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The residually weakly primitive geometries of HS

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Abstract

We announce the end of the classification of all firm and residually connected geometries satisfying the conditions $(IP)_2$ and $(2T)_1$ and on which the Higman-Sims group HS acts flag-transitively and residually weakly primitively. We state some facts regarding the results. In particular, we construct two rank three geometries which are extensions of the Hoffman-Singleton graph. These are the first known extensions of the Hoffman-Singleton graph. The complete list of geometries is available as a supplement to this paper.

1 Introduction

The Higman-Sims group HS is the seventh smallest sporadic group. It has 44,352,000 elements. Using a series of MAGMA [1] programs described in [4], we classified all firm and residually connected geometries satisfying the conditions $(IP)_2$ and $(2T)_1$ and on which the Higman-Sims group HS acts flag-transitively and residually weakly primitively. If a geometry satisfies these conditions we say it is a $RWPRI + (2T)_1$ geometry. Such a classification has already been obtained for the six smallest sporadic groups and for J_3. We refer to [5] (resp. [4], [6], [8], [10], [11], [9]) for the geometries of M_{11}, M_{12}, J_1, J_2, M_{22}, M_{23} and J_3. This work is part of a project initiated by Buekenhout in the 1980's. It aims to find a unified geometric interpretation of all the finite simple groups. We refer to [2] for a survey of this project. The complete list of geometries obtained for HS is available online as a supplement to this paper [12]. We take out of the list two nice rank three geometries which are extensions of the Hoffman-Singleton graph. We give here geometric constructions for these geometries.

Name	Order	#Conj.Classes	#Subgroups	Rk 2	Rk 3	Rk 4	Rk 5	Rk 6
M_{11}	7920	39	8651	0	9	11	0	0
M_{12}	95040	147	214871	5	11	17	1	0
J_1	175560	40	158485	11	1	2	0	0
M_{22}	443520	156	941627	1	20	12	9	0
J_2	604800	146	1104344	5	16	3	0	0
M_{23}	10200960	204	17318406	0	3	16	10	0
HS	44352000	589	149985646	4	20	41	8	1
J_3	50232960	137	71564248	5	17	0	0	0

Table 1: The eight sporadic groups explored so far.

The paper is organised as follows. In section 2, we state the results of the classification. In section 3, we summarize the results obtained so far the the eight smallest sporadic groups. Finally, in section 4, we construct two geometries of rank three which are extensions of the Hoffman-Singleton graph, answering partially the open problem 5.3.9 of [3].

2 The results

Using our set of programs, we obtained 1, 6, 30, 41, 8, 1, 0 $RWPRI + (2T)_1$ geometries of rank 1, 2, 3, 4, 5, 6 and ≥ 7. The complete list is available online as a supplement to this paper (see [12]). For every geometry Γ, we give its diagram, the automorphism and correlation groups, the Borel subgroup, the kernels. We mention when a geometry is a truncation of a higher rank geometry. We also say when a geometry can be constructed from another using a construction described in Theorem 4.1 of [14] and that Pasini calls doubling [15].

Two rank two geometries are truncations of geometries of higher rank. Ten rank three geometries are truncations of geometries of higher rank. This yields 0, 4, 20, 41, 8, 1, 0 geometries of rank 1, 2, 3, 4, 5, 6, and ≥ 7 that are not truncations of a geometry of higher rank. For a construction of the only rank six geometry found, we refer to [13].

3 The eight sporadic groups analysed so far

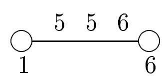
As we mentioned in the introduction, all $RWPRI + (2T)_1$ geometries are known for the eight smallest sporadic groups. No geometry of rank at least seven was found for any of these groups.

Table 1 gives for each of the eight sporadic groups explored so far its order, the number of conjugacy classes of subgroups, the number of subgroups, and the number of geometries obtained in rank 2, 3, 4, 5 and 6 without counting those that appear as truncations of higher rank geometries for the same group.

4 Two rank three geometries extending the Hoffman-Singleton graph

The Hoffman-Singleton graph HoSi is the unique rank two geometry over the following diagram.

HoSi



It has 50 vertices, 175 edges and a flag-transitive automorphism group isomorphic to $U_3(5):2$. The Higman-Sims group has maximal subgroups isomorphic to $U_3(5):2$. Therefore, a natural question arises : are there geometries for the Higman-Sims group which have the Hoffman-Singleton graph as a rank two residue ? The answer is yes and we construct in this section two such geometries with a linear diagram. One of them is a HoSi.HoSi* geometry and the other is a HoSi.c* geometry.

4.1 A HoSi.c* geometry

G. Higman constructed in 1967 a group of degree 176 having the same order and character table as the Higman-Sims sporadic group HS (see [7]). Charles Sims proved in [16] that the group of G. Higman is isomorphic to HS.

G. Higman obtained HS as the automorphism group of a “geometry” whose objects are a set $\mathcal{P}$ of 176 points, a set $\mathcal{Q}$ of 176 subsets of $\mathcal{P}$ he calls “quadrics” and a set of 1100 subsets of $\mathcal{P}$ he calls “conics”. The quadrics (respectively conics) are special subsets of 50 (respectively 8) points of $\mathcal{P}$. He proves that his geometry has the following properties (as given in [7]).

(i) There are 176 points, 1100 conics and 176 quadrics.

(ii) Each quadric contains 50 points and each point is on 50 quadrics.

(iii) Each conic contains 8 points and lies on 8 quadrics.

(iv) Through any two points there pass just two conics; any quadric through both points contains at least one of the conics; and just two quadrics contain both conics.

(v) On any two quadrics there lie just two conics; any point on both quadrics lies on at least one of the conics; and there are just two points lying on both conics.

(vi) A conic S determines a one to one correspondence between the points q on it and the quadrics Q through it, such that, if q corresponds to Q,

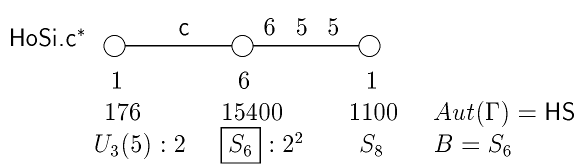
(a) The conics S' meeting S in two points, one of which is q, lie on Q;

(b) The conics S' lying on two quadrics through S, one of which is Q, contain q.

The automorphism group of the geometry is transitive on conics, on incident point-quadric pairs, and on non-incident point-quadrics pairs. Moreover, it is doubly transitive on points and quadrics.

The stabilizer of a conic is a group isomorphic to S_8.

The HoSi.c* geometry is constructed in the following way. The points are the elements of $\mathcal{P}$. The lines are the pairs of points of $\mathcal{P}$. The planes are the conics. Incidence is symmetrised inclusion.
By property (iii), the residue of a point contains $\frac{1100\times 8}{176} = 50$ planes. It obviously contains 175 lines. By property (iv), this residue has a structure of a graph, namely, there are just two planes containing a line. This graph is isomorphic to the Hoffman-Singleton graph.
By property (iii), the residue of a plane contains 8 points and 28 pairs of points which form a complete graph.
The residue of a line contains two points and, by property (iv), two planes. It is a generalized digon. The following picture gives the diagram of the geometry. The automorphism group and correlation group are obvious.

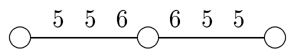


This geometry is number 3.2 in [12].

4.2 A HoSi.HoSi* geometry

We replace the planes of the HoSi.c* geometry by the set $\mathcal{Q}$ of quadrics. Incidence is symmetrized inclusion except for between lines and planes : a plane is incident to a line l if it contains l and the two conics containing l.

There are 176 planes and 15400 lines. Hence each plane contains $\frac{15400\cdot 2}{176} = 175$ lines. Moreover, by property (i), each plane contains 50 points and each point is incident to 50 planes. The duality exchanging points and planes stabilizes the lines. Thus the residue of a point is isomorphic to the residue of a plane. Moreover, by property (v), there are exactly two planes containing a line in the residue of a point. So this residue has a structure of a graph. It is isomorphic to the Hoffman-Singleton graph. Finally, the residue of a line consists in two points and two planes that form a digon. The diagram of the geometry is given below.



1	6	1	$Cor(\Gamma) = Aut(\mathsf{HS})$
176	15400	176	$Aut(\Gamma) = \mathsf{HS}$
$U_3(5):2$	$\boxed{S_6}:2^2$	$U_3(5):2$	$B = S_6$

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