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Compatibly ordered Room squares

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Abstract

Let $S = \{\infty\} \cup Z_n$ be a set of $n+1$ elements called symbols. A *compatibly ordered Room square* of side n (on symbol set S) is an $n \times n$ array, F, that satisfies the following properties:

(i) every cell of F either is empty or contains an ordered pair of distinct symbols from S;

(ii) the cell (s, s) contains the ordered pair (∞, s) for $s \in Z_n$;

(iii) the set of the first components of all ordered pairs in row s is the same as that in column s, for all $s \in S$;

(iv) F is a Room square when each ordered pair (x, y) is viewed as an unordered pair $\{x, y\}$.

In this article, we will show that there exists a compatibly ordered Room square of side n if and only if n is odd and $n \neq 3, 5$.

1 Introduction and background

Room squares were named after T.G. Room who published a paper [6] in 1955 in which he proved that Room squares of side three and five do not exist and constructed a Room square of side seven.

An orthogonal Steiner triple system of order n (briefly OSTS(n)) was originally introduced in 1968 by O'Shaughnessy [10] as a method of constructing Room squares.

Room frames are of fundamental importance in recursive constructions for Room squares. The first Room frame constructed in the literature was one of type 2^5. It was presented by Wallis in [11]. Note that a Room frame of type 1^n is equivalent to a Room square of side n.

The existence problem for Room squares has been given quite a lot of attention by various authors (see [3]). The problem was completely settled in 1974 (see [11]). That is,

Theorem 1.1 *A Room square of side n exists if and only if n is odd and $n \neq 3$ or 5.*

An orthogonal group-divisible design (briefly OGDD) is a generalization of an OSTS suggested in 1991 by Stinson and Zhu [8]. OGDDs are of fundamental importance in recursive constructions for OSTSs.

The existence problem for orthogonal Steiner triple systems has been given quite a lot of attention by various authors (see [1, 8]) since it was posed by O'Shaughnessy in 1968. The problem was completely settled in 1994 (see [1]). That is,

Theorem 1.2 *An orthogonal Steiner triple system of order n exists if and only if $n \equiv 1, 3 \pmod 6$, $n \geq 7$ and $n \neq 9$.*

While OGDDs are useful tools in constructions of OSTSs, finding when they exist is an interesting problem in itself.

To establish recursive constructions for OGDDs, the author [12] in 1996 introduced the concept of a cyclic-OGDD and renamed it as a compatibly ordered-OGDD in [13] as follows.

Definition 1.3 (x, y, z) is defined to be $\{(x, y), (y, z), (z, x)\}$ and is called a *cyclically ordered* 3-subset. A *compatibly ordered orthogonal group-divisible design* (briefly, compatibly ordered-OGDD) $(X, \mathcal{G}, \mathcal{A}, \mathcal{B})$ is a set X and a partition $\mathcal{G}$ of X into classes (usually called *groups*), and two sets $\mathcal{A}$ and $\mathcal{B}$ of cyclically ordered 3-subsets of X (usually called *blocks*), so that $(X, \mathcal{G}, \mathcal{C}, \mathcal{D})$ is an orthogonal group-divisible design, and if (a, b) appears in a block of $\mathcal{A}$ then (a, b) appears in a block of $\mathcal{B}$, where $\mathcal{C} = \{\{x, y, z\} : (x, y, z) \in \mathcal{A}\}$ and $\mathcal{D} = \{\{x, y, z\} : (x, y, z) \in \mathcal{B}\}$.

A compatibly ordered orthogonal Steiner triple system (briefly, compatibly ordered-OSTS(v)) can be viewed as a compatibly ordered-OGDD of type 1^v.

Example 1.4 A compatibly ordered orthogonal Steiner triple system of order 7 is

$\mathcal{A} = \{(0, 1, 3), (1, 2, 4), (2, 3, 5), (3, 4, 6), (4, 5, 0), (5, 6, 1), (6, 0, 2)\}$;

$\mathcal{B} = \{(0, 2, 3), (1, 3, 4), (2, 4, 5), (3, 5, 6), (4, 6, 0), (5, 0, 1), (6, 1, 2)\}$.

Theorem 1.5 [12, 13] *There exists a compatibly ordered-OSTS(n) and a compatibly ordered-OGDD of type 2^n for $n = 6k + 1$.*

We can obtain an array F from a compatibly ordered-OSTS(n) as follows.

Construction 1.6 An array F is obtained from a compatibly ordered-OSTS(n) of $(X, \mathcal{A}, \mathcal{B})$ as follows.

Let ∞ be a symbol not in X. We construct an array with rows and columns indexed by X. First place (∞, x) in cell (x, x), for all $x \in X$. Then, for every pair $(x, y) \in \mathcal{A}$, place (x, y) in cell (r, c), where $(x, y, r) \in \mathcal{A}$ and $(x, y, c) \in \mathcal{B}$.

Figure 1.1

$(\infty,0)$	$(2,6)$	$(4,5)$		$(1,3)$		
	$(\infty,1)$	$(3,0)$	$(5,6)$		$(2,4)$	
		$(\infty,2)$	$(4,1)$	$(6,0)$		$(3,5)$
$(4,6)$			$(\infty,3)$	$(5,2)$	$(0,1)$	
	$(5,0)$			$(\infty,4)$	$(6,3)$	$(1,2)$
$(2,3)$		$(6,1)$			$(\infty,5)$	$(0,4)$
$(1,5)$	$(3,4)$		$(0,2)$			$(\infty,6)$

Example 1.7 An array obtained by Construction 1.6 from the compatibly ordered-OSTS(7) of Example 1.4 is displayed in Figure 1.1.

Theorem 1.8 *An array F obtained by Construction 1.6 has the following properties:*

(i) Every cell of F either is empty or contains an ordered pair of distinct symbols from $\{\infty\} \cup X$.

(ii) The cell (s,s) contains the ordered pair (∞,s) for $s \in X$.

(iii) F is a Room square when each ordered pair (x,y) is viewed as an unordered pair $\{x,y\}$.

(iv) The set of the first components of all ordered pairs in row s is the same as that in column s, for all $s \in X$.

(v) The set of the second components of all ordered pairs in row s is the same as that in column s, for all $s \in X$.

Proof Let $(X,\mathcal{A},\mathcal{B})$ be a compatibly ordered-OSTS(n). It is easy to see that the properties (i) and (ii) hold. The property (iii) holds since a compatibly ordered-OSTS(n) is also OSTS(n) when each block (x,y,z) is viewed as $\{x,y,z\}$ and the properties (iv) and (v) hold since if (a,b) appears in a block of $\mathcal{A}$ then (a,b) appears in a block of $\mathcal{B}$.

It is clear that property (iv) implies property (v).

Definition 1.9 A *compatibly ordered Room square of side n* is an $n \times n$ array, F, that satisfies the above properties (i)–(iv).

Similarly, we define the concept of a compatibly ordered Room frame.

In this article, we will show that there exists a compatibly ordered Room square of order n if and only if n is odd and $n \neq 3, 5$.

2 Direct constructions

The most useful technique for the direct construction of Room squares has been the technique of orthogonal starters. This technique was introduced in the mathematical literature in 1968 by Stanton and Mullin in [7]. Similarly we can define compatibly ordered orthogonal starters and compatibly ordered orthogonal frame starters for construction of compatibly ordered Room squares.

In this section, G will stand for an additive abelian group of order g. If g is odd, let $m = (g-1)/2$.

Definition 2.1 Let g be odd. An ordered starter in G is a set of ordered pairs $S = \{(s_i, t_i) : 1 \le i \le m\}$, which satisfies the following properties:

(i) S is a partition of $G \setminus \{0\}$;

(ii) The difference set of pairs in S is $G \setminus \{0\}$.

Definition 2.2 Let $S = \{(s_i, t_i) : 1 \le i \le m\}$ and $T = \{(u_i, v_i) : 1 \le i \le m\}$ be two ordered starters in G. Then S and T are said to be compatibly ordered orthogonal starters if

(i) $s_i - t_i = u_i - v_i$ for all i;

(ii) $s_i \ne u_i$ for all i;

(iii) $s_i - u_i \ne s_j - u_j$ for all $i \ne j$;

(iv) $\{s_i : 1 \le i \le m\} = \{u_i : 1 \le i \le m\}$.

Lemma 2.3 *If there exist two compatibly ordered orthogonal starters in G, then there exists a compatibly ordered Room square of side g.*

Example 2.4

$$S_1 = \{(2,3), (4,6), (1,5)\}, S_2 = \{(4,5), (1,3), (2,6)\}$$

are two compatibly ordered orthogonal starters in Z_7.

A compatibly ordered Room square of side 7 obtained from those is displayed in Figure 2.1.

Definition 2.5 Let S be an ordered starter in G. Then S is said to be a compatibly ordered strong starter if

(i) $s_i + t_i \ne s_j + t_j$ for all $i \ne j$;

Figure 2.1

$(\infty,0)$			$(4,6)$		$(2,3)$	$(1,5)$
$(2,6)$	$(\infty,1)$			$(5,0)$		$(3,4)$
$(4,5)$	$(3,0)$	$(\infty,2)$			$(6,1)$	
	$(5,6)$	$(4,1)$	$(\infty,3)$			$(0,2)$
$(1,3)$		$(6,0)$	$(5,2)$	$(\infty,4)$		
	$(2,4)$		$(0,1)$	$(6,3)$	$(\infty,5)$	
		$(3,5)$		$(1,2)$	$(0,5)$	$(\infty,6)$

(ii) $s_i + t_i \neq 0$ for all i;

(iii) $\{s_i : 1 \leq i \leq m\} = \{-t_i : 1 \leq i \leq m\}$.

Example 2.6 $S = \{(3,4),(9,7),(11,14),(2,6),(8,13),(1,10),(5,12)\}$ is a compatibly ordered strong starter of Z_{15}.

Lemma 2.7 *If S is a compatibly ordered strong starter in G, then S and $-S$ are compatibly ordered orthogonal.*

Definition 2.8 Let H be a subgroup of order h of G, where $g - h$ is even. An ordered frame starter in $G \setminus H$ is a set of ordered pairs S, which satisfies the following properties:

(i) $S = \{(s_i, t_i) : 1 \leq i \leq (g-h)/2\}$;

(ii) $\{s_i, t_i : 1 \leq i \leq (g-h)/2\} = G \setminus H$;

(iii) $\{\pm(s_i - t_i) : 1 \leq i \leq (g-h)/2\} = G \setminus H$.

Similarly, we define the concepts of a compatibly ordered strong frame starter and compatibly ordered orthogonal frame starters.

Lemma 2.9 *Let* $S = \{(1,2),(10,12),(3,6),(7,11),(14,9),(5,15),(4,13)\}$*;*
$T = \{(13,12),(3,5),(14,17),(6,10),(7,2),(16,4),(8,15),(1,11)\}$.
Then S is a compatible ordered strong frame starter of type 2^8 and T is a compatible ordered strong frame starter of type 2^9.

Theorem 2.10 *Let G be an additive abelian group of order $n = 2k+1$. Then a strong starter can be arranged to a compatibly ordered strong starter.*

Proof. Let $S = \{\{s_i, t_i\} : 1 \leq i \leq k\}$ be a strong starter. For a pair $\{x, y\} \in S$, we say x is a partner of y and y is a partner of x . If a pair $\{x, y\} \in S$ is arranged to (x, y), we say that x is assigned white color and y is assigned black color. We take an element in $G \setminus \{0\}$, say, g_1, we assign g_1 white color and its partner, say, g_2 black

Figure 3.1

$(\infty,0)$	(4,3)	(8,7)	(2,1)	(6,5)				
(8,3)	$(\infty,1)$	(4,6)				(5,7)	(2,0)	
(4,7)	(8,6)	$(\infty,2)$			(1,0)			(3,5)
(6,1)	(2,7)		$(\infty,3)$			(4,0)	(8,5)	
(2,5)			(8,0)	$(\infty,4)$			(6,3)	(7,1)
		(3,0)	(6,7)		$(\infty,5)$	(8,2)	(1,4)	
	(5,0)			(8,1)	(3,7)	$(\infty,6)$		(4,2)
		(1,5)		(2,3)	(8,4)		$(\infty,7)$	(6,0)
			(4,5)	(7,0)	(6,2)	(3,1)		$(\infty,8)$

color; then we assign $-g_2$ white color and its partner, say, g_3 black color; then we assign $-g_3$ white color and its partner black color and so on. Suppose that g_{2u} is the first element which is equal to $-g_1$. If $u = k$, that is all right; if $u < k$, repeat the above process in the remaining pairs.

Theorem 2.11 *Let G be an additive abelian group of order g and H be a subgroup of order h of G, where $g-h$ is even. A strong frame starter in $G \setminus H$ can be arranged to a compatibly ordered strong frame starter.*

Proof. The proof is similar to that of Theorem 2.10.

3 A compatibly ordered Room square of side 9

We need to construct a compatibly ordered Room square of side 9. First we hope to construct it by starters. However, with the help of a computer, we prove that

Lemma 3.1 *There do not exist two compatibly ordered orthogonal starters in $G = Z_9$ or $G = Z_3 \times Z_3$.*

With the help of a computer, we obtain a compatibly ordered Room square of side 9, that is:

Lemma 3.2 *There is a compatibly ordered Room square of side 9, displayed in Figure 3.1.*

4 Recursive constructions

The following construction is a variation of that for Room squares [9].

Theorem 4.1 [Fundamental Frame Construction] *Let w be a function from X to non-negative integers with $w(x)$ (we say w is a weighting) and $w_A = w(x_1)+w(x_2)+$*

$\ldots + w(x_r)$ *for* $A = \{x_1, x_2, \ldots, x_r\} \subseteq X$.

Suppose $(X, \mathcal{G}, \mathcal{A})$ *is a GDD with* $\lambda = 1$ *and for every block* $A \in \mathcal{A}$, *there is a compatibly ordered Room frame with type* $\{w(x) : x \in A\}$. *Then there is a compatibly ordered Room frame with type* $\{w_G : G \in \mathcal{G}\}$.

The following construction is a variation of that for Room squares [3].

Theorem 4.2 [Filling in-Holes Construction] *Suppose that there is a compatibly ordered Room frame of type* $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$, *and let* $w = 0$ *or* 1. *For* $1 \leq i \leq k$, *suppose there is a compatibly ordered Room square of side* $t_i + w$. *Then there is a compatibly ordered Room square of side* n *for* $n = t_1 + t_2 + \ldots + t_k + w$.

From Lemma 2.9, we have the following theorem by applying Theorem 4.1 and Theorem 4.2.

Theorem 4.3 *Suppose that there exists a TD*$(9, m)$. *If there exists a compatibly ordered Room square of side* u *for* $u = 2s + 1, 2m + 1$, *then there exists a compatibly ordered Room square of side* n *for* $n = 16m + 2s + 1$, *where* $0 \leq s \leq m$.

By applying Theorem 4.3 with $m = 59$ and $s = 51$ we have:

Lemma 4.4 *If there exists a compatibly ordered Room square of side* u *for* $u = 103$ *and* 119, *then there exists a compatibly ordered Room square of side* 1047.

5 Conclusion

The following theorem is due to Greig [4].

Theorem 5.1 *Let* OQ_7 *be the set of all odd prime powers not less than* 7. *Then the PBD closure* $B(OQ_7)$ *contains all odd integers greater than* 861 *except possibly* 1047.

The following results can be found in [3].

Theorem 5.2 [Mullin-Nemeth] *For any* $q \in OQ_7$ *there is a strong starter of order* q *except* $q = 9$ *and* q *is a Fermat prime.*

Theorem 5.3 [Chong-Chan] *If* q *is a Fermat prime and* $q > 5$, *then there is a strong starter of order* q.

Theorem 5.4 [Dinitz and Stinson] *There is a strong starter of order odd* n *with* $7 \leq n \leq 999$.

It follows from Theorem 4.1 that compatibly ordered Room squares are PBD-closed. Therefore by combining the above results we have:

Theorem 5.5 *There exists a compatibly ordered Room square of side n if and only if n is odd and $n \neq 3, 5$.*

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