ● ● ● in the diagrams represent even number of vertices with the first, third, fifth, and so on ... vertices are in the defining set S.

The assignment of colors 1,2, or 3 (having $\chi = 3$) to the vertices in the defining set S depends on the positions of it in the first row and in indicated column of the diagrams. We assign the colors to the defining vertices (i.e. vertices of S) in the first row before we do the assignment of colors to the column.

To get the assignment of colors for the defining vertices in the first row, we proceed as follows. Make a one to one correspondence between all the vertices in that row with the colors 1,2, or 3 in repeating pattern 1,2,3,1,2,3,1,2,... with the first vertex in the row corresponds to color 1. This means that if the second vertex in the row is a defining vertex, then this defining vertex is assigned with color 2; and if, for example there are 11 vertices in the row and the last vertex is a defining vertex, then this defining vertex is given color 2 (the 11^{th} cell in the pattern). Note that non-defining vertices in the first row are not given any color; their colors are forced. (See Table 1b)

The assignment of colors for the defining vertices in the column (shown in the diagram) is performed as follows. Here we also make a one to one correspondence between all the vertices in that column with the colors 1,2, or 3 in repeating pattern 1,2,3,1,2,3,1,2,... with the first vertex in the column corresponds to color of the vertex in the first row which has been corresponded above. This means that if, for example the vertex (not necessarily a defining vertex) in the first row is corresponded to color 2 (in the first row assignment above), then the second vertex in the column corresponds to color 3; the third vertex corresponds to color 1, the fourth vertex corresponds to color 2, and so on. This means that if the second vertex in the column is a defining vertex, then this defining vertex is assigned with color 3. Note that non-defining vertices in this column are not given any color; their colors are forced. (See Table 1b).

It is easy to see that in each diagram in the table, the defining vertices in S **force** the color of non-defining vertices in the first row and also the the color of non-defining vertices in the column shown in the diagram. Subsequently, the colors of the first row and the colors of the given column, force the color of all the other vertices in $C_m \times C_n$. Therefore, the given set S is a defining set for $C_m \times C_n$. For example, the defining set S for $C_{13} \times C_{13}$ is shown in Figures 2 and the unique extension of the colors of S to a 3-coloring of the vertices of $C_{13} \times C_{13}$ is given in Figure 3. □

1		**3**		**2**		**1**		**3**		**2**		
											1	
											3	
											2	
											1	
											3	
											2	

Figure 2: A Defining Set S for $C_{13} \times C_{13}$

1	2	**3**	1	**2**	3	**1**	2	**3**	1	**2**	3	2
2	3	1	2	3	1	2	3	1	2	3	**1**	3
3	1	2	3	1	2	3	1	2	3	1	2	1
1	2	3	1	2	3	1	2	3	1	2	**3**	2
2	3	1	2	3	1	2	3	1	2	3	1	3
3	1	2	3	1	2	3	1	2	3	1	**2**	1
1	2	3	1	2	3	1	2	3	1	2	3	2
2	3	1	2	3	1	2	3	1	2	3	**1**	3
3	1	2	3	1	2	3	1	2	3	1	2	1
1	2	3	1	2	3	1	2	3	1	2	**3**	2
2	3	1	2	3	1	2	3	1	2	3	1	3
3	1	2	3	1	2	3	1	2	3	1	**2**	1
2	3	1	2	3	1	2	3	1	2	3	1	3

Figure 3: A Unique Extension of S to a 3-Coloring of $C_{13} \times C_{13}$

Conjecture 1. *If at least one of the numbers m and n is odd, then*

$$d(C_m \times C_n, \chi) = \left\lfloor \frac{m+n-1}{2} \right\rfloor.$$

Remark. By using a computer program, we have found that $d(C_4 \times C_5, 3) = 4$ and $d(C_5 \times C_5, 3) = 4$, which support Conjecture 1.

3 Defining number of $C_m \times P_n$

If m is even, then $C_m \times P_n$ is a bipartite graph. Thus, in this case $d(C_m \times P_n, \chi) = 1$. If m is odd, with the similar argument in the proof for $C_m \times C_n$, we have a lower bound and an upper bound for $d(C_m \times P_n, \chi)$, as follows.

Theorem 3. *For each odd number m and each positive integer n, we have*

$$d(C_m \times P_n, \chi) \geq \max\left\{\left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{n+1}{2} \right\rceil\right\}.$$

Theorem 4. *For each odd number m and each positive integer n, we have*

$$d(C_m \times P_n, \chi) \leq \left\lfloor \frac{m+n}{2} \right\rfloor.$$

Proof. To show this statement we present a defining set of size $\left\lfloor \frac{m+n}{2} \right\rfloor$ for $C_m \times P_n$ which is similar to the one given in the proof of Theorem 2. The defining sets are as in the Table 2a and the assignment of colors for the defining vertices is shown in Table 2b. □

Conjecture 2. *For each odd number m and each positive integer n, we have,*

$$d(C_m \times P_n, \chi) = \left\lfloor \frac{m+n}{2} \right\rfloor.$$

Remark. By using a computer program, we found that $d(C_3 \times P_3, 3) = 3$, $d(C_3 \times P_5, 3) = 4$, and $d(C_5 \times P_5, 3) = 5$, which are consistent with Conjecture 2.

Table 1a: Defining Sets For $C_m \times C_n$

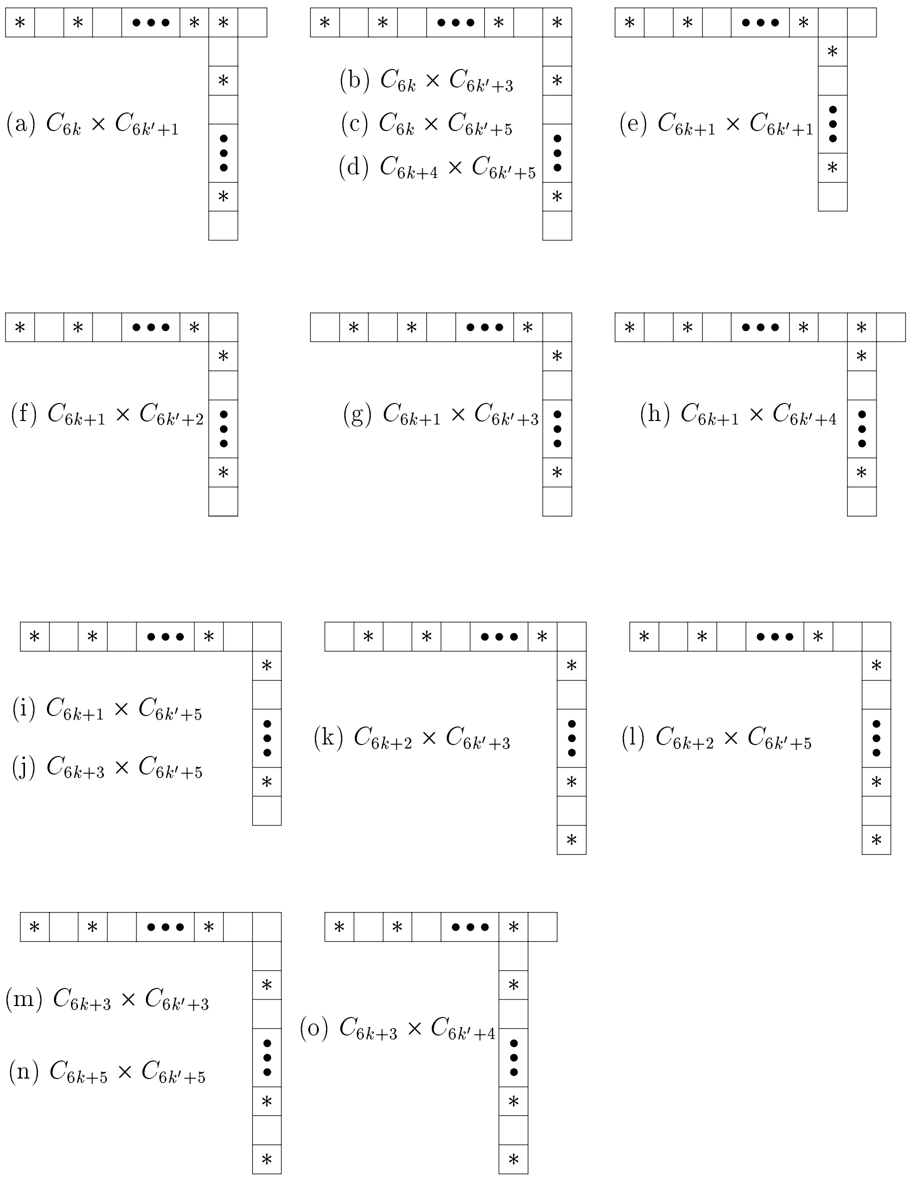

Table 1b: Color Assignment for Defining Sets of $C_m \times C_n$

$1 \star 3 \star \bullet \bullet \bullet 2\ 3 \star$ $\star$ 2 $\star$ $\bullet$ $\bullet$ $\bullet$ 1 $\star$ (a) $C_{6k} \times C_{6k'+1}$	$1 \star 3 \star \bullet \bullet \bullet 1 \star 3$ $\star$ 2 $\star$ $\bullet$ $\bullet$ $\bullet$ 1 $\star$ (b) $C_{6k} \times C_{6k'+3}$	$1 \star 3 \star \bullet \bullet \bullet 3 \star 2$ $\star$ 1 $\star$ $\bullet$ $\bullet$ $\bullet$ 3 $\star$ (c) $C_{6k} \times C_{6k'+5}$	$1 \star 3 \star \bullet \bullet \bullet 3 \star 2$ $\star$ 1 $\star$ $\bullet$ $\bullet$ $\bullet$ 1 $\star$ (d) $C_{6k+4} \times C_{6k'+5}$
$1 \star 3 \star \bullet \bullet \bullet 2 \star \star$ 1 $\star$ $\bullet$ $\bullet$ $\bullet$ 2 $\star$ (e) $C_{6k+1} \times C_{6k'+1}$	$1 \star 3 \star \bullet \bullet \bullet 1 \star$ 3 $\star$ $\bullet$ $\bullet$ $\bullet$ 1 $\star$ (f) $C_{6k+1} \times C_{6k'+2}$	$\star 2 \star 1 \star \bullet \bullet \bullet 2 \star$ 1 $\star$ $\bullet$ $\bullet$ $\bullet$ 2 $\star$ (g) $C_{6k+1} \times C_{6k'+3}$	$1 \star 3 \star \bullet \bullet \bullet 1 \star 3 \star$ 1 $\star$ $\bullet$ $\bullet$ $\bullet$ 2 $\star$ (h) $C_{6k+1} \times C_{6k'+4}$
$1 \star 3 \star \bullet \bullet \bullet 3 \star \star$ 3 $\star$ $\bullet$ $\bullet$ $\bullet$ 1 $\star$ (i) $C_{6k+1} \times C_{6k'+5}$	$1 \star 3 \star \bullet \bullet \bullet 3 \star \star$ 3 $\star$ $\bullet$ $\bullet$ $\bullet$ 3 $\star$ (j) $C_{6k+3} \times C_{6k'+5}$	$\star 2 \star 1 \star \bullet \bullet \bullet 2 \star$ 1 $\star$ $\bullet$ $\bullet$ $\bullet$ 2 $\star$ 1 (k) $C_{6k+2} \times C_{6k'+3}$	$1 \star 3 \star \bullet \bullet \bullet 3 \star \star$ 3 $\star$ $\bullet$ $\bullet$ $\bullet$ 1 $\star$ 3 (l) $C_{6k+2} \times C_{6k'+5}$
$1 \star 3 \star \bullet \bullet \bullet 1 \star \star$ $\star$ 2 $\star$ $\bullet$ $\bullet$ $\bullet$ 3 $\star$ 2 (m) $C_{6k+3} \times C_{6k'+3}$	$1 \star 3 \star \bullet \bullet \bullet 3 \star \star$ $\star$ 1 $\star$ $\bullet$ $\bullet$ $\bullet$ 1 $\star$ 3 (n) $C_{6k+5} \times C_{6k'+5}$	$1 \star 3 \star \bullet \bullet \bullet 3 \star$ $\star$ 2 $\star$ $\bullet$ $\bullet$ $\bullet$ 3 $\star$ 2 (o) $C_{6k+3} \times C_{6k'+4}$	

Table 2a: Defining Sets For $C_m \times P_n$

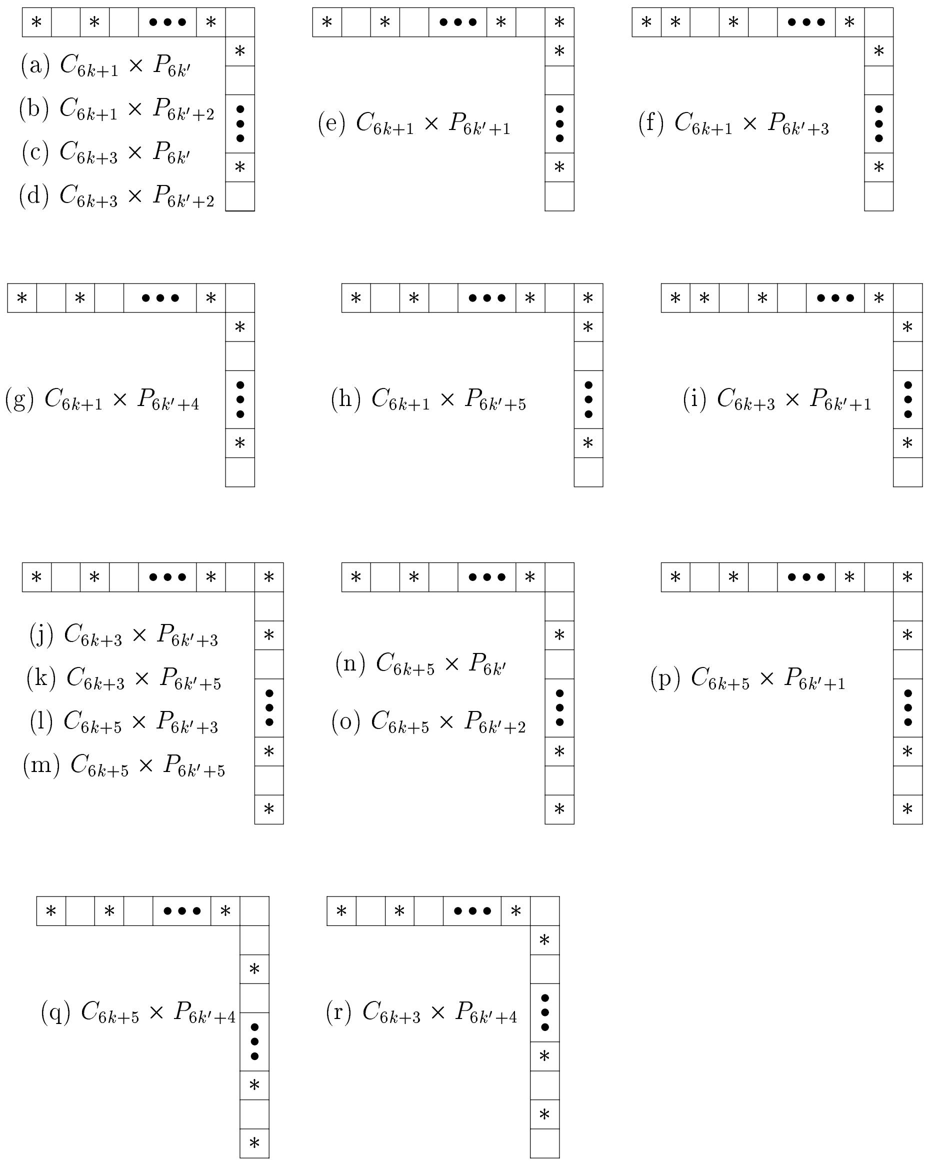

Table 2b: Color Assignment for Defining Sets of $C_m \times P_n$

1 ⋆ 3 ⋆ • • • 2 ⋆ 1 ⋆ • • • 2 ⋆ (a) $C_{6k+1} \times P_{6k'}$	1 ⋆ 3 ⋆ • • • 1 ⋆ 3 ⋆ • • • 1 ⋆ (b) $C_{6k+1} \times P_{6k'+2}$	1 ⋆ 3 ⋆ • • • 2 ⋆ 1 ⋆ • • • 1 ⋆ (c) $C_{6k+3} \times P_{6k'}$	1 ⋆ 3 ⋆ • • • 1 ⋆ 3 ⋆ • • • 3 ⋆ (d) $C_{6k+3} \times P_{6k'+2}$
1 ⋆ 3 ⋆ • • • 2 ⋆ 1 2 ⋆ • • • 3 ⋆ (e) $C_{6k+1} \times P_{6k'+1}$	1 2 ⋆ 1 ⋆ • • • 2 ⋆ 1 ⋆ • • • 2 ⋆ (f) $C_{6k+1} \times P_{6k'+3}$	1 ⋆ 3 ⋆ • • • 3 ⋆ 2 ⋆ • • • 3 ⋆ (g) $C_{6k+1} \times P_{6k'+4}$	1 ⋆ 3 ⋆ • • • 3 ⋆ 2 3 ⋆ • • • 1 ⋆ (h) $C_{6k+1} \times P_{6k'+5}$
1 2 ⋆ 1 ⋆ • • • 3 ⋆ 2 ⋆ • • • 2 ⋆ (i) $C_{6k+3} \times P_{6k'+1}$	1 ⋆ 3 ⋆ • • • 1 ⋆ 3 ⋆ 2 ⋆ • • • 3 ⋆ 2 (j) $C_{6k+3} \times P_{6k'+3}$	1 ⋆ 3 ⋆ • • • 3 ⋆ 2 ⋆ 1 ⋆ • • • 2 ⋆ 1 (k) $C_{6k+3} \times P_{6k'+5}$	1 ⋆ 3 ⋆ • • • 1 ⋆ 3 ⋆ 2 ⋆ • • • 2 ⋆ 1 (l) $C_{6k+5} \times P_{6k'+3}$
1 ⋆ 3 ⋆ • • • 3 ⋆ 2 ⋆ 1 ⋆ • • • 1 ⋆ 3 (m) $C_{6k+5} \times P_{6k'+5}$	1 ⋆ 3 ⋆ • • • 2 ⋆ ⋆ 2 ⋆ • • • 2 ⋆ 1 (n) $C_{6k+5} \times P_{6k'}$	1 ⋆ 3 ⋆ • • • 1 ⋆ ⋆ 1 ⋆ • • • 1 ⋆ 3 (o) $C_{6k+5} \times P_{6k'+2}$	1 ⋆ 3 ⋆ • • • 2 ⋆ 1 ⋆ 3 ⋆ • • • 3 ⋆ 2 (p) $C_{6k+5} \times P_{6k'+1}$
1 ⋆ 3 ⋆ • • • 3 ⋆ ⋆ 3 ⋆ • • • 3 ⋆ 2 (q) $C_{6k+5} \times P_{6k'+4}$	1 ⋆ 3 ⋆ • • • 3 ⋆ 2 ⋆ • • • 3 ⋆ 2 ⋆ (r) $C_{6k+3} \times P_{6k'+4}$		

References

[1] A.D. Keedwell, Critical sets and critical partial latin squares. In: *Graph Theory, Combinatorics, Algorithms and Applications* (Proc. Third China–USA Internat. Conf., Beijing, June 1–5, 1993), 111–124, Singapore: World Scientific Publ. Co. 1994.

[2] A.D. Keedwell, Critical sets for latin squares, graphs and block designs: a survey. *Congressus Numeratium* **113** (1996), 231–245.

[3] M. Mahdian and E.S. Mahmoodian, A characterization of uniquely 2-list colorable graphs. *Ars Combinatoria* **51** (1999), 295–305.

[4] E.S. Mahmoodian, Some problems in graph colorings. In: S. H. Javadpour and M. Radjabalipour, editors, *Proc.* 26*th Annual Iranian Math. Conference*, 215–218, Kerman: Iranian Mathematical Society, University of Kerman, 1995.

[5] E.S. Mahmoodian, R. Naserasr and M. Zaker, Defining sets in vertex coloring of graphs and latin rectangles. *Discrete Mathematics* **167/168** (1997), 451–460.

[6] R. Naserasr, E.S. Mahmoodian, M. Mahdian and F. Harary, On defining sets of vertex coloring of the cartesian product of a cycle with a complete graph. In: Y, Alavi, D.R. Lick, and A. Schwenk, editors: *Combinatorics, Graph Theory and Algorithms*, Kalamazoo, Mi, USA: New Issues Press 1999.

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