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Characterisations of intersection graphs by vertex orderings

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Abstract

Characterisations of interval graphs, comparability graphs, co-comparability graphs, permutation graphs, and split graphs in terms of linear orderings of the vertex set are presented. As an application, it is proved that interval graphs, co-comparability graphs, AT-free graphs, and split graphs have bandwidth bounded by their maximum degree.

1 Introduction

We consider finite, simple and undirected graphs G with vertex set $V(G)$, edge set $E(G)$, and maximum degree $\Delta(G)$. The *complement* of G is the graph $\overline{G}$ with vertex set $V(G)$ and edge set $\{vw : v, w \in V(G), vw \notin E(G)\}$. A *vertex ordering* of G is a total order $(v_1, v_2, \dots, v_n)$ of $V(G)$. Let $\mathcal{S}$ be a finite family of sets. The *intersection graph* of $\mathcal{S}$ has vertex set $\mathcal{S}$ and edge set $\{AB : A, B \in S, A \cap B \neq \emptyset\}$. This paper presents characterisations of a number of popular intersection graphs in terms of vertex orderings. In particular, we consider interval graphs in Section 2, comparability and co-comparability graphs in Section 3, AT-free graphs in Section 4, and chordal graphs in Section 5.

In a vertex ordering $(v_1, v_2, \dots, v_n)$ of a graph G, the *width* of an edge $v_iv_j \in E(G)$ is $|i-j|$. The maximum width of an edge is the *width* of the ordering. The *bandwidth* of G is the minimum width of a vertex ordering of G. Bandwidth is a ubiquitous concept with numerous applications (see [2]). Obviously the bandwidth of G is at least $\frac{1}{2}\Delta(G)$. As an application of our results, we prove upper bounds on the bandwidth of many intersection graphs G in terms of $\Delta(G)$.

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2 Interval Graphs

An *interval graph* is the intersection graph of a finite set of closed intervals in $\mathbb{R}$. Jamison and Laskar [12] (and later Olariu [18]) characterised interval graphs as follows.

Theorem 1 ([12]). *A graph G is an interval graph if and only if G has a vertex ordering $(v_1, v_2, \ldots, v_n)$ such that*

$$\forall\, i < j < k,\; v_i v_k \in E(G) \;\Rightarrow\; v_i v_j \in E(G)\ . \tag{1}$$

A similar result by Gilmore and Hoffman [9] states that G is an interval graph if and only if there is an ordering of the maximal cliques of G such that for each vertex v, the maximal cliques containing v appear consecutively.

Theorem 1 implies the following result of Fomin and Golovach [7].

Corollary 1 ([7]). *Every interval graph G has bandwidth at most $\Delta(G)$.*

Proof. In the vertex ordering $(v_1, v_2, \ldots, v_n)$ from Theorem 1, the width of an edge $v_i v_k \in E(G)$ is $|\{v_i v_j \in E(G) : i < j \leq k\}| \leq \deg(v_i) \leq \Delta(G)$. ☐

A *proper interval graph* is the intersection graph of a finite set $\mathcal{S}$ of closed intervals in $\mathbb{R}$ such that $A \not\subset B$ for all $A, B \in \mathcal{S}$. The following characterisation is due to Jamison and Laskar [12] (and later Looges and Olariu [17]).

Theorem 2 ([12]). *A graph G is a proper interval graph if and only if G has a vertex ordering $(v_1, v_2, \ldots, v_n)$ such that,*

$$\forall\, i < j < k,\; v_i v_k \in E(G) \;\Rightarrow\; v_i v_j \in E(G) \;\wedge\; v_j v_k \in E(G)\ . \tag{2}$$

It is easily seen that the bandwidth of a proper interval graph is one less than the maximum clique size. Moreover, Kaplan and Shamir [13] proved that the bandwidth of any graph G equals the minimum, taken over all proper interval supergraphs G' of G, of the bandwidth of G'.

Note that Hell and Huang [11] recently characterised the 'interval bigraphs' in terms of the existence of vertex orderings that avoid certain forbidden patterns.

3 Comparability Graphs

Let $(P, \preceq)$ be a poset. The *comparability graph* of $(P, \preceq)$ has vertex set P, and distinct elements are adjacent if and only if they are comparable under $\preceq$. We have the following well-known characterisation of comparability graphs.

Theorem 3. *The following are equivalent for a graph G:*
(a) *G is a comparability graph,*
(b) *G has a vertex ordering $(v_1, v_2, \ldots, v_n)$ such that,*

$$\forall\, i < j < k,\; v_i v_j \in E(G) \;\wedge\; v_j v_k \in E(G) \;\Rightarrow\; v_i v_k \in E(G)\ . \tag{3}$$

Proof. Let G be the comparability graph of a poset $(V(G), \preceq)$. A linear extension of $\preceq$ satisfies (3). Given a vertex ordering that satisfies (3), define $v_i \prec v_j$ whenever $v_iv_j \in E(G)$ and $i < j$. Thus $(V(G), \preceq)$ is a poset, and G is a comparability graph. □

A *co-comparability graph* is a complement of a comparability graph. As illustrated in Figure 1, a *function diagram* is a set $\{c_i : 1 \leq i \leq n\}$, where each c_i is a curve $\{(x, f_i(x)) : 0 \leq x \leq 1\}$ for some function $f_i : [0, 1] \to \mathbb{R}$. If each c_i is a line segment we say $\{c_i : 1 \leq i \leq n\}$ is *linear*.

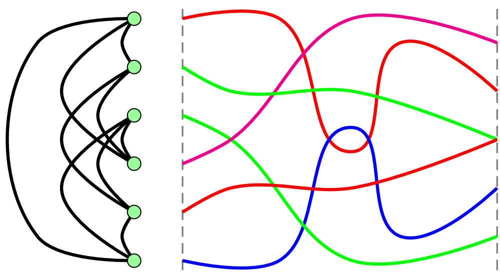

Figure 1: A vertex ordering of the intersection graph of a function diagram.

Theorem 4. *The following are equivalent for a graph G:*
(a) *G is a co-comparability graph,*
(b) *G is the intersection graph of a function diagram, and*
(c) *G has a vertex ordering $(v_1, v_2, \dots, v_n)$ such that,*

$$\forall\, i < j < k,\ v_iv_k \in E(G) \ \Rightarrow\ v_iv_j \in E(G) \ \vee\ v_jv_k \in E(G)\ . \tag{4}$$

Proof. Kratochvíl *et al.* [16] and Golumbic *et al.* [10] independently proved that (a) and (b) are equivalent.

We now prove that (c) implies (a). Suppose that G has a vertex ordering $(v_1, v_2, \dots, v_n)$ satisfying (4). Define $v_i \prec v_j$ if $i < j$ and $v_iv_j \notin E(G)$. Obviously $\prec$ is antisymmetric. Suppose $v_i \prec v_j$ and $v_j \prec v_k$. Then $i < k$ and $v_iv_k \notin E(G)$, as otherwise (4) fails. That is, $v_i \prec v_k$. Thus $\prec$ is transitive, and $(V(G), \preceq)$ is a poset, whose comparability graph is $\overline{G}$. Therefore G is a co-comparability graph.

We now prove that (b) implies (c). Let G be the intersection graph of a function diagram $\{c_i : 1 \leq i \leq n\}$ with corresponding functions $\{f_i : 1 \leq i \leq n\}$. Re-index so that $f_i(0) \leq f_{i+1}(0)$ for all $1 \leq i \leq n-1$. Associate a vertex v_i with each function f_i. Consider an edge $v_iv_k \in E(G)$ and a vertex v_j with $i < j < k$. There is a region S bounded by c_i, c_k, and the line $X = 0$, such that c_j intersects the closed interior of S and the closed exterior of S. Thus c_j intersects the boundary of S. Since f_j is a function on $[0, 1]$, c_j intersects the boundary of S at a point on c_i or c_k. Thus $c_i \cap c_j \neq \emptyset$ or $c_j \cap c_k \neq \emptyset$. Hence $v_iv_k \in E(G)$ or $v_jv_k \in E(G)$. That is, the vertex ordering $(v_1, v_2, \dots, v_n)$ satisfies (4). Note that we could have ordered the vertices with respect to any fixed value of $x_0 \in [0,1]$, and in general, there are many vertex orderings that satisfy (4). □

Corollary 2. *Every co-comparability graph G has bandwidth at most $2\Delta(G) - 1$.*

Proof. In the vertex ordering $(v_1, v_2, \dots, v_n)$ from Theorem 4, the width of an edge $v_i v_k \in E(G)$ is at most $|\{v_i v_j \in E(G) : i < j < k\}| + |\{v_j v_k \in E(G) : i < j < k\}| + 1 \leq (\deg(v_i) - 1) + (\deg(v_k) - 1) + 1 \leq 2\Delta(G) - 1$. □

It is interesting to ask whether Corollary 2 is tight. It is easily seen that the complete bipartite graph $K_{n,n}$, which is a co-comparability graph with maximum degree n, has bandwidth $3n/2$.

Let π be a permutation of $\{1, 2, \dots, n\}$. Let $\pi^{-1}(i)$ denote the position of i in π. The *permutation graph* associated with π has vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_i v_j : (i-j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0\}$. The following characterisations of permutation graphs can be derived from results of Dushnik and Miller [4] and Baker *et al.* [1]. Part (e) is proved as in Theorems 3 and 4.

Theorem 5 ([1, 4]). *The following are equivalent for a graph G:*
(a) *G is a permutation graph,*
(b) *G is the intersection graph of a linear function diagram,*
(c) *G is a comparability graph and a co-comparability graph,*
(d) *G is the comparability graph of a two-dimensional poset,*
(e) *G has a vertex ordering that simultaneously satisfies* (3) *and* (4).

4 AT-free Graphs

An *asteroidal triple* in a graph consists of an independent set of three vertices such that each pair is joined by a path that avoids the neighbourhood of the third. A graph is *asteroidal triple-free* (*AT-free*) if it contains no asteroidal triples.

Lemma 1. *Every AT-free graph G has bandwidth at most $3\Delta(G)$.*

Proof. A *caterpillar* is a tree for which a path (called the spine) is obtained by deleting all the leaves. Let $(v_1, v_2, \dots, v_m)$ be the spine of a caterpillar T. The vertex ordering of T obtained by inserting the leaves adjacent to each v_i immediately after v_i has bandwidth at most $\Delta(T)$.

Kloks *et al.* [15] proved that every (connected) AT-free graph G has a spanning caterpillar subgraph T, and adjacent vertices in G are at distance at most four in T. Moreover, for any edge $vw \in E(G)$ with v and w at distance four in T, both v and w are leaves of T. Consider the above vertex ordering of T to be a vertex ordering of G. The bandwidth is at most $3\Delta(T) \leq 3\Delta(G)$. □

5 Chordal Graphs

A *chord* of a cycle C is an edge not in C connecting two vertices in C. A graph is *chordal* if every induced cycle on at least four vertices has at least one chord. The following famous characterisation of chordal graphs is due to Dirac [3], Fulkerson and Gross [8], and Rose [19].

Theorem 6 ([3, 8, 19]). *The following are equivalent for a graph G:*
(a) *G is chordal,*
(b) *G is the intersection graph of subtrees of a tree, and*
(c) *G has a vertex ordering $(v_1, v_2, \dots, v_n)$ such that,*

$$\forall\, i < j < k,\ v_iv_j \in E(G)\ \wedge\ v_iv_k \in E(G)\ \Rightarrow\ v_jv_k \in E(G)\ . \tag{5}$$

A vertex ordering that satisfies (5) is called a *perfect elimination* vertex ordering.

A chord xy in an even cycle C is *odd* if the distance in C between x and y is odd. A graph is *strongly chordal* if it is chordal and every even cycle on at least six vertices has an odd chord. Farber [5] characterised the strongly chordal graphs as follows.

Theorem 7 ([5]). *The following are equivalent for a graph G:*
(a) *G is strongly chordal,*
(c) *G has a perfect elimination vertex ordering $(v_1, v_2, \dots, v_n)$ such that,*

$$\forall\, i < j < k < \ell,\ v_iv_k \in E(G)\ \wedge\ v_iv_\ell \in E(G)\ \ \wedge\ v_jv_k \in E(G)\ \Rightarrow\ v_jv_\ell \in E(G)\ .$$

It is not possible to bound the bandwidth of every chordal or every strongly chordal graph G in terms of $\Delta(G)$. For example, the bandwidth of the complete binary tree on n vertices is $\approx n/\log n$ [20].

A graph G is a *split graph* if $V(G) = K \cup I$, where K induces a complete graph of G, and I is an independent set of G.

Theorem 8. *The following are equivalent for a graph G:*
(a) *G is a split graph,*
(b) *G is chordal and $\overline{G}$ is chordal,*
(c) *G has a vertex ordering $(v_1, v_2, \dots, v_n)$ simultaneously satisfying* (5) *and*

$$\forall\, i < j < k,\ v_iv_j \in E(G)\ \Rightarrow\ v_jv_k \in E(G)\ \vee\ v_iv_k \in E(G)\ , \tag{6}$$

(d) *G has a vertex ordering $(v_1, v_2, \dots, v_n)$ such that,*

$$\forall\, i < j < k,\ v_iv_j \in E(G)\ \Rightarrow\ v_jv_k \in E(G)\ . \tag{7}$$

Proof. Földes and Hammer [6] proved that (a) and (b) are equivalent.

Observe that (d) implies (c) trivially. We now prove that (a) implies (d). Let G be a split graph with $V(G) = K \cup I$, where K induces a complete subgraph and I is an independent set. Let $m = |I|$. Consider a vertex ordering $(v_1, v_2, \dots, v_n)$ of G where $I = \{v_1, v_2, \dots, v_m\}$ and $K = \{v_{m+1}, v_{m+2}, \dots, v_n\}$. Suppose that $1 \leq i < j < k \leq n$

and $v_iv_j \in E(G)$. There is no edge with both endpoints in I. Thus $j \geq m+1$, and both $v_j, v_k \in K$. Hence $v_jv_k \in E(G)$, and $(v_1, v_2, \ldots, v_n)$ satisfies (7).

It remains to prove that (c) implies (b). Let $(v_1, v_2, \ldots, v_n)$ be a vertex ordering of a graph G satisfying (5) and (6). By Theorem 6, G is chordal. Equation (6) is equivalent to:

$$\forall\, i < j < k,\ v_jv_k \in E(\overline{G}) \ \wedge\ v_iv_k \in E(\overline{G}) \ \Rightarrow\ v_iv_j \in E(\overline{G}) \ . \tag{8}$$

That is, $(v_n, v_{n-1}, \ldots, v_1)$ is a perfect elimination vertex ordering of $\overline{G}$. By Theorem 6, $\overline{G}$ is chordal. □

We have the following bounds on the bandwidth of split graphs. Note that Kloks *et al.* [14] studied the computational complexity of determining the bandwidth of split graphs.

Theorem 9. *Every split graph G has bandwidth at most $\Delta(G)\,(\Delta(G)+2)$. For all $\Delta \geq 2$ there is a split graph G with $\Delta(G) = \Delta$, and G has bandwidth at least $\Delta(G)^2/12$.*

Proof. First we prove the upper bound. Let G be a split graph with $V(G) = K \cup I$, where K induces a complete subgraph, and I is an independent set. The result is trivial if $K = \emptyset$. Now assume that $K \neq \emptyset$. Let I_0 be the set of isolated vertices in G. Consider a vertex ordering π in which the vertices in I_0 precede all other vertices. Let $I_1 = I \setminus I_0$. Regardless of the order of $I_1 \cup K$, the bandwidth of π is at most $|I_1| + |K| - 1$. Thus it suffices to prove that $|I_1| + |K| \leq \Delta(G)\,(\Delta(G)+2) + 1$.

If $I_1 = \emptyset$ then π has bandwidth $\Delta(G)$. Now assume that $I_1 \neq \emptyset$. Let a be the average degree of vertices in I_1. Thus $1 \leq a \leq |K|$. For each vertex $v \in K$, let $b_v = \deg(v) - |K| + 1$. That is, b_v is the number of edges between v and I_1. Let $b = \sum_{v\in K} b_v/|K|$. Thus $a|I_1| = b|K|$, which implies that

$$|I_1| + |K| \ = \ \frac{b|K|}{a} + |K| \ = \ \frac{(b+a)|K|}{a} \ .$$

Now $\Delta(G)$ is at least the average degree of the vertices in K. That is, $\Delta(G) \geq |K| - 1 + b$. Hence

$$\frac{|I_1| + |K|}{(\Delta(G)+1)^2} \ \leq \ \frac{(b+a)|K|}{a(|K|+b)^2} \ .$$

Since $a \leq |K|$,

$$\frac{|I_1| + |K|}{(\Delta(G)+1)^2} \ \leq \ \frac{|K|}{a(|K|+b)} \ .$$

Since $a \geq 1$ and $b \geq 0$,

$$|I_1| + |K| \ \leq \ (\Delta(G)+1)^2 \ = \ \Delta(G)\,(\Delta(G)+2) + 1 \ ,$$

as required.

Now we prove the lower bound. Given Δ, let $n = \lfloor \Delta/2 \rfloor$. Let G be the split graph with $V(G) = K \cup I$, where K is a complete graph on n vertices, and I is an independent set on $n(\Delta - n + 1)$ vertices, such that every vertex in K is adjacent to $\Delta - n + 1$ vertices in I, and every vertex in I is adjacent to exactly one vertex in K. Clearly G has diameter 3, maximum degree Δ, and $n + n(\Delta - n + 1) = n(\Delta + n - 2)$ vertices. It is easily seen that every connected graph with n' vertices and diameter d' has bandwidth at least $(n' - 1)/d'$ [2]. Thus G has bandwidth at least

$$\frac{1}{3}\left(n(\Delta - n + 2) - 1\right) \;=\; \frac{1}{3}\left(\left\lfloor\frac{\Delta}{2}\right\rfloor\left\lceil\frac{\Delta+2}{2}\right\rceil - 1\right) \;\geq\; \frac{\Delta^2}{12}\ .$$

□

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