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Quadrangularity and strong quadrangularity in tournaments

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Abstract

The pattern of a matrix M is a $(0,1)$-matrix which replaces all non-zero entries of M with a 1. A directed graph is said to support M if its adjacency matrix is the pattern of M. If M is an orthogonal matrix, then a digraph which supports M must satisfy a condition known as quadrangularity. We look at quadrangularity in tournaments and determine for which orders quadrangular tournaments exist. We also look at a more restrictive necessary condition for a digraph to support an orthogonal matrix, and give a construction for tournaments which meet this condition.

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1 Introduction

A *directed graph* or *digraph*, D, is a set of vertices $V(D)$ together with a set of ordered pairs of the vertices, $A(D)$, called arcs. If (u, v) is an arc in a digraph, we say that u beats v or u dominates v, and typically write this as $u \to v$. If $v \in V(D)$ then we define the *outset* of v by,

$$O_D(v) = \{u \in V(D) : (v, u) \in A(D)\}.$$

That is, $O_D(v)$ is all vertices in D which v beats. Similarly, we define the set of all vertices in D which beat v to be the *inset* of v, written,

$$I_D(v) = \{u \in V(D) : (u, v) \in A(D)\}.$$

The *closed outset* and *closed inset* of a vertex v are $O_D[v] = O_D(v) \cup \{v\}$ and $I_D[v] = I_D(v) \cup \{v\}$ respectively. The *in-degree* and *out-degree* of a vertex v are $d_D^-(v) = |I_D(v)|$ and $d_D^+(v) = |O_D(v)|$ respectively. When it is clear to which digraph v belongs, we will drop the subscript. The minimum out-degree (in-degree) of D is the smallest out-degree (in-degree) of any vertex in D and is represented by $\delta^+(D)$ ($\delta^-(D)$). Similarly, the maximum out-degree (in-degree) of D is the largest out-degree (in-degree) of any vertex in D and is represented by $\Delta^+(D)$ ($\Delta^-(D)$).

A *tournament* T is a directed graph with the property that for each pair of distinct vertices $u, v \in V(T)$ exactly one of (u, v), (v, u) is in $A(T)$. An n-*tournament* is a tournament on n vertices. If T is a tournament and $W \subseteq V(T)$ we denote by $T[W]$ the subtournament of T induced on W. The *dual* of a tournament T, which we denote by T^r, is the tournament on the same vertices as T with $x \to y$ in T^r if and only if $y \to x$ in T. If $X, Y \subseteq V(T)$ such that $x \to y$ for all $x \in X$ and $y \in Y$, then we write $X \Rightarrow Y$. If $X = \{x\}$ or $Y = \{y\}$ we write $x \Rightarrow Y$ or $X \Rightarrow y$ respectively for $X \Rightarrow Y$. A vertex $s \in V(T)$ such that $s \Rightarrow V(T) - s$ is called a *transmitter.* Similarly a *receiver* is a vertex t of T such that $V(T) - t \Rightarrow t$.

We say that a tournament is *regular* if every vertex has the same out-degree. A tournament is called *near regular* if the largest difference between the out-degrees of any two vertices is 1. Let S be a subset of $\{1, 2, \ldots, 2k\}$ of order k such that if $i, j \in S$, $i + j \not\equiv 0 \pmod{2k+1}$. The tournament on $2k+1$ vertices labeled $0, 1, \ldots, 2k$, with $i \to j$ if and only if $j - i \pmod{2k+1} \in S$ is called a *rotational tournament* with *symbol* S. If $p \equiv 3 \pmod 4$ is a prime and S is the set of quadratic residues modulo p, then the rotational tournament whose symbol is S is called the *quadratic residue tournament* of order p, denoted QR_p. We note that $|O(x) \cap O(y)| = |I(x) \cap I(y)| = k$ for all distinct $x, y \in V(QR_p)$ where $p = 4k+3$. For more on tournaments the reader is referred to [2], [6], and [7].

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be n-vectors over some field (while the following definitions hold over any field, we are interested only in those of characteristic 0). We use $\langle x, y\rangle$ to denote the usual euclidean inner product of x and y. We say that x and y are *combinatorially orthogonal* if $|\{i : x_i y_i \neq 0\}| \neq 1$. Observe, this is a necessary condition for x and y to be orthogonal, for if there were a unique i so that $x_i y_i \neq 0$, then $\langle x, y \rangle = x_i y_i \neq 0$. We say a matrix M is combinatorially orthogonal if every two rows of M are combinatorially orthogonal and every two columns of M are combinatorially orthogonal. In [1], Beasley, Brualdi and Shader study matrices with the combinatorial orthogonality property to obtain a lower bound on the number of non-zero entries in a fully indecomposable orthogonal matrix.

Let M be an $n \times n$ matrix. The *pattern* of M is the $(0,1)$-matrix whose i, j entry is 1 if and only if the i, j entry of M is non-zero. If D is the directed graph whose adjacency matrix is the pattern of M, we say that D *supports* M or that D *is the digraph of* M. We say a digraph D is *out-quadrangular* if for all distinct $u, v \in V(D)$, $|O(u) \cap O(v)| \neq 1$. Similarly, if for all distinct $u, v \in V(D)$, $|I(u) \cap I(v)| \neq 1$, we say D is *in-quadrangular.* If D is both out-quadrangular and in-quadrangular, then we say D is *quadrangular.* It is easy to see that if D is the digraph of M, then D is quadrangular if and only if M is combinatorially orthogonal. So, if D is the digraph of an orthogonal matrix, D must be quadrangular. In [3], Gibson and Zhang study an equivalent version of quadrangularity in undirected graphs. In [5], Lundgren, Severini and Stewart study quadrangularity in tournaments. In the following section we expand on the results in [5], and in section 3 we consider another necessary condition for a digraph to support an orthogonal matrix.

2 Known orders of quadrangular tournaments

In this section we determine for exactly which n there exists a quadrangular tournament on n vertices. We first need some results from [5].

Theorem 2.1 [5] *Let T be an out-quadrangular tournament and choose $v \in V(T)$. Let W be the subtournament of T induced on the vertices of $O(v)$. Then W contains no vertices of out-degree* 1*.*

Theorem 2.2 [5] *Let T be an in-quadrangular tournament and choose $v \in V(T)$. Let W be the subtournament of T induced on $I(v)$. Then W contains no vertices of in-degree* 1*.*

Corollary 2.1 [5] *If T is an out-quadrangular tournament with $\delta^+(T) \geq 2$, then $\delta^+(T) \geq 4$.*

Corollary 2.2 [5] *If T is a quadrangular tournament with $\delta^+(T) \geq 2$ and $\delta^-(T) \geq 2$, then $\delta^+(T) \geq 4$ and $\delta^-(T) \geq 4$.*

Note that the only tournament on 4 vertices with no vertex of out-degree 1 is a 3-cycle together with a receiver. Similarly, the only tournament on 4 vertices with no vertex of in-degree 1 is a 3-cycle with a transmitter. Thus, if a quadrangular tournament T has a vertex v of out-degree 4, $T[O(v)]$ must be a 3-cycle with a receiver, and if u has in-degree 4, $T[I(u)]$ must be a 3-cycle with a transmitter.

Theorem 2.3 *There does not exist a quadrangular near regular tournament of order* 10.

Proof. Suppose T is such a tournament and pick a vertex x with $d^+(x) = 5$. So $d^-(x) = 4$. Therefore $I(x)$ must induce a subtournament comprised of a 3-cycle, and a transmitter. Call this transmitter u. If a vertex y in $O(x)$ has $O(y) = I(x)$, then $|O(y) \cap O(w)| = 1$ for all $w \neq u$ in $I(x)$. This contradicts T being quadrangular, so $O(y) \neq I(x)$ for any $y \in O(x)$. Since every vertex in $O(x)$ beats at most 3 vertices outside of $O(x)$, and since T is near regular we have that $\delta^+(T[O(x)]) \geq 1$. Thus, by Theorem 2.1, we have $\delta^+(T[O(x)]) \geq 2$. This means that $T[O(x)]$ must be the regular tournament on 5 vertices.

Consider the vertex u which forms the transmitter in $T[I(x)]$. Since u beats $I[x]-u$, and T is near regular, u can beat at most one vertex in $O(x)$. If $u \to z$ for any $z \in O(x)$, then $|O(u) \cap O(x)| = |\{z\}| = 1$ which contradicts T being quadrangular. Thus, $z \to u$ for all $z \in O(x)$.

Since T is near regular, it has exactly 5 vertices of out-degree 5, one of which is x. So, there can be at most four vertices in $O(x)$ with out-degree 5. Thus, there exists some vertex in $O(x)$ with out-degree 4, call it v. Since $x \to v$, v beats 2 vertices in $O(x)$ and $v \to u$ there is exactly one vertex $r \in I(x) - u$ such that $v \to r$. Since $O(u) = I[x] - u$, we have $|O(v) \cap O(u)| = |\{r\}| = 1$. Therefore, T is not quadrangular, and so such a tournament does not exist. □

Given a digraph D, and set $S \subseteq V(D)$, we say that S is a *dominating set* in D if each vertex of D is in S or dominated by some vertex of S. The size of a smallest dominating set in D is called the *domination number* of D, and is denoted by $\gamma(D)$. In [5] a relationship is shown to hold in certain tournaments between quadrangularity and the domination number of a subtournament.

Lemma 2.1 *If T is a tournament on 8 vertices with $\gamma(T) \geq 3$ and $\gamma(T^r) \geq 3$, then T is near regular. Further, if $d^-(x) = 3$, then $I(x)$ induces a 3-cycle, and if $d^+(y) = 3$, then $O(y)$ induces a 3-cycle.*

Proof. Let T be such a tournament. If T has a vertex a with $d^-(a) = 0$ or 1, then $I[a]$ would form a dominating set of size 1 or 2 respectively. If T had a vertex b with $I(b) = \{u, v\}$, where $u \to v$, then $\{u, b\}$ forms a dominating set of size 2. So $d_T^-(x) \geq 3$ for all $x \in V(T)$. Similarly, $d_{T^r}^-(x) \geq 3$ for all $x \in V(T)$. Thus,

$$3 \leq d_{T^r}^-(x) = d_T^+(x) = 8 - 1 - d_T^-(x) \leq 7 - 3 = 4$$

for all $x \in V(T)$. That is $3 \leq d_T^+(x) \leq 4$ for all $x \in V(T)$, and T is near regular. Now, pick $x \in V(T)$ with $d^-(x) = 3$. If $I(x)$ induces a transitive triple with transmitter u, then $\{u, x\}$ would form a dominating set in T. Thus, $I(x)$ must induce a 3-cycle. By duality we have that $O(y)$ induces a 3-cycle for all y with $d^+(y) = 3$. □

Up to isomorphism there are 4 tournaments on 4 vertices, and exactly one of these is strongly connected. We refer to this tournament as the *strong 4-tournament,* and note that it is also the only tournament on 4 vertices without a vertex of out-degree 3 or 0.

Lemma 2.2 *Suppose T is a tournament on 8 vertices with $\gamma(T) \geq 3$ and $\gamma(T^r) \geq 3$. Then if $x \in V(T)$ with $d^+(x) = 4$, $O(x)$ induces the strong 4-tournament.*

Proof. By Lemma 2.1, T is near regular so pick $x \in V(T)$ with $d^+(x) = 4$, and let W be the subtournament induced on $O(x)$. If there exists $u \in V(W)$ with $d^+_W(u) = 0$, then since $d^+_T(u) \geq 3$, $u \Rightarrow I(x)$ and $\{u, x\}$ forms a dominating set in T. This contradicts $\gamma(T) \geq 3$, so no such u exists. Now assume there exists a vertex $v \in V(W)$ with $d^+_W(v) = 3$. If $d^+_T(v) = 4$, then $v \to y$ for some $y \in I(x)$. So, $I(v) = I[x] - y$. However, $I(v) = I[x] - y$ forms a transitive triple, a contradiction to Lemma 2.1. So $d^+_T(v) = 3$. Now, since $\delta^+(W) > 0$, the vertices of $W - v$ all have out-degree 1 in W. If some $z \in V(W) - v$ had $d^+_T(z) = 4$, then $z \Rightarrow I(x)$ and $\{x, z\}$ would form a dominating set of size 2. Therefore, all $z \in V(W)$ have $d^+_T(z) = 3$. Since T is near regular, this implies that every vertex of $I[x]$ must have out-degree 4. Further, since $d^+_T(v) = 3$, $O(v) \subseteq O(x)$ and so $I(x) \Rightarrow v$. So, each vertex of $I(x)$ dominates x, v and another vertex of $I(x)$. Thus, each vertex of $I(x)$ dominates a unique vertex of $O(x) - v$. Further each vertex of $O(x) - v$ has in-degree 4 in T and so must be dominated by a unique vertex of $I(x)$. So label the vertices of $I(x)$ as y_1, y_2, y_3 and the vertices of $O(x) - v$ as w_1, w_2, w_3 so that $y_i \to w_i$, and $w_i \to y_j$ for $i \neq j$. Since $I(x)$ and $O(x) - v$ form 3-cycles we may also assume that $y_1 \to y_2 \to y_3$, $y_3 \to y_1$ and $w_1 \to w_2 \to w_3$ and $w_3 \to w_1$. So, $O(w_1) = \{w_2, y_2, y_3\}$ which forms a transitive triple a contradiction to Lemma 2.1. Hence, no such v exists and $1 \leq \delta^+(W) \leq \Delta^+(W) \leq 2$ and W is the strong 4-tournament. □

Theorem 2.4 *Let T be a tournament on 8 vertices. Then $\gamma(T) \leq 2$ or $\gamma(T^r) \leq 2$.*

Proof. Suppose to the contrary that T is a tournament on 8 vertices with $\gamma(T) \geq 3$ and $\gamma(T^r) \geq 3$. By Lemma 2.1 we know that T is near regular. Let W be the subtournament of T induced on the vertices of out-degree 4. We can always choose x in W with $d^-_W(x) \geq 2$. So pick $x \in V(T)$ with $d^+_T(x) = 4$ so that it dominates at most one vertex of out-degree 4. By Lemma 2.2, $O(x)$ induces the strong 4-tournament. By our choice of x, at least one of the vertices with out-degree 2 in $T[O(x)]$ has out-degree 3 in T. Call this vertex x_1. Label the vertices of $O(x_1) \cap O(x)$ as x_2 and x_3 so that $x_2 \to x_3$, and label the remaining vertex of $O(x)$ as x_0. Note since $T[O(x)]$ is the strong 4-tournament, we must have $x_3 \to x_0$ and $x_0 \to x_1$. Since $d^+_T(x_1) = 3$, x_1 must dominate exactly one vertex in $I(x)$, call it y_1. Recall $I(x)$ must induce a 3-cycle by Lemma 2.1, so we can label the remaining vertices of $I(x)$ as y_2 and y_3 so that $y_1 \to y_2 \to y_3$ and $y_3 \to y_1$. Note since $O(x_1) \cap I(x) = \{y_1\}$, $y_2 \to x_1$ and $y_3 \to x_1$. Also, by Lemma 2.1, $O(x_1)$ forms a 3-cycle, so $x_3 \to y_1$ and $y_1 \to x_2$.

Now, assume that $y_1 \to x_0$. Then $O(y_1) = \{x_0, x_2, x, y_2\}$. Now, since $O(x_3) \cap O(x) = \{x_0\}$, $d^+_T(x_3) = 3$ or else $x_3 \Rightarrow I(x)$ and $\{x, x_3\}$ forms a dominating set of size 2. So, x_3 dominates exactly one of y_2 or y_3. If $x_3 \to y_2$ then $y_3 \to x_3$ and since $y_3 \to x_1$, $\{y_1, y_3\}$ forms a dominating set of size 2. So, assume $x_3 \to y_3$ and $y_2 \to x_3$. Then $x, y_3, x_1, x_3 \in O(y_2)$ and $\{y_2, y_1\}$ forms a dominating set of size 2. Thus $x_0 \to y_1$.

Since $d^+(x_3) = 3$, if $x_3 \to y_3$ then $O(x_3) = \{y_1, y_3, x_0\}$. However, $y_3 \to y_1$ and $x_0 \to y_1$ so $O(x_3)$ forms a transitive triple, a contradiction to Lemma 2.1. Thus $y_3 \to x_3$. Since $d^+_T(y_3) \leq 4$ and $y_1, x, x_1, x_3 \in O(y_3)$, these are all the vertices in $O(y_3)$. So, $x_0 \to y_3$.

If $x_0 \to y_2$ then $x_0 \Rightarrow I(x)$ and $\{x, x_0\}$ form a dominating set of size 2, so $y_2 \to x_0$. So, $x_0, y_3, x \in O(y_2)$ and $y_1, x_2, x_3 \in O(x_1)$, and so $\{y_2, x_1\}$ forms a dominating set of size 2. Therefore, such a tournament cannot exist. □

Theorem 2.5 *No tournament T on 9 vertices with $\delta^+(T) \geq 2$ is out-quadrangular.*

Proof. Suppose to the contrary T is such a tournament. Since T is out-quadrangular, and $\delta^+(T) \geq 2$, by Corollary 2.1, $\delta^+(T) \geq 4$. Since the order of T is 9, this means T must be regular. Pick a vertex $x \in V(T)$. Then $O(x)$ must induce a subtournament which is a 3-cycle together with a receiver. Call the receiver of this subtournament y. Since T is regular, $d^+(y) = 4$. Since $I(y) = O[x] - y$, this means $O(y) = I(x)$. So, $O(y) = I(x)$ must induce a subtournament which is a 3-cycle together with a receiver vertex. Call this receiver z. Since $d^+(z) = 4$, $y \to z$ and $I(x) - z$ dominate z, $O(z) = O[x] - y$. Now, $x \Rightarrow O(x) - y$ and $O(x) - y$ is a 3-cycle so $T[O(z)]$ must contain a vertex of out-degree 1. Hence, by Theorem 2.1, T is not out-quadrangular. Thus, no such tournament exists.

□

Corollary 2.3 *No tournament T on 9 vertices with $\delta^-(T) \geq 2$ is in-quadrangular.*

Proof. Let T be a tournament on 9 vertices with $\delta^-(T) \geq 2$. Then T^r is not out-quadrangular by Theorem 2.5. Thus T is not in-quadrangular. □

We now state a few more results from [5].

Theorem 2.6 [5] *Let T be a tournament on 4 or more vertices with a vertex x of out-degree 1, say $x \to y$. Then, T is quadrangular if and only if*

1. $O(y) = V(T) - \{x, y\}$,
2. $\gamma(T - \{x, y\}) > 2$,
3. $\gamma((T - \{x, y\})^r) > 2$.

Theorem 2.7 [5] *Let T be a tournament on 3 or more vertices with a transmitter s and receiver t. Then T is quadrangular if and only if both $\gamma(T - \{s, t\}) > 2$ and $\gamma((T - \{s, t\})^r) > 2$.*

Theorem 2.8 [5] *Let T be a tournament with a transmitter s and no receiver. Then T is quadrangular if and only if, $\gamma(T - s) > 2$, $T - s$ is out-quadrangular, and $\delta^+(T - s) \geq 2$.*

Corollary 2.4 [5] *Let T be a tournament with a receiver t and no transmitter. Then T is quadrangular if and only if $\gamma((T-t)^r) > 2$, $T-t$ is in-quadrangular, and $\delta^-(T-t) \geq 2$.*

Corollary 2.5 *No quadrangular tournament of order 10 exists.*

Proof. By Corollaries 2.2 and 2.4, and by Theorems 2.6, 2.7 and 2.8, a quadrangular tournament T must satisfy one of the following.

1. $\delta^+(T) \geq 4$ and $\delta^-(T) \geq 4$, and hence T is near regular.
2. T has a transmitter s and receiver t such that $\gamma(T - \{s,t\}) > 2$ and $\gamma((T - \{s,t\})^r) > 2$.
3. T contains an arc (x, y) such that $O(y) = I(x) = V(T) - \{x, y\}$ and $\gamma(T - \{x,y\}) > 2$ and $\gamma((T - \{x,y\})^r) > 2$.
4. T has a transmitter s and $T - s$ is out-quadrangular with $\delta^+(T-s) \geq 2$.
5. T has a receiver t and $T - t$ is in-quadrangular with $\delta^-(T-t) \geq 2$.

Note, Theorem 2.3 implies that case 1 is impossible. If 2 or 3 were satisfied, then there would be a tournament on 8 vertices such that it and its dual have domination number at least 3, which contradicts Theorem 2.4. If 4 were satisfied, then $T-s$ would be of order 9 and out-quadrangular, a contradiction to Theorem 2.5. Similarly, 5 contradicts Corollary 2.3. Thus, no quadrangular tournament on 10 vertices exists.

□

The following result shows how to construct quadrangular tournaments of order 11 and higher.

Theorem 2.9 *If $n \geq 11$, then there exists a quadrangular tournament on n vertices.*

Proof. Let T' be a tournament with $V(T') = \{1, 2, \ldots, k\}$ which satisfies $\delta^+(T') \geq 2$, $\delta^-(T') \geq 2$, and $|O(u) \cap O(v)| \geq 2$ and $|I(u) \cap I(v)| \geq 2$ for all distinct $u, v \in V(T')$. Note, the smallest tournament which satisfies these requirements is QR_{11}. Let $n \geq 11$, and let $a_1, a_2, \ldots, a_k$ be a sequence of positive integers which satisfies $\sum_{i=1}^{k} a_i = n$. Let $T_1, T_2, \ldots, T_k$ be tournaments of orders $a_1, a_2, \ldots, a_k$ respectively. Construct the tournament T on n vertices as follows. Start with a set V of n vertices, and partition V into sets $S_1, S_2, \ldots, S_k$ of size $a_1, a_2, \ldots, a_k$ respectively. Place arcs in each S_i to form T_i. Now, add arcs such that $S_i \Rightarrow S_j$ if and only if $i \to j$ in T'. We claim that the resulting tournament, T, is quadrangular.

Choose distinct vertices u and v of T. We consider two cases. First, if $u, v \in S_i$ for some i, then since $\delta^+(T') \geq 2$, there exists at least two sets S_j and S_m such that $S_i \Rightarrow S_j$ and $S_i \Rightarrow S_m$. So,

$$|O(u) \cap O(v)| \geq |S_j| + |S_m| = a_j + a_k \geq 2.$$

Similarly, since $\delta^-(T') \geq 2$, $|I(u) \cap I(v)| \geq 2$.

Now assume that $u \in S_i$ and $v \in S_j$ for $i \neq j$. Then, since $|O_{T'}(i) \cap O_{T'}(j)| \geq 2$, there exist at least two sets S_m and S_p such that $S_i \Rightarrow S_m \cup S_p$ and $S_j \Rightarrow S_m \cup S_p$. So,

$$|O(u) \cap O(v)| \geq |S_m| + |S_p| = a_m + a_p \geq 2.$$

Similarly, since $|I_{T'}(i) \cap I_{T'}(j)| \geq 2$, $|I(u) \cap I(v)| \geq 2$. Thus T is a quadrangular tournament of order $n \geq 11$. □

We now characterize those n for which there exists a quadrangular tournament of order n.

Theorem 2.10 *There exists a quadrangular tournament of order n if and only if $n = 1, 2, 3, 9$ or $n \geq 11$.*

Proof. Note that the single vertex, the single arc, and the 3-cycle are all quadrangular. Now, recall that the smallest tournament with domination number 3 is QR_7 (for a proof of this see [4]). Further, QR_7 is isomorphic to its dual, so $\gamma(QR_7^r) = 3$. This fact together with Theorems 2.6 and 2.7 tell us that the smallest quadrangular tournament, T, on $n \geq 4$ vertices with $\delta^+(T) = \delta^-(T) = 0$ or $\delta^+(T) = 1$ or $\delta^-(T) = 1$ has order 9.

Theorem 2.8 and Corollary 2.4 together with the fact that QR_7 is the smallest tournament with domination number 3 imply that a quadrangular tournament with just a transmitter or receiver must have at least 8 vertices. However, QR_7 is the only tournament on 7 vertices with domination number 3 and a quick check shows that QR_7 is neither out-quadrangular nor in-quadrangular. So, QR_7 together with a transmitter or receiver is not quadrangular, and hence any quadrangular tournament with just a transmitter or receiver must have order 9 or higher.

Corollary 2.2 states that if $\delta^+(T) \geq 2$ and $\delta^-(T) \geq 2$, then $\delta^+(T) \geq 4$ and $\delta^-(T) \geq 4$. The smallest tournament which meets these requirements is a regular tournament on 9 vertices. Thus, there are no quadrangular tournaments of order $4, 5, 6, 7$ or 8. The result now follows from Corollary 2.5 and Theorem 2.9. □

It turns out that quadrangularity is a common (asymptotic) property in tournaments as the following probabilistic result shows.

Theorem 2.11 *Almost all tournaments are quadrangular.*

Proof. Let $P(n)$ denote the probability that a random tournament on n vertices contains a pair of distinct vertices x and y so that $|O(x) \cap O(y)| = 1$. We now give an over-count for the number of labeled tournaments on n vertices which contain such a pair, and show $P(n) \to 0$ as $n \to \infty$.

There are $\binom{n}{2}$ ways to pick distinct vertices x and y, and the arc between them can be oriented so that $x \to y$ or $y \to x$. There are $n-2$ vertices which can play the role of z where $\{z\} = O(x) \cap O(y)$. For each $w \notin \{x, y, z\}$ there are 3 ways to orient the arcs from x and y to w, namely $w \Rightarrow \{x, y\}$, $w \to x$ and $y \to w$, or $w \to y$

and $x \to w$. Also, there are $n-3$ such w. The arcs between all other vertices are arbitrary. So there are $2^{\binom{n-2}{2}}$ ways to finish the tournament. When orienting the remaining arcs we may double count some of these tournaments, so all together there are at most

$$2\binom{n}{2}(n-2)3^{n-3}2^{\binom{n-2}{2}}$$

tournaments containing such a pair of vertices. Now, there are $2^{\binom{n}{2}}$ total labeled tournaments so,

$$\begin{aligned}
0 \le P(n) &\le \frac{2\binom{n}{2}(n-2)3^{(n-3)}2^{\binom{n-2}{2}}}{2^{\binom{n}{2}}} \\
&= \frac{n(n-1)(n-2)3^{(n-3)}2^{\binom{n-2}{2}}}{2^{\binom{n-2}{2}+n-2+n-1}} \\
&= \frac{n(n-1)(n-2)3^{n-3}}{2^{2n-3}} \\
&= \frac{n(n-1)(n-2)3^{n-3}}{2^{2(n-3)}2^3} \\
&= \frac{n(n-1)(n-2)}{8}\left(\frac{3}{4}\right)^{n-3} \\
&= \frac{\frac{1}{8}n(n-1)(n-2)}{(\frac{4}{3})^{n-3}}.
\end{aligned}$$

Since this value tends to 0 as n tends to ∞, it must be that $P(n) \to 0$ as $n \to \infty$.

From duality we have that the probability that vertices x and y exist such that $|I(x) \cap I(y)| = 1$ also tends to 0 as n tends to ∞. Thus, the probability that a tournament is not quadrangular tends to 0 as n tends to ∞. That is, almost all tournaments are quadrangular. □

3 Strong Quadrangularity

In this section we define a stronger necessary condition for a digraph to support an orthogonal matrix, and give a construction for a class of tournaments which satisfy this condition. Let D be a digraph. Let $S \subseteq V(D)$ be such that for all $u \in S$, there exists $v \in S$ such that $O(u) \cap O(v) \neq \emptyset$, and let $S' \subseteq V(D)$ be such that for all $u \in S'$, there exists $v \in S'$ such that $I(u) \cap I(v) \neq \emptyset$. We say that D is *strongly quadrangular* if for all such sets S and S',

(i) $$\left|\bigcup_{u,v\in S}(O(u) \cap O(v))\right| \ge |S|,$$

(ii) $$\left|\bigcup_{u,v\in S'}(I(u) \cap I(v))\right| \ge |S'|.$$

In [8], Severini showed that strong quadrangularity is a necessary condition for a digraph to support an orthogonal matrix. To see that this is in fact a more restrictive condition consider the following tournament. Let T be a tournament with $V(T) = \{0,1,2,3,4,5,6,x,y\}$ so that $\{0,1,2,3,4,5,6\}$ induce the tournament QR_7, $x \to y$ and $O(y) = I(x) = V(T) - \{x,y\}$. In the previous section we saw that T is quadrangular. Now consider the set of vertices $S = \{0,1,5\}$. Since each of $0,1,5$ beat x, we have that for all $u \in S$, there exits $v \in S$ so that $O(u) \cap O(v) \neq \emptyset$. Also,

$$\begin{aligned} \left| \bigcup_{u,v \in S} (O(u) \cap O(v)) \right| &= |(O(0) \cap O(1)) \cup (O(0) \cap O(5)) \cup (O(1) \cap O(5))| \\ &= |\{2,x\} \cup \{2,x\} \cup \{2,x\}| \\ &= 2 \\ &< |S|. \end{aligned}$$

So T is not strongly quadrangular. We now construct a class of strongly quadrangular tournaments, but first observe the following lemma.

Lemma 3.1 *Let T be a tournament on $n \geq 4$ vertices. Then there must exist distinct $a, b \in V(T)$ such that $O(a) \cap O(b) \neq \emptyset$.*

Proof. Pick a vertex a of maximum out-degree in T. As, $n \geq 4$, $d^+(a) \geq 2$. Pick a vertex b of maximum out-degree in the subtournament W induced on $O(a)$. As $d^+(a) \geq 2$, $d^+_W(b) \geq 1$. Thus, $|O(a) \cap O(b)| = d^+_W(b) \geq 1$. □

Theorem 3.1 *Pick $l \geq 1$. Let T' be a strong tournament on the vertices $\{1,2,\ldots,l\}$, and let $T_1, T_2, \ldots, T_l$ be regular or near-regular tournaments of order $k \geq 5$. Construct a tournament T on kl vertices as follows. Let V be a set of kl vertices. Partition the vertices of V into l subsets $V_1, \ldots, V_l$ of size k and place arcs to form copies of $T_1, T_2, \ldots, T_l$ on $V_1, \ldots, V_l$ respectively. Finally, add arcs so that $V_i \Rightarrow V_j$ if and only if $i \to j$ in T'. Then the resulting tournament, T, is a strongly quadrangular tournament.*

Proof. Pick $S \subseteq V(T)$. Define the set

$$A = \{V_i : \exists u, v \in S \ni u \neq v \text{ and } u, v \in V_i\},$$

and define the set

$$B = \{V_i : \exists! u \in S \ni u \in V_i\}.$$

Let $\alpha = |A|$, and $\beta = |B|$. Then, since each V_i has k vertices, $k\alpha + \beta \geq |S|$. Consider the subtournaments of T' induced on the vertices corresponding to A and B. These are tournaments and so must contain a Hamiltonian path. Let $A_1, \ldots, A_\alpha$ and $B_1, \ldots, B_\beta$ be the elements of A and B respectively, labeled such that $A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_\alpha$ and $B_1 \Rightarrow B_2 \Rightarrow \cdots \Rightarrow B_\beta$. By definition of A, each A_i contains at least two vertices of S, and so if $x, y \in S$ and $x, y \in A_i$, $i \leq \alpha - 1$, then $A_{i+1} \subseteq O(x) \cap O(y)$. Thus,

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k(\alpha - 1).$$

We now consider three cases depending on β.

First assume that $\beta \geq 2$. Consider the vertices of S in B. We see that if $x, y \in S$ and $x \in B_i$ and $y \in B_{i+1}$ then $O(y) \cap B_{i+1} \subseteq O(x) \cap O(y)$. Thus, $|O(x) \cap O(y)| \geq \lfloor \frac{k-1}{2} \rfloor$, and so

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k(\alpha - 1) + \left\lfloor \frac{k-1}{2} \right\rfloor (\beta - 1) \geq k(\alpha - 1) + 2\beta - 2 \geq k(\alpha - 1) + \beta.$$

Now, since T' is a tournament, either $A_1 \Rightarrow B_1$ or $B_1 \Rightarrow A_1$. If $A_1 \Rightarrow B_1$, then for vertices $x, y \in A_1$ we know $B_1 \subseteq O(x) \cap O(y)$. Since no vertex of B_1 had been previously counted, we have that

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k(\alpha - 1) + \beta + k = k\alpha + \beta \geq |S|.$$

So, assume that $B_1 \Rightarrow A_1$. Then for the single vertex of S in B_1, u, and a vertex v of S in A_1 $O(v) \subseteq O(u) \cap O(v)$. This adds $\lfloor \frac{k-1}{2} \rfloor$ vertices which were not previously counted. Also, since T' is strong, some $A_i \Rightarrow V_j$ for some $V_j \notin A$. We counted at most $\lfloor \frac{k-1}{2} \rfloor$ vertices in V_j before, and since A_i contains at least two vertices x, y from S these vertices add at least $k - \lfloor \frac{k-1}{2} \rfloor$ vertices which were not previously counted, so

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k(\alpha - 1) + \beta + \left\lfloor \frac{k-1}{2} \right\rfloor + \left(k - \left\lfloor \frac{k+1}{2} \right\rfloor \right) = k\alpha + \beta \geq |S|.$$

Now assume that $\beta = 1$. Since T' is strong we know that $A_i \Rightarrow V_j$ for some $V_j \notin A$. So,

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k\alpha.$$

Now, if $|S| \leq k\alpha$, then we are done, so assume that $|S| = k\alpha + 1$. So, for every $A_i \in A$, $A_i \subseteq S$. So by Lemma 3.1 we can find two vertices of S in A_1 which beat a common vertex of A_i, adding one more vertex to our count, and

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k\alpha + 1 \geq |S|.$$

For the last case, assume that $\beta = 0$. Then since T' is strong we once again have that some $A_i \Rightarrow V_j$ for some $V_j \notin A$. Thus,

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k\alpha \geq |S|.$$

Note that the dual of T' will again be strong, and the dual of each T_i will again be regular or near regular. Thus, by appealing to duality in T we have that for all $S \subseteq V(T)$,

$$\left| \bigcup_{u,v \in S} I(u) \cap I(v) \right| \geq |S|,$$

and so T is a strongly quadrangular tournament.

□

Recall that strong quadrangularity is a necessary condition for a digraph to support an orthogonal matrix. To emphasize this, consider the strongly quadrangular tournament, T, which the construction in the previous theorem gives on 15 vertices. For this tournament, T_1, T_2 and T_3 are all regular of order 5, and T' is the 3-cycle. Note that up to isomorphism, there is only one regular tournament on 5 vertices, so without loss of generality, assume that T_1, T_2 and T_3 are the rotational tournament with symbol $\{1, 2\}$. We now show that T cannot be the digraph of an orthogonal matrix.

Let J_5 denote the 5×5 matrix of all 1s, O_5 the 5×5 matrix of all 0s and set

$$RT_5 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then the adjacency matrix M of T is

$$M = \begin{pmatrix} RT_5 & J_5 & O_5 \\ O_5 & RT_5 & J_5 \\ J_5 & O_5 & RT_5 \end{pmatrix}.$$

Now, suppose to the contrary that there exists an orthogonal matrix U whose pattern is M. Let R_i and C_i denote the i^{th} rows and columns of U respectively for each $i = 1, \ldots, 15$, and let $U_{i,j}$ denote the i, j entry of U. Observe from the pattern of U that the only entries of U which contribute to $\langle C_i, C_j \rangle$ for $i = 1, \ldots, 5$, $j = 6, \ldots, 10$ are in the first five rows. So, $\langle C_1, C_j \rangle = U_{4,1}U_{4,j} + U_{5,1}U_{5,j}$ for $j = 6, \ldots, 10$. Thus, since $0 = \langle C_1, C_j \rangle$ for each $j \neq 1$,

$$U_{4,1} = \frac{-U_{5,1}U_{5,6}}{U_{4,6}} = \frac{-U_{5,1}U_{5,7}}{U_{4,7}} = \frac{-U_{5,1}U_{5,8}}{U_{4,8}} = \frac{-U_{5,1}U_{5,9}}{U_{4,9}} = \frac{-U_{5,1}U_{5,10}}{U_{4,10}}.$$

Since $U_{5,1} \neq 0$ this gives,

$$-\frac{U_{4,1}}{U_{5,1}} = \frac{U_{5,6}}{U_{4,6}} = \frac{U_{5,7}}{U_{4,7}} = \frac{U_{5,8}}{U_{4,8}} = \frac{U_{5,9}}{U_{4,9}} = \frac{U_{5,10}}{U_{4,10}}.$$

So, the vectors $(U_{4,6}, \ldots, U_{4,10})$ and $(U_{5,6}, \ldots, U_{5,10})$ are scalar multiples of each other. Now, note that for $j = 6, \ldots, 10$, we have $0 = \langle C_2, C_j \rangle = U_{1,2}U_{1,j} + U_{5,2}U_{5,j}$. So, by applying the same argument, we see that $(U_{5,6}, U_{5,7}, U_{5,8}, U_{5,9}, U_{5,10})$ is a scalar multiple of $(U_{1,6}, U_{1,7}, U_{1,8}, U_{1,9}, U_{1,10})$. So, $(U_{4,6}, U_{4,7}, U_{4,8}, U_{4,9}, U_{4,10})$ is a scalar multiple of $(U_{1,6}, U_{1,7}, U_{1,8}, U_{1,9}, U_{1,10})$. Now, from the pattern of U we see that only the 6^{th} through 10^{th} columns contribute to $\langle R_1, R_4\rangle$. So, since linearly dependent nonzero vectors cannot be orthogonal,

$$\langle R_1, R_4\rangle = \langle (U_{1,6}, U_{1,7}, U_{1,8}, U_{1,9}, U_{1,10}), (U_{4,6}, U_{4,7}, U_{4,8}, U_{4,9}, U_{4,10})\rangle \neq 0.$$

This contradicts our assumption that U is orthogonal. So, T is not the digraph of an orthogonal matrix.

4 Conclusions

The problem of determining whether or not there exist tournaments (other than the 3-cycle) which support orthogonal matrices has proved to be quite difficult. As we have seen in sections 2 and 3, for large values of n we can almost always construct examples of tournaments which meet our necessary conditions. Knowing that almost all tournaments are quadrangular and having a construction for an infinite class of strongly quadrangular tournaments, one may believe that there will exist a tournament which supports an orthogonal matrix. However, attempting to find an orthogonal matrix whose digraph is a given tournament has proved to be a difficult task. In general, aside from the 3-cycle, the existence of a tournament which supports an orthogonal matrix is still an open problem. We conclude this section with a result that may lead one to believe non-existence is the answer to this problem.

Theorem 4.1 *Other than the* 3*-cycle, there does not exist a tournament on* 10 *or fewer vertices which is the digraph of an orthogonal matrix.*

Proof. By Theorem 2.10 there exists a quadrangular n-tournament for $n \leq 10$ if and only if n is 1, 2, 3 or 9. Note, in the case $n = 1$ and $n = 2$, the only tournaments are the single vertex and single arc, both of whose adjacency matrices have a column of zeros. Since orthogonal matrices have full rank, these cannot support an orthogonal matrix. When $n = 3$, the 3-cycle is the only quadrangular tournament. The adjacency matrix for this tournament is a permutation matrix and hence orthogonal. Now consider $n = 9$. By Theorem 2.5, if T is quadrangular, $\delta^+(T) \leq 1$. If $\delta^+(T) = 0$, then T's adjacency matrix will have a row of zeros, and T cannot be the digraph of an orthogonal matrix. So we must have $\delta^+(T) = 1$. So by Theorem 2.6, T has an arc (x, y) with $O(y) = I(x) = V(T) - \{x, y\}$ and $\gamma(T - \{x, y\}) > 2$. The only 7-tournament with domination number greater than 2 is QR_7, thus $T - \{x, y\} = QR_7$. However, in section 3 we observed that this tournament is not strongly quadrangular. Thus, other than the 3-cycle, no tournament on 10 or fewer vertices can be the digraph of an orthogonal matrix. □

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