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Possible cardinalities for locating-dominating codes in graphs

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Abstract

Consider a connected undirected graph $G = (V, E)$, a subset of vertices $C \subseteq V$, and an integer $r \geq 1$; for any vertex $v \in V$, let $B_r(v)$ denote the ball of radius r centered at v, i.e., the set of all vertices linked to v by a path of at most r edges. If for all vertices $v \in V \setminus C$, the sets $B_r(v) \cap C$ are all nonempty and different, then we call C an r-locating-dominating code.

It is known that the cardinality of a minimum r-locating-dominating code C in any connected undirected graph G having a given number, n, of vertices satisfies the inequalities $|C| \leq n - 1$ and $|C| + 2^{|C|} \geq n + 1$, and that these lower and upper bounds can be achieved. Here, we prove that any in-between value can also be reached by $|C|$.

1 Introduction

Given a connected undirected graph $G = (V, E)$ and an integer $r \geq 1$, we define $B_r(v)$, the *ball* of radius r centered at $v \in V$, by

$$B_r(v) = \{x \in V : d(x, v) \leq r\},$$

where $d(x, v)$ denotes the number of edges in any shortest path between v and x. Whenever $d(x, v) \leq r$, we say that x and v *r-cover* each other (or simply *cover* if there is no ambiguity). A set $X \subseteq V$ covers a set $Y \subseteq V$ if every vertex in Y is covered by at least one vertex in X.

A *code* C is a nonempty set of vertices, and its elements are called *codewords*. For each vertex $v \in V$, we denote by

$$K_{C,r}(v) = C \cap B_r(v)$$

the set of codewords which r-cover v. Two vertices v_1 and v_2 with $K_{C,r}(v_1) \neq K_{C,r}(v_2)$ are said to be *r-separated*, or *separated*, by code C.

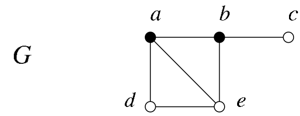

Figure 1: A graph G admitting no 1-identifying code. Black vertices form a 1-locating-dominating code.

A code C is called *r-locating-dominating*, or *locating-dominating*, if the sets $K_{C,r}(v), v \in V \setminus C$, are all nonempty and distinct [8]. It is called *r-identifying*, or *identifying*, if the same is true for all $v \in V$ [10]. In other words, in the latter case all vertices must be covered and pairwise separated by C, in the former case only the noncodewords need to be covered and separated.

Remark 1. For given graph $G = (V, E)$ and integer r, there exists an r-identifying code $C \subseteq V$ if and only if

$$\forall v_1, v_2 \in V \ (v_1 \neq v_2),\ B_r(v_1) \neq B_r(v_2).$$

Indeed, if for all $v_1, v_2 \in V$, $B_r(v_1)$ and $B_r(v_2)$ are different, then $C = V$ is r-identifying. Conversely, if for some $v_1, v_2 \in V$, $B_r(v_1) = B_r(v_2)$, then for any code $C \subseteq V$, we have $K_{C,r}(v_1) = K_{C,r}(v_2)$. For instance, there is no r-identifying code in a complete graph. See also Example 1 below.

Remark 2. For given graph $G = (V, E)$ and integer r, an r-locating-dominating code always exists (simply take $C = V$), and any r-identifying code is r-locating-dominating.

Example 1. Consider the graph G in Figure 1. We see that $B_1(a) = \{a, b, d, e\}$, $B_1(b) = \{a, b, c, e\}$, $B_1(c) = \{b, c\}$, $B_1(d) = \{a, d, e\}$, $B_1(e) = \{a, b, d, e\}$; consequently, because $B_1(a) = B_1(e)$, there is no 1-identifying code in G (cf. Remark 1 above). On the other hand, $C = \{a, b\}$ is 1-locating-dominating, since the sets $K_{C,1}(c) = \{b\}$, $K_{C,1}(d) = \{a\}$, and $K_{C,1}(e) = \{a, b\}$, are all nonempty and different.

Definition 1. A graph is said to be *r-identifiable* if it admits at least one r-identifying code.

The motivations come, for instance, from fault diagnosis in multiprocessor systems. Such a system can be modeled as a graph where vertices are processors and edges are links between processors. Assume that at most one of the processors is malfunctioning and we wish to test the system and locate the faulty processor. For this purpose, some processors (constituting the code) will be selected and assigned the task of testing their neighbourhoods (i.e., the vertices at distance at most r). Whenever a selected processor (= a codeword) detects a fault, it sends an alarm signal, saying that one element in its neighbourhood is malfunctioning. We require that we can uniquely tell the location of the malfunctioning processor based only on the information which ones of the codewords gave the alarm, and in this case an identifying code is what we need.

If the selected codewords are assumed to work without failure, or if their only task is to test their neighbourhoods (i.e., they are not considered as processors anymore) and we assume that they perform this simple task without failure, then we shall search for locating-dominating codes. These codes can also be considered for modeling the protection of a building, the rooms of which are the vertices of a graph.

Locating-dominating codes were introduced in [8], identifying codes in [10], and they constitute now a topic of their own: both were studied in a large number of various papers, investigating particular graphs or families of graphs (such as planar graphs, certain infinite regular grids, or the n-cube), dealing with complexity issues, or using heuristics such as the noising methods for the construction of small codes. See, e.g., [3], [4], [5], [9], [11], [12], [13], and references therein, or [14].

In this paper, we concentrate on locating-dominating codes and continue the investigation started in [6]; it is known that, for all $r \geq 1$, the cardinality of a minimum r-locating-dominating code C in any connected, undirected graph G having a given number, n, of vertices satisfies the inequalities

$$|C| \leq n-1 \text{ and } |C| + 2^{|C|} \geq n+1,$$

and that these lower and upper bounds can be achieved, provided that n is large enough, see [6, Sec. 6]. Here, we prove that any in-between value for $|C|$ can also be reached. Note that allowing disconnected graphs would only make things easier.

2 Previous Results and Constructions

The bounds in themselves are easy to establish.

Theorem 1 *Let $r \geq 1$ and $n \geq 2$ be two integers. Let $G = (V, E)$ be a connected, undirected graph with n vertices. If $C \subseteq V$ is r-locating-dominating, then $|C| + 2^{|C|} \geq n+1$, and if C is minimum, then $|C| \leq n-1$.*

Proof. Because for all vertices $v \in V \setminus C$, the sets $K_{C,r}(v)$ must be nonempty and distinct, we have: $2^{|C|} - 1 \geq n - |C|$. The fact that not all vertices are necessary in a minimum code is obvious, because G is connected. △

A complete graph on n vertices is one example of a graph where $n-1$ vertices are necessary for a locating-dominating code.

The following theorem states that, for n large enough with respect to r, the lower bound on $|C|$ can also be achieved. Its proof can be found in [6].

Theorem 2 *Let $r \geq 1$ and n be integers such that $n \geq 2^{2r+1} + \lceil \log_2(n+1) \rceil$. There exists a connected graph with n vertices admitting an r-locating-dominating code C with size satisfying $|C| + 2^{|C|} = n+1$.*

The following lemma, also from [6], was used in the proof of Theorem 2. We shall use it, together with Theorem 3, in Section 3.

Lemma 1 *Let $G = (V, E)$ be r-identifiable, with $V = \{v_1, \dots, v_n\}$, and $C = \{c_1, \dots, c_m\} \subseteq V$ be a code. Let $G^* = (V^*, E^*)$ be defined as follows: $C^* = \{c_1^*, \dots, c_m^*\}$ is a set of new vertices (not belonging to V), and*

$$V^* = V \cup C^*,\ \ E^* = E \cup \{\{c_i^*, v_j\} : \{c_i, v_j\} \in E\} \cup \{\{c_i, c_i^*\} : 1 \leq i \leq m\}.$$

The code C is r-identifying in G if and only if the code C^* is r-locating-dominating in G^*.

Theorem 3 *(Theorem 16 in [7]) Let $r \geq 1$ and $c \geq 5r^2 + 5r + 1$. For n between $c+1$ and $2^c - 1$, there exists a connected graph G with n vertices, such that any minimum r-identifying code in G contains c elements.*

3 Achieving All Values

In this section we prove our main result: given an integer $r \geq 1$ and an integer n sufficiently large with respect to r, for any integer c satisfying $c \leq n - 1$ and $c + 2^c \geq n + 1$, there exists a connected graph with n vertices admitting a minimum r-locating-dominating code of size c. Or the other way round: given an integer $r \geq 1$ and an integer c sufficiently large with respect to r, for any integer n between $c+1$ and $2^c + c - 1$, there exists a connected graph with n vertices admitting a minimum r-locating-dominating code of size c.

We will proceed in three steps: first from $2^c + c - 1$ to $2c + 1$ (Theorem 4), second from $2c$ to $c + 2r + 1$ (Theorem 5), third from $c + 2r$ to $c + 1$ (Theorem 6).

Theorem 4 *Let $r \geq 1$ and $c \geq 5r^2 + 5r + 1$. For n satisfying $2^c + c - 1 \geq n \geq 2c + 1$, there exists a connected graph G with n vertices, such that any minimum r-locating-dominating code in G contains c elements.*

Proof. By Theorem 3, for n_0 satisfying

$$c + 1 \leq n_0 \leq 2^c - 1, \tag{1}$$

there is a connected graph G with n_0 vertices, such that any minimum r-identifying code in G contains c elements. Therefore, using Lemma 1, we can construct a graph G^* with $n = n_0 + c$ vertices, admitting an r-locating-dominating code C^* of size $c^* = c$. In the construction of Lemma 1, for all couples (c_i, c_i^*), $1 \leq i \leq c$,

$$B_r(c_i) = B_r(c_i^*),$$

so c_i or c_i^* must be a codeword; thus any r-locating-dominating code in G^* contains at least c codewords, and we see that the code C^* is minimum. Moreover, by (1), $2c + 1 \leq n \leq 2^c + c - 1$. △

Theorem 5 *Let $r \geq 1$ and $c \geq 2r + 1$. For n satisfying $2c \geq n \geq c + 2r + 1$, there exists a connected graph G with n vertices, such that any minimum r-locating-dominating code in G contains c elements.*

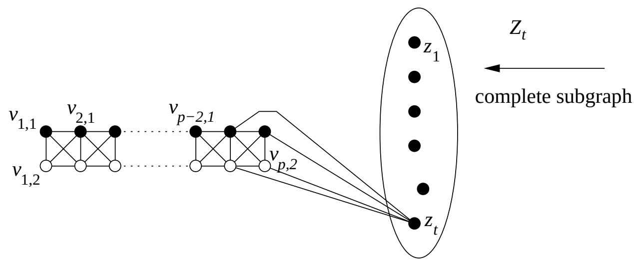

Figure 2: A partial representation of the graph G of Theorem 5, with a minimum r-locating-dominating code. Codewords are in black.

Proof. Let $p \in [2r+1, c]$, $t = c - p \geq 0$, and $G(p,t) = (V(p,t), E(p,t))$ be the following graph (see Figure 2):

$$V(p,t) = Y_p \cup Z_t \text{ with } Y_p = \{v_{i,1}, v_{i,2} : 1 \leq i \leq p\}, Z_t = \{z_i : 1 \leq i \leq t\},$$

$$E(p,t) = \{\{v_{i,1}, v_{i,2}\} : 1 \leq i \leq p\} \cup$$
$$\{\{v_{i,1}, v_{i+1,1}\}, \{v_{i,1}, v_{i+1,2}\}, \{v_{i,2}, v_{i+1,1}\}, \{v_{i,2}, v_{i+1,2}\} : 1 \leq i \leq p-1\} \cup$$
$$\{\{z_i, z_j\} : 1 \leq i < j \leq t\} \cup$$
$$\{\{z_i, v_{p-1,1}\}, \{z_i, v_{p-1,2}\}, \{z_i, v_{p,1}\}, \{z_i, v_{p,2}\} : 1 \leq i \leq t\}.$$

Note that the balls of radius r centered at the vertices in $Z_t \cup \{v_{p,1}, v_{p,2}\}$ are all and the same set.

We claim that $G(p,t)$ contains a minimum r-locating-dominating code of size $c = p + t$. First, the code

$$C = \{v_{i,1} : 1 \leq i \leq p\} \cup Z_t,$$

of size $p + t$, is r-locating-dominating, thanks to the inequality $p \geq 2r + 1$.

Now let C be a minimum r-locating-dominating code in $G(p,t)$. The remark following the definition of $E(p,t)$ shows that in $Z_t \cup \{v_{p,1}, v_{p,2}\}$, at least $t+1$ vertices are codewords. But, since in $Y_p \setminus \{v_{p,1}, v_{p,2}\}$ the vertices $v_{i,1}$, $v_{i,2}$ are "clones" of each other, at least one of them must belong to C, for $1 \leq i \leq p-1$, whence at least $p+t$ codewords in C.

Finally, $G(p,t)$ has $2p + t = c + p$ vertices, with $2r + 1 \leq p \leq c$. △

In order to prove Theorem 6, we need the following four lemmas, where, for $q \geq 1$, $G_q = (V_q, E_q)$ is the tree defined by

$$V_q = \{v_i : 1 \leq i \leq q\} \cup \{x, y\},$$

$$E_q = \{\{v_i, v_{i+1}\} : 1 \leq i \leq q-1\} \cup \{\{x, v_1\}, \{y, v_1\}\},$$

and γ_q is the cardinality of a minimum r-locating-dominating code in G_q.

Lemma 2 *For any $q \geq 1$,*

$$0 \leq \gamma_{q+1} - \gamma_q \leq 1. \tag{2}$$

Proof. 1) $\gamma_q \leq \gamma_{q+1}$. Let C_{q+1} be a minimum r-locating-dominating code in G_{q+1}: $|C_{q+1}| = \gamma_{q+1}$. If $v_{q+1} \notin C_{q+1}$, then $C_{q+1} \subseteq V_q$ is r-locating-dominating in V_q, and $\gamma_q \leq \gamma_{q+1}$.

So we assume that $v_{q+1} \in C_{q+1}$, and consider $C_q = C_{q+1} \setminus \{v_{q+1}\}$. Two things can prevent C_q from being locating-dominating in G_q:
– some vertex in V_q which was covered by v_{q+1} is not covered by C_q;
– or v_{q-r} and v_{q-r+1}, which were separated by v_{q+1}, are not separated by any element in C_q (with a slight abuse of notation, v_{q-r} can represent here x — or y); in this case, $q \geq r$.

Now if $q \geq r$, we can simultaneously meet these two requirements, by taking v_{q-r+1} as a codeword in C_q. If $q < r$, adding one arbitrary noncodeword to C_q is sufficient.

2) $\gamma_{q+1} \leq \gamma_q + 1$. Let C_q be a minimum r-locating-dominating code in G_q. Then $C_q \cup \{v_{q+1}\}$ is r-locating-dominating in G_{q+1}. △

Lemma 3

$$\textit{(i) If } r = 1, \ \ \gamma_1 = 2; \tag{3}$$

$$\textit{if } r = 1, \ \textit{ for any } q \geq 1, \quad \gamma_q \leq 2 + \left\lceil \frac{2q}{5} \right\rceil. \tag{4}$$

$$\textit{(ii) If } r \geq 2, \ \textit{for any } q \leq r, \quad \gamma_q = q + 1. \tag{5}$$

$$\textit{(iii) If } r \geq 2, \ \textit{if } r + 1 \leq q \leq 2r, \quad \gamma_q \leq q. \tag{6}$$

$$\textit{(iv) If } r \geq 2, \ \textit{for any } q \geq 2r + 1, \quad \gamma_q \leq 2 + \left\lceil \frac{q + 7r + 6}{3} \right\rceil. \tag{7}$$

Proof. (i) The case $q = 1$ is trivial. In $V_q \setminus \{x, y\}$, which is a chain with q vertices, $q \geq 1$, it is shown in [12] that a minimum 1-locating-dominating code has size $\lceil \frac{2q}{5} \rceil$.

(ii) If $r \geq 2$, $q \leq r$, all balls of radius r are all and the same set, therefore one has to take as codewords all $q + 2$ vertices but one.

(iii) We deliberately do not try to optimize the construction and simply take

$$C = \{x\} \cup \{v_i : 1 \leq i \leq q - 1\},$$

for an r-locating-dominating code in G_q, that is, all vertices but two.

(iv) In $V_q \setminus \{x, y\}$, $q \geq 2r + 1$, $r \geq 2$, it is shown in [2] (see also [1]) that a minimum r-locating-dominating code has size at most $\lceil \frac{q+7r+6}{3} \rceil$. △

Lemma 4 *The set $\{q + 2 - \gamma_q : q \geq 1\}$ is equal to $I\!N^*$.*

Proof. By (5) and (3), $q + 2 - \gamma_q = 1$ for all $q \leq r$;
by (2), for $q \geq 1$, $(q + 3 - \gamma_{q+1}) - (q + 2 - \gamma_q) = 0$ or 1;
and by (7) and (4), $q + 2 - \gamma_q \geq q - \lceil \frac{q+7r+6}{3} \rceil$ or $q + 2 - \gamma_q \geq q - \lceil \frac{2q}{5} \rceil$.

Therefore the sequence $(q+2-\gamma_q)$ starts at one, possibly increases by one only, and is bounded from below by a quantity going to infinity with q. △

Next, starting from G_q, we construct the tree $G_{q,s} = (V_{q,s}, E_{q,s})$ in the following way, for $s \geq 0$:

$$V_{q,s} = V_q \cup \{w_i : 1 \leq i \leq s\},\ \ E_{q,s} = E_q \cup \{\{v_1, w_i\} : 1 \leq i \leq s\}.$$

Lemma 5 *For $q \geq 1$, $s \geq 0$, any minimum r-locating-dominating code in $G_{q,s}$ has size $s + \gamma_q$.*

Proof. Let C_0 be a minimum r-locating-dominating code in G_q, of size γ_q, and let $C_1 = C_0 \cup \{w_i : 1 \leq i \leq s\}$. Obviously, C_1 is an r-locating-dominating code in $G_{q,s}$, with size $\gamma_q + s$.

Now let C_2 be a minimum r-locating-dominating code in $G_{q,s}$. For any two vertices z_1, z_2 in $\{x, y\} \cup \{w_i : 1 \leq i \leq s\}$, we have: $B_1(z_1) \setminus \{z_1\} = B_1(z_2) \setminus \{z_2\} = \{v_1\}$. Therefore, among these $s+2$ vertices in $G_{q,s}$, at least $s+1$ must belong to C_2, and since they all play the same role, we can assume, without loss of generality, that $\{w_i : 1 \leq i \leq s\} \cup \{x\} \subset C_2$.

Let $C_3 = C_2 \setminus \{w_i : 1 \leq i \leq s\}$. We claim that C_3 is r-locating-dominating in G_q: the fact that the vertices $w_i \in C_2 \setminus C_3$ play exactly the same role as the codeword $x \in C_2 \cap C_3$ with respect to the vertices in $V_q \setminus \{x\}$, shows that the noncodewords in V_q are still r-covered and r-separated by C_3.

Finally $\gamma_q \leq |C_3| = |C_2| - s$ shows that $|C_2| \geq \gamma_q + s$. △

We are now ready to prove Theorem 6.

Theorem 6 *Let $r \geq 1$ and $c \geq \frac{9r+13}{2}$. For n satisfying $c + 2r \geq n \geq c+1$, there exists a tree G with n vertices, such that any minimum r-locating-dominating code in G contains c elements.*

Proof. Let j be such that $1 \leq j \leq 2r$. By Lemma 4, there exists an integer m such that $j = m + 2 - \gamma_m$. To prove that $\gamma_m \leq c$, we distinguish between two cases, $r = 1$ and $r \neq 1$.

1) $r = 1$. In this case, by (4),

$$\gamma_m \leq 3 + \frac{2m}{5} = 3 + \frac{2(j-2+\gamma_m)}{5} \leq \frac{2\gamma_m}{5} + \frac{4r+11}{5} = \frac{2\gamma_m}{5} + 3$$

$$\Longrightarrow \gamma_m \leq 5 \leq 11 \leq c.$$

2) $r \geq 2$. If $m \leq 2r$, then $\gamma_m \leq m+1 \leq 2r+1 \leq c$, so we can assume that $m \geq 2r+1$. In this case, by (7),

$$\gamma_m \leq 3 + \frac{m+7r+6}{3} = 3 + \frac{j+\gamma_m+7r+4}{3} \leq \frac{\gamma_m}{3} + \frac{9r+13}{3}$$

$$\Longrightarrow \gamma_m \leq \frac{9r+13}{2} \leq c.$$

Now that we have established that $\gamma_m \leq c$, we set $s = c - \gamma_m$, $s \geq 0$, and we use the construction of $G_{q,s}$, just before Lemma 5, with $q = m$. We obtain a graph with

$$n = m + 2 + s = (j - 2 + \gamma_m) + 2 + (c - \gamma_m) = c + j$$

vertices, $c+1 \leq n \leq c+2r$, in which there is a minimum r-locating-dominating code of size $\gamma_m + s = c$. △

Theorems 4–6 show that for c large enough, all intermediate integer values between the lower and upper bounds can be achieved.

Theorem 7 *Let $r \geq 1$ and $c \geq 5r^2 + 5r + 1$. For n between $c + 1$ and $2^c + c - 1$, there exists a connected graph G with n vertices, such that any minimum r-locating-dominating code in G contains c elements.*

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