AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **33** (2005), Pages 335–347

Transitive hyperovals in finite projective planes

Angelo Sonnino

Dipartimento di Matematica
Università della Basilicata
Contrada Macchia Romana
85100 Potenza, ITALY
sonnino@unibas.it

Abstract

Let Ω be a hyperoval in a projective plane π of even order n, and G the collineation group of π preserving Ω. If G acts transitively on the points of Ω, then Ω is a *transitive* hyperoval. By a deep result due to M. Biliotti and G. Korchmáros [1], if Ω is transitive and $|G|$ is divisible by 4, then either $n = 2, 4$ and Ω is a hyperconic, or $n = 16$ and $|G| \leq 144$. In this paper, it is shown that the case $n = 16$ with $|G| = 144$ only occurs when $\pi \cong PG(2, 16)$ and Ω is the Lunelli-Sce-Hall hyperoval.

1 Introduction

Let π be a projective plane of even order n. A hyperoval of π is a set of $n + 2$ points no three of which are collinear, see [9]. The classical example is the *regular* hyperoval in $\mathrm{PG}(2, n)$ consisting of the points of an irreducible conic $\mathcal{C}$ together with the nucleus of $\mathcal{C}$. A hyperoval Ω of a projective plane π of even order n is said to be *transitive* if the collineation group G of π preserving Ω acts transitively on the points of Ω. Transitive hyperovals in desarguesian planes are completely classified: They only exist for $n = 2, 4$ and 16. In both planes $\mathrm{PG}(2, 2)$ and $\mathrm{PG}(2, 4)$, hyperovals are regular and transitive. In $\mathrm{PG}(2, 16)$, besides regular hyperovals, there is one more class of hyperovals. The latter were found by Lunelli and Sce [12] in 1958, but it was M. Hall Jr. in 1975 who pointed out that the Lunelli-Sce hyperovals are transitive. Later on G. Korchmáros [11] and (independently S. Payne and J.E. Conklin [13].) showed that no transitive hyperoval exists in $\mathrm{PG}(2, n)$ with $n > 16$. In 1987 Biliotti and Korchmáros [1] raised the transitive hyperoval problem by asking whether a non-desarguesian projective plane can contain a transitive hyperoval. They tackled this problem by using deep results on irreducible collineation groups containing involutory elations. Their main result was the following theorem

Theorem 1.1. *Let π be a projective plane of even order $n \geq 8$ containing a transitive hyperoval Ω. If the order of the collineation group G preserving Ω is divisible by 4, then $n = 16$ and $|G|$ divides 144.*

In the present paper we set up a computer aided approach to decide whether the only projective plane of order 16 containing a transitive hyperoval is the Desarguesian plane.

Theorem 1.2. *Let π be a projective plane of order* 16 *containing a transitive hyperoval Ω. If the order of the collineation group G preserving Ω is equal to* 144, *then $\pi \cong \mathrm{PG}(2,16)$ and Ω is the Lunelli-Sce-Hall hyperoval.*

In 1991 Cherowitzo [6] gave an exhaustive classification of hyperovals in each of the 8 translation planes of order 16. Later on. in 1996, Penttila, Royle and Simpson [14] extended this classification to each of the 22 known projective planes of order 16. From [14, Table 2], the above Theorem 1.2 can be deduced for the known projective planes of order 16 by observing that the only case in which the full collineation group of the hyperoval has order 144 is the Lunelli-Sce-Hall hyperoval in $\mathrm{PG}(2,16)$.

2 Transitive hyperovals in a projective plane of order 16

Let π be a projective plane of even order 16 containing a hyperoval Ω. Let G be a collineation group of π preserving Ω which acts transitively on Ω. Assume that 4 divides the order of G. By [1, Lemma 2.9 and Proposition 3.1], the subgroup S generated by the nine involutory elations in G is a Frobenius group of order 18 which fixes no point, or line but preserves a subplane π_0 of order 4 disjoint from Ω. Let K be the subgroup of G which fixes π_0 pointwise. Then K is either trivial or it has order 2. In the latter case, the Baer involution generating K acts on Ω as an odd permutation. Furthermore, one of the following two cases occurs, see [1, Section 3]:

i) K is trivial, and $G = O(S) \rtimes E$ where E is isomorphic to a 2-subgroup of $P\Gamma U(3,4)$ of order at least 4.

ii) K has order 2. Let $G_0 = G \cap Alt_\Omega$. Then $G = K \times G_0$ and $G_0 = O(S) \rtimes E$ where E is either a cyclic group of order 2, 4 or 8, or is a quaternion group.

If $\pi \cong \mathrm{PG}(2,16)$ and Ω is the Lunelli- Sce hyperoval, then ii) occurs with E a cyclic group of order 8.

A deeper analysis requires some properties of the configuration C of centers and axes of the nine involutory elations in G. Clearly, C is contained in the invariant subplane π_0 which is a desarguesian subplane $\mathrm{PG}(2,4)$.

Lemma 2.1. *C is a unital in π_0 consisting of all points of a non-degenerate Hermitian curve. For any point $A \in \pi_0$, the axis of the involutory elation in G whose centre is A is the tangent to C in π_0.*

Proof. Let φ_i with $i = 1,2$ denote the involutory elation whose centre is P_i. The conjugate $\varphi_3 = \varphi_1\varphi_2\varphi_1$ of φ_2 is another involutory elation in G. Its centre P_3 is the image of P_2 under φ_1. Hence P_3 lies on the line ℓ joining P_1 to P_2. Note that $\varphi_3 = \varphi_2\varphi_1\varphi_2$ also holds as $\varphi_1\varphi_2$ has order 3. Therefore, φ_3 is distinct from both φ_1 and φ_2. Further, ℓ is preserved by each of φ_1, φ_2 and φ_3. In particular, the collineation group generated by $\langle \varphi_1, \varphi_2, \varphi_3 \rangle$ is a dihedral group of order 6 and preserves ℓ. Note that every dihedral subgroup of order 6 is maximal in the Frobenius group S of order 18 generated by all involutory elation in G. Hence, if G contained another involutory elation preserving ℓ, then S would preserve ℓ. But then the centre of every involutory elation would lie in ℓ. Since ℓ has 5 points, and G contains 9 involutory elations, this would yield that some points in ℓ would be the centre of two distinct involutory elations in G. A contradiction, because G preserves the hyperoval Ω. This proves the assertion. It turns out that every line of π_0 meets C in either 0, or 1, or 3 points. Actually, as C has size 9, no line of π_0 is disjoint from C. Sets of size 9 meeting every line in either 1, or 3 points are unitals in $\mathrm{PG}(2,4)$. Therefore, C can be viewed as a Hermitian curve of π_0. Since every involutory elation in G induces an involutory elation in π_0, the second assertion follows of the basic properties of the action of the projective unitary group, see [10, Theorem 2.49]. □

Theorem 2.2. *Case i) does not occur.*

Proof. By Lemma 2.1 G is a subgroup of $\mathrm{P\Gamma U}(3,4)$ viewed as the collineation group of π_0 preserving C. Since $\mathrm{P\Gamma U}(3,4)$ has order 432, its Sylow 2-subgroups have order 16. In case i), G has order 144 and it acts on π_0 faithfully. Hence E is a Sylow 2-subgroup of $\mathrm{P\Gamma U}(3,4)$. But this is impossible because $\mathrm{P\Gamma U}(3,4)$ contains some Baer-involutions, while in case i) every involution in G is an elation. □

Theorem 2.3. *The group G has exactly 19 involutions, 9 are elations, and 10 are Baer-involutions. For a point $P \in \pi$ let ν_P denote the number of involutions in G fixing P. Then*

$$\nu_P = \begin{cases} 7 \text{ for } P \in \pi_0 \setminus C \\ 3 \text{ for } P \in C \\ 1 \text{ for } P \notin \pi_0. \end{cases} \tag{2.1}$$

Furthermore, no point outside π_0 is fixed by an element of G of order 3.

Proof. Let $k \in G$ be the Baer-involution fixing π_0 pointwise. Note that the product of k with a non-trivial element $g_0 \in G_0$ has order 2 if and only if g_0 is an involution, and if this is the case the product $k' = kg_0$ is a Baer-involution. Since G_0 has 9 involutions, it turns out G has exactly 19 involutions, 9 of them are elations, say $e_1, \ldots, e_9$ and the others, that is $k, ke_1, \ldots, ke_9$ are Baer-involutions.

If $P \in C$, then the stabiliser of G_0 is either a cyclic group, or a quaternion group of order 8. Hence, there is a unique involutory elation fixing P, say e_i. Therefore, the involutions in G fixing P are e_i, k and ke_i. Note that the Baer involution ke_i fixes each of the five points of π_0 lying on the axis of e_i, but no other point in π_0. Therefore, the Baer subplane π_i of ke_i meets π_0 in five points.

If $P \in \pi_0$ but $P \notin C$, then the stabiliser of G_0 is a dihedral group of order six. Hence, there are exactly three involutory elations in G fixing P, say e_i, e_j, e_m. Therefore, the involutions in G fixing P are seven, namely, e_i, e_j, e_m, ke_i, ke_j, ke_m and k.

If $P \notin \pi_0$, let ℓ denote the unique line of π_0 through P, and ℓ^* the set of the point of ℓ outside π_0. If ℓ contains only one point from C, then Lemma 2.1 shows that ℓ is the axis of an involutory elation in G. Clearly, such an involution fixes P. If ℓ meets C in three points, then the subgroup G_ℓ of G preserving ℓ has order 12, and it is the direct product of a dihedral group D_3 of order 6 by K. Let C_3 the cyclic subgroup of D_3 of order 3. Since every non-trivial element in C_3 is the product of two distinct involutory elations whose centres are on ℓ, the fixed points of C_3 are also on ℓ, see [10, Exercise 4.21].

Let Δ denote the (possibly empty) set of all fixed points of C_3 lying on ℓ^*. Since C_3 is a normal subgroup of G_ℓ, both Δ and its complementary set $\Delta^* = \ell^* \setminus \Delta$ are left invariant by G_ℓ. Since G_ℓ has order 12, a subgroup of order 4 preserves both Δ and Δ^*. Hence, if neither Δ nor Δ^* has size divisible by 4, P is fixed by some involution in G_ℓ. Since ℓ has 12 points outside π_0, we also have that $12 - |\Delta|$ is divisible by 3.

If Δ is non-empty, $\Delta = \ell^*$ follows. Then, C_3 fixes at least 12 lines through the pole of ℓ with respect to the unitary polarity associated to C. But this is impossible, no chord of Ω is fixed by a collineation of order larger than 2.

Therefore, C_3 fixes no point outside π_0, and we are left with the case in which Δ is empty.

If G_P be trivial, then the orbit $o(P)$ of P under G has maximum size 144. On the other hand, the number of the points of π outside π_0 which are uncovered by the axes of the involutory elations is equal to 144. This implies that Baer subplanes of the Baer involutions in G are covered by π_0 together with the axes of the involutory elations in G. To show that this cannot actually occur, denote by π_i the Baer subplane of ke_i, for $i = 1, \ldots, 9$. For a point $P \in \pi_i$ let e_j be the involution whose axis ℓ_j passes through P. Since ℓ_j is the unique line of π_0 through P, ke_i preserves ℓ_j. Therefore, $ke_iej = ejke_i$. Since k is in the center of G, this yields that e_i and e_j commute, which is only possible when $i = j$. Hence, $P \in \ell_i$ which yields that the points of π_i lie on ℓ_i. A contradiction, because π_i a subplane.

If G_P is non-trivial, then P is fixed by an involution, as the only prime divisors of $|G|$ are 2 and 3, but no element of order 3 fixes P. Therefore, every point of π is fixed by an involution of G.

As noted before, every Baer subplane π_i meets π_0 in five points, and hence it has 16 points outside π_0. Since G has 9 Baer involutions distinct from k, the total number of points fixed by these 9 Baer involutions is at most 144. Actually, equality holds since these 9 Baer-subplanes must cover each of the 144 points that do not lie on either π_0 or on the axes of the involutory elations in G. Therefore, every point outside π_0 is fixed by just one involution of G. □

Theorem 2.4. *Let $P \in \pi$ be any point not lying either on π_0, or on the axes of the involutory elations in G. Then the stabiliser G_P of P under G has order 2.*

Proof. By Theorem 2.3, G_P has order a power of 2. Let $g \in G_P$ be a non-trivial collineation. If g is an involution, then Theorem 2.3 yields that g is the unique Baer involution fixing P. Otherwise, a power h of g is an involution, but not an elation as $h \in G_P$ while, by hypothesis, P is not fixed by any involutory elation in G. Therefore, h is a Baer involution, and hence it has no fixed point on Ω. It turns out that h induces an odd permutation on Ω which is not possible as h is a power of g. □

A straightforward counting argument shows that 144 is the total number of points which do not lie either on π_0, or on Ω, or on the axes of the involutory elations in G. Hence Theorem 2.4 has the following corollary.

Corollary 2.5. *The points not lying either on π_0, or on Ω, or on the axes of the involutory elations in G split into two orbits under G each of length* 72*.*

Theorem 2.6. *Let P be a point on the axis ℓ of an involutory elation $e \in G$. If P does not lie either on π_0, or Ω, then G_P has order either* 2*,or* 4*, or* 8*. The latter case occurs for exactly two points P on ℓ.*

Proof. There is just one elation axis through P. Hence G_P is a subgroup of the stabiliser H of ℓ in G. Therefore, $G_P = H_P$. Since G has 9 involutory elations, H has order 16. There are 10 points on ℓ which do not lie either on Ω or π_0. The subset ℓ' of ℓ consisting of such 10 points is left invariant by G_ℓ. Note that H contains the Baer involution k. Since k is fix point free on ℓ', no point on ℓ' is fixed by H. Therefore, H has at least an orbit of length 2 on ℓ'. If P is in such an orbit, then H_P has order 8. The other possibilities for the size of an orbit of H are 4 and 8. In the latter case, ℓ' splits into two orbits, one of size 2 and another of size 8. □

Now look at the action of G on Ω. Let Sym_Ω denote the symmetric group on Ω. By Theorem 2.2, we may assume that case ii) occurs.

Theorem 2.7. *Up to conjugacy in* Sym_Ω *there are only two possibilities for G.*

Proof. Since $\mathrm{Sym}(\Omega)$ has only one class of fixed-point-free involutions, the Baer involution in $k \in G$ may be assumed to act on Ω as

$$z = (9,10)(1,11)(2,12)(3,13)(4,14)(5,15)(6,16)(7,17)(8,18).$$

The centraliser Z of z in Sym_Ω has order $185794560 = 2^{17} \cdot 3^4 \cdot 5 \cdot 7$, and a Sylow 3-subgroup S_3 generated by the following permutations:

$$\begin{aligned} t_1 &= (1,18,7)(8,17,11), \\ t_2 &= (2,14,5)(4,15,12), \\ t_3 &= (3,9,6)(10,16,13), \\ t_4 &= (1,10,15,7,13,12,18,16,4)(2,8,6,14,11,9,5,17,3). \end{aligned}$$

The normaliser N of S_3 in Z has order 648. A computer aided exhaustive search shows that N contains exactly 45 subgroups of order 18 up conjugacy in N, six of them transitive on Ω, namely:

$$\begin{aligned}
U_1 = \langle &(1,18,7)(2,5,14)(3,9,6)(4,12,15)(8,17,11)(10,16,13),\\
&(1,2)(3,13)(4,17)(5,18)(6,16)(7,14)(8,15)(9,10)(11,12),\\
&(1,16,4)(2,8,3)(5,17,9)(6,14,11)(7,10,15)(12,18,13)\rangle\\
U_2 = \langle &(1,18,7)(2,5,14)(3,9,6)(4,12,15)(8,17,11)(10,16,13),\\
&(1,2)(3,10)(4,8)(5,7)(6,16)(9,13)(11,12)(14,18)(15,17),\\
&(1,10,15,7,13,12,18,16,4)(2,8,6,14,11,9,5,17,3)\rangle\\
U_3 = \langle &(1,18,7)(2,5,14)(3,9,6)(4,12,15)(8,17,11)(10,16,13),\\
&(1,2)(3,10)(4,8)(5,7)(6,16)(9,13)(11,12)(14,18)(15,17),\\
&(1,10,15)(2,17,3)(4,18,16)(5,11,9)(6,14,8)(7,13,12)\rangle\\
U_4 = \langle &(1,18,7)(2,5,14)(3,9,6)(4,12,15)(8,17,11)(10,16,13),\\
&(1,3,4,8,16,2)(5,7,9,15,17,10)(6,12,11,13,14,18)\rangle\\
U_5 = \langle &(1,18,7)(2,5,14)(3,9,6)(4,12,15)(8,17,11)(10,16,13),\\
&(1,9,15,11,10,5)(2,7,3,12,17,13)(4,8,16,14,18,6)\rangle\\
U_6 = \langle &(1,18,7)(2,5,14)(3,9,6)(4,12,15)(8,17,11)(10,16,13),\\
&(1,9,15,17,13,2,18,6,4,11,10,5,7,3,12,8,16,14)\rangle.
\end{aligned}$$

Further, U_5 and U_6 are abelian, while U_2 has a cyclic Sylow 3-subgroup. Therefore, the Frobenius subgroup S of G of order 18 acts on Ω as one of the remaining subgroups, that is, U_1, U_3, U_4 up to conjugacy in N. As a matter of fact, U_1 and U_4 are conjugate in Z. Also, the normaliser of U_1 in Z has order 72 only. So, U_3 is the only possibility for the action of S on Ω, up to conjugacy in the centraliser of k. Therefore, the permutation group induced by G on Ω is a subgroup of the normaliser M of U_3 in Z.

By an exhaustive computer aided search [8], we find out that in the group M there are, up to conjugacy in Sym_Ω, just two groups of order 144 containing exactly 19 involutions, namely

$$\begin{aligned}
G_1 = \langle &(1,8,6)(2,4,5)(3,7,9)(10,17,13)(11,16,18)(12,15,14),\\
&(1,7,2)(3,5,6)(4,8,9)(10,18,14)(11,12,17)(13,16,15),\\
&(1,17,6,10,8,13)(2,12,5,14,4,15)(3,18,9,16,7,11),\\
&(2,4,7,3)(5,6,9,8)(11,17,15,13)(12,18,16,14),\\
&(2,8,7,6)(3,5,4,9)(11,16,15,12)(13,14,17,18)\rangle
\end{aligned}$$

$$\begin{aligned}
G_2 = \langle &(1,8,6)(2,4,5)(3,7,9)(10,17,13)(11,16,18)(12,15,14),\\
&(1,7,2)(3,5,6)(4,8,9)(10,18,14)(11,12,17)(13,16,15),\\
&(1,17,6,10,8,13)(2,12,5,14,4,15)(3,18,9,16,7,11),\\
&(1,7,3,4,6,5,2,9)(10,18,16,12,13,15,14,11)\rangle
\end{aligned}$$

□

Theorem 1.2 is a corollary to the following two results that will be proved in Section 4.

Theorem 2.8. *No collineation group of π preserving Ω can act on Ω as G_1,*

Theorem 2.9. *If a collineation group of π preserving Ω acts on Ω as G_2, then π is the desarguesian plane.*

3 Abstract (or Buekenhout) hyperovals

The computational approach we apply to prove Theorem 1.2 is based on the following generalisation of the concept of a hyperoval due to F. Buekenhout, see [3, 4, 5].

An abstract hyperoval is a pair $(\Omega, \mathcal{F})$ where Ω is a nonempty set of *points* and $\mathcal{F}$ is a quasi sharply 2-transitive set of involutory permutations over Ω. Every permutation in $\mathcal{F}$ is an *involution* of $(\Omega, \mathcal{F})$, and $\mathcal{F}$ is the *involution-set* of $(\Omega, \mathcal{F})$. If Ω has size $n+2$ then the abstract hyperoval $(\Omega, \mathcal{F})$ has order n. Note that n is even and that $|\mathcal{F}| = n^2 - 1$. Further, every involution is fixed-point-free.

Two abstract hyperovals $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are isomorphic if there is a pair of bijective maps (α, f) with $\alpha : \Omega \mapsto \Omega'$ and $f : \mathcal{F} \mapsto \mathcal{F}'$, such that

$$\phi(\omega) = \omega' \quad \text{if and only if} \quad f(\phi)(\alpha(\omega) = \alpha(\omega'),$$

for every $\omega \in \Omega$ and $\phi \in \mathcal{F}$

There is a natural way to derive an abstract hyperoval from a projective hyperoval. Let Ω be a hyperoval in a projective plane π of even order n. Each point $P \in \pi$ outside Ω defines an involutory permutation ϕ_P on Ω as suggested in Figure 1: two distinct points $Q, Q' \in \Omega$ correspond under ϕ_P, that is, $\phi_P(Q) = Q'$ and $\phi_P(Q') = Q$, if and only if P, Q, Q' are collinear. The point P is the *centre* of ϕ_P. Let $\mathcal{F}$ be the set of all such involutory permutations ϕ_P. It is easily seen that $(\Omega, \mathcal{F})$ is an abstract hyperoval of order n. There exist abstract hyperovals that cannot be obtained from a projective hyperoval by means of the above procedure, see Faina [7]. On the other hand, it has been conjectured that no abstract hyperoval can derive from two non isomorphic projective planes. Actually, this conjecture has been solved so far only for abstract hyperovals that derive from conics. Here the case of the Lunelli-Sce-Hall is investigated.

Lemma 3.1. *Let π be a projective plane of order* 16 *containing a hyperoval Ω. If $(\Omega, \mathcal{F})$ isomorphic to the abstract hyperoval deriving from the Lunelli-Sce-Hall hyperoval, then $\pi \cong \mathrm{PG}(2, 16)$ and Ω is the Lunelli-Sce-Hall hyperoval.*

Proof. Let C be the Lunelli-Sce-Hall hyperoval in $\mathrm{PG}(2, 16)$. We have to prove that if each chord and each tangent of C meets a point-set $\mathcal{L}$ of size 18 in exactly one point, then $\mathcal{L}$ consists of all points of an external line to C. The key observation is that every point in C can be viewed as nucleus of $\mathcal{L}$. Therefore, $\mathcal{L}$ has at least 18 nuclei. Then the assertion follows from a result of Blokhuis and Wilbrink, see [2]. □

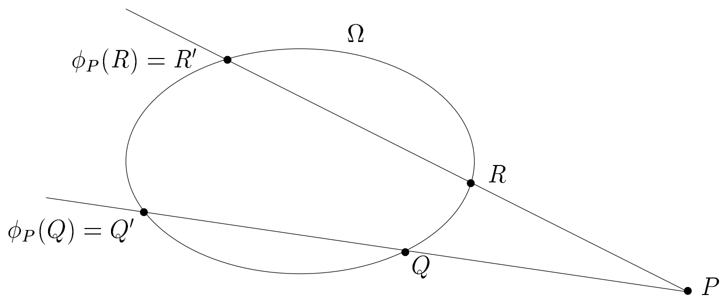

Figure 1: The involutory permutation ϕ_P

Let $\mathcal{J}$ be the set of all fixed-point-free involutions in Sym_Ω. Clearly, the involution-set of any abstract hyperoval is contained in $\mathcal{J}$. However, sharply 2-transitivity is inconsistent with many subsets of $\mathcal{J}$, as the following lemma shows.

Lemma 3.2. *Let $i, j \in \mathcal{J}$ be two distinct involutory permutations. Suppose that their product ij has more then two fixed points. Then no abstract hyperoval exists whose involution-set contains both i and j.*

Lemma 3.2 leads to a useful definition.

Definition 3.3. Two involutory permutations $i, j \in \mathcal{J}$ are *consistent* if ij has at most two fixed points.

An automorphism of $(\Omega, \mathcal{F})$ is a permutation $g \in \mathrm{Sym}_\Omega$ such that $g \circ f \circ g^{-1} \in \mathcal{F}$ for every $f \in \mathcal{F}$. Set

$$G = \{\, g \in \mathrm{Sym}_\Omega \mid g \circ f \circ g^{-1} \in \mathcal{F} \text{ for every } f \in \mathcal{F}\,\}.$$

Clearly, G is a group. Note that if $(\Omega, \mathcal{F})$ derives from a hyperoval Ω of a projective plane π, then every collineation of π preserving Ω gives rise to an automorphism of $(\Omega, \mathcal{F})$. Also, such a collineation is uniquely determined by the corresponding automorphism, as the identity is the only collineation of π that fixes Ω pointwise. In particular, any collineation group of π preserving Ω can be viewed as an automorphism group of $(\Omega, \mathcal{F})$.

If G is transitive on $\mathcal{F}$, then $(\Omega, \mathcal{F})$ is called a transitive abstract hyperoval. Since G maps $\mathcal{J}$ onto itself, $\mathcal{J}$ is partitioned in G-orbits, that is, in orbits under G. Such an orbit is either disjoint from $\mathcal{F}$, or it is contained in $\mathcal{F}$. Actually, some G-orbits cannot be contained at all in the involution-set of any abstract hyperoval. This follows from Lemma 3.2.

Lemma 3.4. *Let $i \in \mathcal{J}$. Suppose that G contains an element g for which $g(i) \neq i$ but $ig(i)$ has more than two fixed points. Then the G-orbit of i is disjoint from the involution-set of any abstract hyperoval whose automorphism group contains G.*

Definition 3.5. Let $i \in \mathcal{J}$. The G-orbit of i is *admissible* if no element $g \in G$ exists such that $g(i) \neq i$ but the product $ig(i)$ has more than two fixed points.

4 The computational approach

In this section a computer aided [8] proof of Theorems 2.8 and 2.9 is described. Set $\Omega = \{1, \ldots, 18\}$.

First, the case G_1 is investigated. As we have seen in the previous section, G_1 can be regarded as a permutation group on Ω. We begin by noting that the centre of G_1 has order two being generated by

$$k := (1,10)(2,14)(3,16)(4,12)(5,15)(6,13)(7,18)(8,17)(9,11)$$

which is a fixed point free involutory permutation. Further, the involutory permutations in G_1 with two fixed points are:

$$\begin{aligned}
e_1 &= (2,7)(3,4)(5,9)(6,8)(11,15)(12,16)(13,17)(14,18),\\
e_2 &= (1,6)(2,3)(4,9)(5,7)(10,13)(11,12)(14,16)(15,18),\\
e_3 &= (1,8)(2,9)(3,5)(4,7)(10,17)(11,14)(12,18)(15,16),\\
e_4 &= (1,9)(2,4)(3,6)(7,8)(10,11)(12,14)(13,16)(17,18),\\
e_5 &= (1,7)(3,8)(4,5)(6,9)(10,18)(11,13)(12,15)(16,17),\\
e_6 &= (1,3)(2,5)(6,7)(8,9)(10,16)(11,17)(13,18)(14,15),\\
e_7 &= (1,5)(2,6)(3,7)(4,8)(10,15)(12,17)(13,14)(16,18),\\
e_8 &= (1,4)(2,8)(5,6)(7,9)(10,12)(11,18)(13,15)(14,17),\\
e_9 &= (1,2)(3,9)(4,6)(5,8)(10,14)(11,16)(12,13)(15,17).
\end{aligned}$$

The remaining involutory permutations in G_1 are the products ke_i for $i = 1, \ldots, 9$. They form a single conjugacy class in G_1.

To prove Theorem 2.8, G_1 is assumed to be a collineation group of a projective plane π of order 16 which preserves a hyperoval acting on it transitively. This hyperoval can be identified by the above set Ω such that the action of G_1 is the same as on Ω. The collineations $e_1, \ldots, e_9$ are involutory elations in G_1. Let $\ell_1, \ldots, \ell_9$ denote their axes. The pointset covered by these axes splits into three subsets which are Ω and π_0 together with a pointset $\mathcal{L}$ of size 90. Also, $ke_1, \ldots, ke_9$ are Baer involutions whose Baer subplanes are denoted by $\pi_1, \ldots, \pi_9$. Let $\overline{\pi}_i = \pi_i \setminus (\pi_i \cap \pi_0)$; then $|\overline{\pi}_i| = 16$. Note that π_0 is invariant under G_1. By Theorem 2.4, the pointset of π is partitioned into $\Omega, \mathcal{L}, \pi_0, \overline{\pi}_1, \ldots, \overline{\pi}_9$. Further, Corollary 2.5 states that $\overline{\pi}_1 \cup \ldots \cup \overline{\pi}_9$ coincides with two point- orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ of G_1 each of size 72 such that neither $\mathcal{O}_1$ nor $\mathcal{O}_2$ contains any π_i entirely. Hence both $\mathcal{O}_1$ and $\mathcal{O}_2$ take a point form each $\overline{\pi}_i$, $1 \le i \le 9$. Therefore, $\overline{\pi}_9$ contains two points, say P and Q, such that $\mathcal{O}_1 = \{P^g \mid g \in G_1\}$, and $\mathcal{O}_2 = \{Q^g \mid g \in G_1\}$.

Now, we focus our attention on ϕ_P, the involution with centre P of the abstract hyperoval $(\Omega, \mathcal{F})$ that derives from Ω. Clearly, $ke_9\phi_P = \phi_P ke_9$. An exhaustive computer search shows that ke_9 commutes with exactly $n_1 = 26785$ involutory permutations of $\mathcal{J}$, and that $n_2 = 26168$ generate an orbit of length 72. But only a few of these, namely $n_3 = 240$, is consistent with Lemma 3.4, and the number of admissible orbits generated by them is 30. Unfortunately, a test based on Lemma

3.2 rules out the possibility of having a pair of consistent admissible orbits. This completes the proof of Theorem 2.8.

The same procedure is used to investigate G_2. The list of the involutory permutations in G_2 with two fixed points is:

$$
\begin{aligned}
e_1 &= (2,7)(3,4)(5,9)(6,8)(11,15)(12,16)(13,17)(14,18),\\
e_2 &= (1,6)(2,3)(4,9)(5,7)(10,13)(11,12)(14,16)(15,18),\\
e_3 &= (1,8)(2,9)(3,5)(4,7)(10,17)(11,14)(12,18)(15,16),\\
e_4 &= (1,3)(2,5)(6,7)(8,9)(10,16)(11,17)(13,18)(14,15),\\
e_5 &= (1,9)(2,4)(3,6)(7,8)(10,11)(12,14)(13,16)(17,18),\\
e_6 &= (1,7)(3,8)(4,5)(6,9)(10,18)(11,13)(12,15)(16,17),\\
e_7 &= (1,4)(2,8)(5,6)(7,9)(10,12)(11,18)(13,15)(14,17),\\
e_8 &= (1,2)(3,9)(4,6)(5,8)(10,14)(11,16)(12,13)(15,17),\\
e_9 &= (1,5)(2,6)(3,7)(4,8)(10,15)(12,17)(13,14)(16,18),
\end{aligned}
$$

and the above defined numbers are $n_1 = 26785$, $n_2 = 26168$, $n_3 = 80$. This time, two pairs of consistent admissible orbits of length 72 are found. They are $\{\mathcal{O}_1, \mathcal{O}_2\}$ and $\{\mathcal{O}'_1, \mathcal{O}'_2\}$ where

- $\mathcal{O}_1 = \{\, p_{11}^g \mid g \in G_2 \,\}$ with
  $p_{11} = (1,2)(3,13)(4,10)(5,16)(6,7)(8,14)(9,18)(11,15)(12,17)$,
- $\mathcal{O}_2 = \{\, p_{11}^g \mid g \in G_2 \,\}$ with
  $p_{12} = (1,3)(2,13)(4,6)(5,17)(7,14)(8,11)(9,18)(10,15)(12,16)$,
- $\mathcal{O}'_1 = \{\, p_{11}^g \mid g \in G_2 \,\}$ with
  $p_{21} = (1,2)(3,14)(4,10)(5,9)(6,15)(7,11)(8,16)(12,17)(13,18)$,
- $\mathcal{O}'_2 = \{\, p_{11}^g \mid g \in G_2\}$ with
  $p_{22} = (1,3)(2,18)(4,5)(6,11)(7,15)(8,14)(9,17)(10,13)(12,16)$.

The normaliser of G_2 in Sym_{18} is a group $N(G_2)$ of order 288 generated by G_2 and $a = (1,9,4,7)(2,5,8,6)(10,11,12,18)(13,14,15,17)$. By straightforward computation, $p_{21} = a^{-1}p_{11}a$ and $a^{-1}p_{12}a = g^{-1}p_{22}g$ with

$$g = (1,9,6,8,4,7,5,2)(10,11,13,17,12,18,15,14).$$

Therefore, it suffices to consider the pair $\{\mathcal{O}_1, \mathcal{O}_2\}$.

The next step is to investigate ϕ_P centred at a point P lying on the axis of an involutory elation in G_2. Since such elations are conjugate under G_2, it suffices to consider one of them, say e_9. The above computer aided procedure gives the following results: The number of elements in $\mathcal{J}$ commuting with e_9 is equal to $n_1 = 5937$. They partitioned into 5937 G_2-orbits: 5448 of length 72, 404 of length 36, 32 of length 24, 26 of length 18, 24 of length 12, 2 of length 9 and 1 of length 1. Those consistent with Lemma 3.4 are listed below. For each of them a representative is given.

Length 72

$$\begin{aligned}
a_1^{(72)} &= (1,2)(3,12)(4,13)(5,6)(7,17)(8,14)(9,11)(10,16)(15,18),\\
a_2^{(72)} &= (1,2)(3,13)(4,15)(5,6)(7,14)(8,10)(9,11)(12,18)(16,17),\\
a_3^{(72)} &= (1,2)(3,14)(4,10)(5,6)(7,13)(8,15)(9,11)(12,18)(16,17),\\
a_4^{(72)} &= (1,2)(3,15)(4,16)(5,6)(7,10)(8,18)(9,11)(12,14)(13,17),\\
a_5^{(72)} &= (1,3)(2,12)(4,10)(5,7)(6,17)(8,15)(9,11)(13,16)(14,18),\\
a_6^{(72)} &= (1,3)(2,15)(4,13)(5,7)(6,10)(8,14)(9,11)(12,18)(16,17);
\end{aligned}$$

length 36

$$\begin{aligned}
a_1^{(36)} &= (1,2)(3,13)(4,10)(5,6)(7,14)(8,15)(9,11)(12,16)(17,18),\\
a_2^{(36)} &= (1,2)(3,14)(4,15)(5,6)(7,13)(8,10)(9,11)(12,16)(17,18),\\
a_3^{(36)} &= (1,4)(2,10)(3,12)(5,8)(6,15)(7,17)(9,11)(13,16)(14,18),\\
a_4^{(36)} &= (1,4)(2,15)(3,17)(5,8)(6,10)(7,12)(9,11)(13,16)(14,18),\\
a_5^{(36)} &= (1,12)(2,16)(3,13)(4,15)(5,17)(6,18)(7,14)(8,10)(9,11),\\
a_6^{(36)} &= (1,12)(2,18)(3,14)(4,15)(5,17)(6,16)(7,13)(8,10)(9,11);
\end{aligned}$$

length 18

$$\begin{aligned}
a_1^{(18)} &= (1,2)(3,4)(5,6)(7,8)(9,11)(10,17)(12,15)(13,18)(14,16),\\
a_2^{(18)} &= (1,4)(2,7)(3,6)(5,8)(9,11)(10,13)(12,18)(14,15)(16,17),\\
a_3^{(18)} &= (1,5)(2,13)(3,7)(4,17)(6,14)(8,12)(9,11)(10,15)(16,18),\\
a_4^{(18)} &= (1,12)(2,15)(3,13)(4,18)(5,17)(6,10)(7,14)(8,16)(9,11),\\
a_5^{(18)} &= (1,14)(2,16)(3,12)(4,15)(5,13)(6,18)(7,17)(8,10)(9,11),\\
a_6^{(18)} &= (1,16)(2,12)(3,15)(4,13)(5,18)(6,17)(7,10)(8,14)(9,11),\\
a_7^{(18)} &= (1,16)(2,17)(3,15)(4,14)(5,18)(6,12)(7,10)(8,13)(9,11);
\end{aligned}$$

length 12

$$\begin{aligned}
a_1^{(12)} &= (1,2)(3,16)(4,8)(5,6)(7,18)(9,11)(10,14)(12,17)(13,15),\\
a_2^{(12)} &= (1,3)(2,6)(4,12)(5,7)(8,17)(9,11)(10,16)(13,14)(15,18),\\
a_3^{(12)} &= (1,4)(2,6)(3,16)(5,8)(7,18)(9,11)(10,12)(13,14)(15,17),\\
a_4^{(12)} &= (1,10)(2,12)(3,18)(4,14)(5,15)(6,17)(7,16)(8,13)(9,11),\\
a_5^{(12)} &= (1,10)(2,13)(3,12)(4,16)(5,15)(6,14)(7,17)(8,18)(9,11),\\
a_6^{(12)} &= (1,10)(2,18)(3,13)(4,17)(5,15)(6,16)(7,14)(8,12)(9,11);
\end{aligned}$$

length 9

$$\begin{aligned}
a_1^{(9)} &= (1,5)(2,6)(3,7)(4,8)(9,11)(10,15)(12,17)(13,14)(16,18),\\
a_2^{(9)} &= (1,15)(2,13)(3,18)(4,17)(5,10)(6,14)(7,16)(8,12)(9,11);
\end{aligned}$$

length 1

$$a_1^{(1)} = (1,10)(2,14)(3,16)(4,12)(5,15)(6,13)(7,18)(8,17)(9,11).$$

Now, all orbits which are not consistent with both $\mathcal{O}_1$ and $\mathcal{O}_2$ are ruled out. There remain only eight orbits namely those containing the involutions

$$a_5^{(72)}, a_1^{(18)}, a_2^{(12)}, a_4^{(12)}, a_5^{(12)}, a_6^{(12)}, a_1^{(9)}, a_1^{(1)}.$$

It turns out that

$$\mathcal{O}_3 = \{\, a_5^{(72)^g} \mid g \in G_2 \,\},$$
$$\mathcal{O}_4 = \{\, a_1^{(18)^g} \mid g \in G_2 \,\},$$
$$\mathcal{O}_5 = \{\, a_1^{(9)^g} \mid g \in G_2 \,\},$$

together with one of the four orbits of length 12 contain all involutions of the abstract hyperoval $(\Omega, \mathcal{F})$. Actually, only one of the orbits of length 12 is consistent with $\mathcal{O}_3$, namely $a_5^{(12)}$. Setting $\mathcal{O}_6 = \{\, a_5^{(12)^g} \mid g \in G_2 \,\}$, we find that the involutions of the abstract hyperoval are exactly those contained in the orbits $\mathcal{O}_i$ for $i = 1, \ldots 6$.

Therefore, there is only one abstract hyperoval $(\Omega, \mathcal{F})$ of order 16 whose automorphism group contains G_2. In [11] Korchmáros proved that G_2 is the collineation group of the Lunelli-Sce-Hall hyperoval. Lemma 3.1 completes the proof of Theorem 1.2.

References

[1] M. Biliotti and G. Korchmáros, Hyperovals with a transitive collineation group, *Geom. Dedicata* **24** (1987), 269–281.

[2] A. Blokhuis and H. A. Wilbrink, A characterization of exterior lines of certain sets of points in PG$(2, q)$, *Geom. Dedicata* **23** (1987), no. 2, 253–254.

[3] F. Buekenhout, Études intrinsèque des ovales, *Rend. Mat. e Appl. (5)* **25** (1966), 333–393.

[4] F. Buekenhout, Ovales et ovales projectifs, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **40** (1966), 46–49.

[5] F. Buekenhout, Plans projectifs à ovoïdes pascaliens, *Arch. Math. (Basel)* **17** (1966), 89–93.

[6] W. E. Cherowitzo, Hyperovals in the translation planes of order 16, *J. Combin. Math. Combin. Comput.* **9** (1991), 39–55.

[7] G. Faina, A new class of 2-transitive involutory permutation sets, *Aequationes Math.* **24** (1982), no. 2-3, 175–178.

[8] The GAP Group, GAP—Groups, Algorithms, and Programming, Version 4.3, 2002, (`http://www.gap-system.org`).

[9] J. W. P. Hirschfeld, *Projective Geometries Over Finite Fields*, second ed., The Clarendon Press Oxford University Press, New York, 1998.

[10] D. R. Hughes and F. C. Piper, *Projective planes*, Springer-Verlag, New York, 1973, GTM Vol. 6.

[11] G. Korchmáros, Gruppi di collineazioni transitivi sui punti di una ovale [$(q+2)$-arco] di $S_{2,q}$, q pari, *Atti Sem. Mat. Fis. Univ. Modena* **27** (1978), 89–105.

[12] L. Lunelli and M. Sce, k-archi completi dei piani desarguesiani di rango 8 e 16, Centro Calc. Num. Politecnico di Milano, 1958.

[13] S.E. Payne and J.E. Conklin, An unusual generalizes quadrangle of order sixteen, *J. Combin. Theory Ser. A* **24** (1978), 50–74.

[14] T. Penttila, G.F. Royle, and M.K. Simpson, Hyperovals in the known projective planes of order 16, *J. Combin. Des.* **4** (1996), no. 1, 59–65.

(Received 5 Apr 2004)