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Triangles in contraction critical 5-connected graphs*

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Abstract

In this paper, we give some examples of contraction critical 5-connected graphs in which there exists one vertex not in any triangle. On the other hand, we prove that for any contraction critical 5-connected graph G there is a vertex x of degree 5 such that each edge incident to x is contained in some triangle of G. This generalizes the result of Ando, Kawarabayashi and Kaneko for the case of 5-connected graphs.

1 Introduction

We only consider finite simple undirected graph. Let k be a positive integer, G a k-connected graph. An edge of G is said to be a k-contractible edge if its contraction yields again a k-connected graph. By Tutte's famous result, any 3-connected graph with order at least 5 has a 3-contractible edge. But for $k \geq 4$, Thomassen ([7]) showed that there are infinitely many k-connected k-regular graphs which do not have a k-contractible edge. So, the contraction critical k-connected graph for $k \geq 4$ was introduced, which is the non-complete k-connected graph without k-contractible edges. The contraction critical 4-connected graphs are characterized, which are two special classes of 4-regular graphs. For $k \geq 5$, the characterization for contraction critical k-connected graphs seems to be very hard. In general, Egawa ([3]) showed that every contraction critical k-connected graph has a vertex of degree at most $\lfloor \frac{5k}{4} \rfloor - 1$. Then, for $4 \leq k \leq 7$ every contraction critical k-connected graph contains a vertex of degree k. Thomassen ([7]) proved that any contraction critical k-connected graph contains one triangle. Mader ([5]) improved this result and obtained that every contraction critical k-connected graph G contains at least $\frac{1}{3}|G|$ triangles, where $|G| = |V(G)|$ is the number of vertices of G. In his same paper, Mader also gave some examples of contraction critical connected graph with higher connectivity in which there is a vertex not in any triangle. Recently, Kriesell ([4]) proved that

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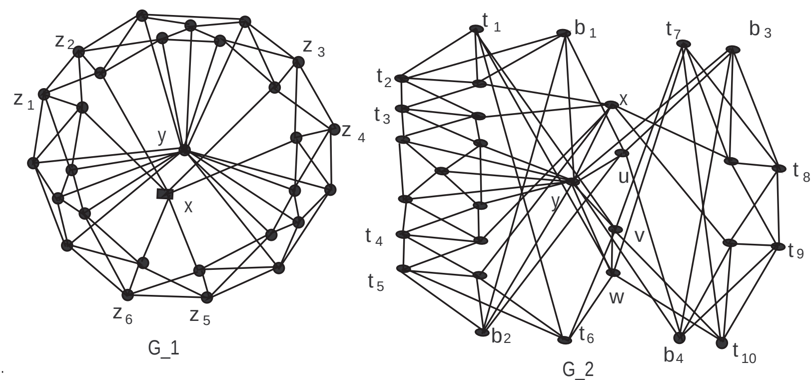

Figure 1:

every contraction critical k-connected graph contains at least $\frac{2}{3}|G|$ triangles. By his improvement, he asked the following question: Is every vertex of a contraction critical 5-connected graph contained in some triangle? But this is not true. In the following, we give two sequences of contraction critical 5-connected graphs in each of which there exists indeed a vertex not in any triangle. The graph G_1 in Figure 1 is a contraction critical 5-connected graph in which the vertex x has degree six and it is not in any triangle. We can modify G_1 to a contraction critical 5-connected graph that contains a vertex of even degree (≥ 6) which is not in any triangle. The graph G_2 in Figure 1 is a contraction critical 5-connected graph in which x has degree seven and it is not in any triangle. We can modify G_2 to a contraction critical 5-connected graph that contains a vertex of odd degree (≥ 5) which is not in any triangle. We show that G_1, G_2 are contraction critical 5-connected graphs in the appendix.

Focusing on a special class of contraction critical k-connected graphs, Ando et al. [1] proved the following result. A k-connected graph G is called minimally k-connected if $G - e$ is not k-connected for each edge e of G.

Theorem 1 *Let G be a minimally contraction critical k-connected graph which does not contain $C_4 + K_1$. Then, G contains a vertex x of degree k such that each edge incident to x is contained in some triangle.*

Here we show that the conclusion of Theorem 1 still holds for any contraction critical 5-connected graph. In fact, we obtain the following theorem.

Theorem 2 *Let G be a contraction critical 5-connected graph. Then, G contains a vertex x of degree 5 such that each edge incident to x is in some triangle.*

For terms not defined here we refer the reader to [2]. Let $G = (V(G), E(G))$ be a graph where $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G. Let $|G| = |V(G)|$ and let $\kappa(G)$ denote the vertex connectivity of G. An edge joining the vertex x, y is written as xy. For $x \in V(G)$, we define $N_G(x) = \{y \in V(G) :$

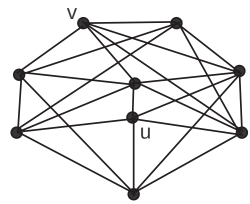

Figure 2: G_3

$xy \in E(G)\}$. By $d_G(x) = |N_G(x)|$ we denote the degree of x. For $F \subset V(G)$, let $N_G(F) = \cup_{x\in F} N_G(x) - F$. A set $T \subseteq V(G)$ is called a separating set of a connected graph G if $G-T$ has at least two connected components. A separating set with $\kappa(G)$ vertices is called a smallest separating set. Let G be a non-complete graph, and let T be a smallest separating set. The union of at least one but not of all the components of $G-T$ is called a $T-$fragment. A fragment of G is a T-fragment for some smallest separating set T. Let $F \subseteq V(G)$ be a T-fragment and let $\overline{F} = V(G) - (F \cup T)$. Then, by definition $\overline{F} \neq \emptyset$ which means that $\overline{F}$ is also a T-fragment. Note that $N_G(F) = T = N_G(\overline{F})$. The set of all smallest separating sets of G is denote by $\mathcal{T}_G$. We often omit the index G if it is clear from the context.

We need more definitions introduced in [5]. For a graph G, let $\mathcal{S}$ be a non-empty set of subsets of $V(G)$. An $\mathcal{S}$-fragment of G is a T-fragment of G for any $T \in \mathcal{T}_G$ such that there is an $S \in \mathcal{S}$ with $S \subseteq T$. An inclusion-minimal $\mathcal{S}$-fragment of G is called an $\mathcal{S}$-end and one of the least vertex numbers is called an $\mathcal{S}$-atom. The following properties of fragments are folklore (for the proof see [5]), we will use them without any further reference .

Let $T, T' \in \mathcal{T}_G$. Let F be a T-fragment of G and let F' be a T'-fragment of G. If $F \cap F' \neq \emptyset$, then $|F \cap T'| \geq |\overline{F'} \cap T|$ and $|F' \cap T| \geq |\overline{F} \cap T'|$. If $F \cap F' \neq \emptyset \neq \overline{F} \cap \overline{F'}$, then both $F \cap F'$ and $\overline{F} \cap \overline{F'}$ are fragments of G, and $N(F \cap F') = (F' \cap T) \cup (T' \cap T) \cup (F \cap T')$. If $F \cap F' \neq \emptyset$ and $F \cap F'$ is not a fragment of G, then $\overline{F} \cap \overline{F'} = \emptyset$, and $|F \cap T'| > |\overline{F'} \cap T|$ and $|F' \cap T| > |\overline{F} \cap T'|$. Also, by the definition, the two end vertices of any edge in a contraction critical k-connected graph are contained in some smallest separating set.

2 Some properties of contraction critical 5-connected graphs

It is proved in [8] that any vertex of a contraction critical 5-connected graph is adjacent to a vertex of degree 5. In fact, from [8] we have the following result.

Lemma 1 ([8]) *Let G be a contraction critical 5-connected graph and F a fragment of G. If $w \in N(F)$ and $N(w) \cap N(F) \neq \emptyset$ and $|\overline{F}| \geq 2$, then $N(w) \cap (F \cup N(F))$ contains a vertex of degree 5.*

By using a more technical method, we ([6]) showed that any vertex in a contraction critical 5-connected graph is adjacent to at least two vertices of degree 5. The bound 'two' is best possible. Indeed, there are some contraction critical 5-connected graphs (see Figure 2, the vertices u, v of G_3) which contain some vertices having only two neighbors of degree 5.

Lemma 2 ([4]) *Let A be a fragment of cardinality 2 in a contraction critical 5-connected graph, and let t_1 and t_2 be two distinct vertices in $N(A)$ such that $|N(t_1)\cap A| = |N(t_2)\cap A| = 1$. Then one of t_1, t_2 has a neighbor of degree 5 in $N(A)-\{t_1,t_2\}$.*

Lemma 3 *Let G be a contraction critical 5-connected graph and let V_5 denote the set of the vertices of degree 5 in G. Let $x \in V(G)$ and let A be a fragment of G such that $x \in N(A)$ and $|A| \geq 3$ and $|\overline{A}| \geq 2$. If $|N(x)\cap A| = 1$, then there exists a vertex $y \in N(x)\cap N(A)\cap V_5$ such that $N(x)\cap A \subseteq N(y)\cap A$ and $|N(y)\cap A| \geq 2$.*

Proof. Let $z \in N(x)\cap A$, and let T be a smallest separating set containing $\{x, z\}$ and F be a T-fragment.

If $A \subseteq T$, as $|A| \geq 3$ and $x \in N(A)\cap T$, then $|T\cap \overline{A}| \leq 1$. So, either $F\cap\overline{A} \neq \emptyset$ or $\overline{F}\cap\overline{A} \neq \emptyset$. Without loss of the generality, we may assume $F\cap\overline{A} \neq \emptyset$. Then $|F\cap N(A)| \geq |A\cap T| = |A| \geq 3$, and thus $|\overline{F}\cap N(A)| \leq 1$. It follows that $\overline{F}\cap\overline{A} = \emptyset$, and hence $|\overline{F}| = 1$. Let $\overline{F} = \{y\}$. Then, $y \in N(A)\cap V_5$ and $A\cup\{x\} \subseteq N(y)$.

If $A \not\subseteq T$, we may assume that $F\cap A \neq \emptyset$. Since $N(x)\cap A = \{z\}$ and $z \in T$, $N(x)\cap(F\cap A) = \emptyset$. So, $|(T\cap A)\cup(T\cap N(A))\cup(F\cap N(A))| > 5$. For otherwise, $F\cap A$ is a fragment such that $x \in N(F\cap A)$, which implies $N(x)\cap(F\cap A) \neq \emptyset$. However, this contradicts the fact that $N(x)\cap(F\cap A) = \emptyset$. It follows that, $\overline{F}\cap\overline{A} = \emptyset$ and $|A\cap T| > |\overline{F}\cap N(A)|$. We show that $|\overline{F}| = |\overline{F}\cap N(A)| = 1$. If $|\overline{F}\cap N(A)| \geq 2$, then $|A\cap T| \geq 3$. Thus $|\overline{A}\cap T| \leq 1$ which implies that $F\cap\overline{A} = \emptyset$, contradicting that $|\overline{A}| \geq 2$. So, $|\overline{F}\cap N(A)| \leq 1$. We show that $\overline{F}\cap A = \emptyset$. Suppose that $\overline{F}\cap A \neq \emptyset$. Then, since $N(x)\cap(F\cap A) = \emptyset$, we have $|(T\cap A)\cup(T\cap N(A))\cup(\overline{F}\cap N(A))| > 5$ which implies that $F\cap\overline{A} = \emptyset$ and $|\overline{A}| = |F\cap\overline{A}| \leq 1$. This contradicts the assumption that $|\overline{A}| \geq 2$. Hence, $\overline{F} \subseteq N(A)$ and $|\overline{F}| = 1$, and so $|A\cap T| \geq 2$. Let $\overline{F} = \{y\}$. Then, $y \in N(A)\cap V_5$ and $(A\cap T)\cup\{x\} \subseteq N(y)$.

Clearly, in both cases we have that $y \in N(x)\cap N(A)\cap V_5$ and $N(x)\cap A \subseteq N(y)\cap A$ and $|N(y)\cap A| \geq 2$. ■

3 Proof of Theorem 2

From now on, we always assume that G is a contraction critical 5-connected graph. V_5 denotes the set of the vertices of degree 5. Suppose that Theorem 2 is not true. Then, each vertex of degree 5 of G is adjacent to at least one edge which is not contained in any triangle. Let $E' \subseteq E(G)$ be the set of such edges that are incident to a vertex of degree 5 and not in any triangle. Let $\mathcal{S} = \{\{x, y\}|xy \in E'\}$. Let A be an $\mathcal{S}$-atom. Let $xy \in E'$, $x, y \in N(A)$ and $d(x) = 5$. Then, $N(x)\cap N(y) = \emptyset$ and hence $|A| \geq 2$.

Assertion 1 $|A| \geq 3$ *and* $A\cap V_5 = \emptyset$.

Proof. At first we show that $|A| \geq 3$. Assume $|A| = 2$ and let $A = \{x_1, y_1\}$. Suppose that $x_1 \in N(x) \cap A$ and $y_1 \in N(y) \cap A$. Since $N(x) \cap N(y) \cap A = \emptyset$, we have $N(x) \cap A = \{x_1\}$ and $N(y) \cap A = \{y_1\}$. Thus, $d(x_1) = 5 = d(y_1)$. Hence, $N(x_1) = (A - \{y\}) \cup \{y_1\}$ and $N(y_1) = (A - \{x\}) \cup \{x_1\}$. By Lemma 2, x or y has a neighbor of degree 5 in $N(A) - \{x, y\}$. We may assume that x has a neighbor of degree 5 in $N(A) - \{x, y\}$. Now it is easy to verify that all edges incident to x_1 are contained in some triangles, a contradiction. Hence, $|A| \geq 3$.

Next we show that $A \cap V_5 = \emptyset$. If $A \cap V_5 \neq \emptyset$, let $z \in A \cap V_5$, then there is an edge $zz' \in E'$ such that $z' \in A \cup N(A)$. Let T be a smallest separating set such that $z, z' \in T$. Then, $T \cap A \neq \emptyset$. By corollary 3 of [5], we have $|A| \leq 2$, contradicting $|A| \geq 3$. This proves Assertion 1. ∎

Since A is an $\mathcal{S}$-atom, we have $|\overline{A}| \geq |A| \geq 3$. Let $z \in N(x) \cap A$, let T be a smallest separating set such that $x, z \in T$ and let F be a T-fragment.

Assertion 2 *$A \not\subseteq T$.*

Proof. Assume $A \subseteq T$. Then, as $x \in N(A) \cap T$ and $|A| \geq 3$, we have $|T \cap \overline{A}| \leq 1$. Thus, either $F \cap \overline{A}$ or $\overline{F} \cap \overline{A}$ is nonempty. We may assume $F \cap \overline{A} \neq \emptyset$, then $|F \cap N(A)| \geq |T \cap A| = |A| \geq 3$, and thus $|\overline{F} \cap N(A)| \leq 1$. It follows $\overline{F} \cap \overline{A} = \emptyset$, and hence $|\overline{F}| = 1$. Let $\overline{F} = \{w\}$. So, $d(w) = 5$ and $|N(w) \cap A| \geq 3$ and $xw \in E(G)$. As $N(w) \cap \overline{A} \neq \emptyset$, we have that $|N(w) \cap N(A)| = 1 = |N(w) \cap \overline{A}|$. Note that $z \in N(x) \cap N(w)$, we have $w \neq y$. Then, $|N(x) \cap N(A)| \geq 2$. As $N(x) \cap \overline{A} \neq \emptyset$, $|N(x) \cap A| = 1$ or $|N(x) \cap A| = 2$ holds. We distinguish two cases.

Case 1. $|N(x) \cap A| = 1$. Then, $N(x) \cap A = \{z\}$. Let $A' = A - \{z\}$. Then, $N(A') = (N(A) - \{x\}) \cup \{z\}$ and A' is a fragment. Note that $w \in N(A')$ and $z \in N(A') \cap N(w)$, we have $N(w) \cap N(A') \neq \emptyset$. Clearly, $|\overline{A'}| \geq |\overline{A}| + 1 \geq 4$. By Lemma1, w has a neighbor of degree 5 in $A' \cup N(A')$, as $N(w) \cap (A' \cup N(A') \subseteq N(w) \cap A$, contradicting that $A \cap V_5(G) = \emptyset$.

Case 2. $|N(x) \cap A| = 2$. Then, $|N(x) \cap A| = 2 = |N(x) \cap N(A)|$, and thus $|N(x) \cap \overline{A}| = 1$. Let $N(x) \cap \overline{A} = \{z'\}$. Note that $N(x) \cap N(A) = \{w, y\}$, by Lemma3, we have $z' \in N(w) \cup N(y)$. Since $N(x) \cap N(y) = \emptyset$, we have $z' \in N(w)$, and thus $N(x) \cap \overline{A} = N(w) \cap \overline{A} = \{z'\}$. It follows that $(N(A) - \{x, w\}) \cup \{z'\}$ is a separating set of cardinality 4, a contradiction. This proves Assertion 2. ∎

As $A \not\subseteq T$. We may assume that $F \cap A \neq \emptyset$.

Assertion 3 *$\overline{F} \subseteq N(A)$, $|T \cap A| > |\overline{F}|$ and $|\overline{F}| = 1$.*

Proof. We first show that $\overline{F} \cap \overline{A} = \emptyset$. Suppose, to the contrary, that $\overline{F} \cap \overline{A} \neq \emptyset$. Then, both $F \cap A$ and $\overline{F} \cap \overline{A}$ are fragments of G. Denote $F_1 = F \cap A$ and $F_2 = \overline{F} \cap \overline{A}$. Clearly, $N(F_1) = (T \cap A) \cup (T \cap N(A)) \cup (F \cap N(A))$ and $N(F_2) = (T \cap \overline{A}) \cup (T \cap N(A)) \cup (\overline{F} \cap N(A))$. By the choice of A, we know that $y \in \overline{F} \cap N(A)$. As $x \in T \cap N(A)$ and $z \in N(x) \cap T$, by Lemma1, we have that $N(x) \cap (F_1 \cup N(F_1))$ has a vertex w of degree 5. By Assertion 1, $w \in N(x) \cap (F \cup T) \cap N(A)$. As $y \in \overline{F} \cap N(A)$, we have $|N(x) \cap N(A)| \geq 2$. Clearly, $N(x) \cap F_1 \neq \emptyset$. Then, $|N(x) \cap A| \geq 2$. Thus, $|N(x) \cap \overline{A}| = 1$ and

$|N(x) \cap N(A)| = 2$. So, $N(x) \cap N(A) = \{y, w\}$. Note that $x \in N(F_2)$, $N(x) \cap F_2 \neq \emptyset$. So, $|N(x) \cap F_2| = 1$. Note that $\{x, y\} \subseteq N(F_2)$, by the choice of A, we have $|F_2| \geq |A| \geq 3$. Clearly, $|\overline{F}_2| \geq 2$. As $N(x) \cap N(F_2) \subseteq \{y, w\}$ and $N(x) \cap N(y) = \emptyset$, by Lemma3, we have that $N(x) \cap F_2 = N(N(x) \cap N(F_2) \cap V_5) \subseteq N(\{y, w\})$, and hence $N(x) \cap F_2 \subseteq N(w) \cap F_2$ and $|N(w) \cap F_2| \geq 2$. Then, $w \in T \cap N(A)$. Now we look at F_1. Since $F_1 \cap V_5 \subseteq A \cap V_5 = \emptyset$, $|F_1| \geq 2$. If $|F_1| = 2$, by noting that $|N(x) \cap F_1| = 1$, then $F_1 \cap V_5 \neq \emptyset$, a contradiction. So, $|F_1| \geq 3$. Note that $z \notin V_5$, we have $N(x) \cap N(F_1) \cap V_5 = \{w\}$. By Lemma3, we also have $N(x) \cap F_1 \subseteq N(w) \cap F_1$ and $|N(w) \cap F_1| \geq 2$. Hence, $|N(w) \cap F_1| = 2 = |N(w) \cap F_2|$. Let $N(x) \cap F_1 = \{z'\}$. Then, $F_1' = F_1 - \{z'\}$ is a fragment and $z' \in N(w)$. As $N(F_1') = (N(F_1) - \{x\}) \cup \{z'\}$, we have $w \in N(F_1')$ and $N(w) \cap N(F_1') \neq \emptyset$. By Lemma1, $N(w) \cap (F_1' \cup N(F_1')) \cap V_5 \neq \emptyset$. By noting that $N(w) \cap A = N(w) \cap (F_1' \cup N(F_1'))$, we have $N(w) \cap A \cap V_5 \neq \emptyset$, a contradiction. Hence, $\overline{F} \cap \overline{A} = \emptyset$.

Now, if $\overline{F} \cap A \neq \emptyset$, then we can similarly deduce that $F \cap \overline{A} = \emptyset$, and thus $\overline{A} \subseteq T$, and so $|N(A)| \geq 2|\overline{A}| + 1 \geq 7$, a contradiction. So, $\overline{F} \cap A = \emptyset$, and hence $\overline{F} \subseteq N(A)$. Next we show that $|T \cap A| > |\overline{F}|$. Otherwise, we have $|T \cap A| = |\overline{F}|$, and thus $F \cap A$ is a fragment and $N(F \cap A) = (T \cap A) \cup (T \cap N(A)) \cup (F \cap N(A))$. By the choice of A, we have $y \in \overline{F}$. As $N(x) \cap N(y) = \emptyset$ and $z \in N(x) \cap T$, we have $|\overline{F}| \geq 2$. If $|\overline{F}| \geq 3$, then $|T \cap A| \geq 3$, and thus $|T \cap \overline{A}| \leq 1$, implying $F \cap \overline{A} = \emptyset$. So, $|\overline{A}| = |T \cap \overline{A}| \leq 1$, a contradiction. Thus, $|\overline{F}| = 2$. Let $\overline{F} = \{y, y'\}$. As $z \notin N(y)$, $d(y) = 5$ and $yy' \in E(G)$, and thus $y'x \notin E(G)$ and $d(y') = 5$. Then, we can similarly deduce as in the proof of Assertion 1 that each edge incident to y' is in some triangle, a contradiction. Hence, $|T \cap A| > |\overline{F}|$. Thus, if $|\overline{F}| \geq 2$, then $|T \cap A| \geq 3$, and so $|T \cap \overline{A}| \leq 1$, implying that $F \cap \overline{A} = \emptyset$ and $|\overline{A}| \leq 1$, a contradiction. So, we have that $|\overline{F}| = 1$. This proves Assertion 3. ■

Let $\overline{F} = \{w\}$. As $z \in N(x) \cap N(w)$, $w \neq y$.

Assertion 4 $|N(w) \cap A| = 2$ *and* $|N(w) \cap N(A)| = 2$ *and* $|N(w) \cap \overline{A}| = 1$.

Proof. At first we show that $|N(w) \cap A| = 2$. By Assertion 3, we have $|N(w) \cap A| = |A \cap T| \geq 2$. Assume that $|N(w) \cap A| \geq 3$. Then we have $|N(w) \cap N(A)| = |N(w) \cap \overline{A}| = 1$. We claim that $|N(x) \cap A| \geq 2$. For Otherwise, $|N(x) \cap A| = 1$, and thus $N(x) \cap A = \{z\}$. Let $A' = A - \{z\}$. Then A' is a fragment. As $z \in N(w)$ and $z \in N(A')$, by Lemma1, we have $N(w) \cap A \cap V_5 \neq \emptyset$. Hence $N(w) \cap A \cap V_5 \neq \emptyset$ which contradicts the fact $A \cap V_5 = \emptyset$. Hence, $|N(x) \cap A| \geq 2$, and thus $|N(x) \cap \overline{A}| = 1$. Note that $N(x) \cap N(A) = \{y, w\}$ and $N(x) \cap N(y) = \emptyset$. Then by Lemma 3, we have $N(x) \cap \overline{A} \subset N(w) \cap \overline{A}$ and $|N(w) \cap \overline{A}| \geq 2$, which contradicts the assumption that $|N(w) \cap \overline{A}| = 1$. Hence, $|N(w) \cap A| = 2$.

Next we show $|N(w) \cap N(A)| = 2$. Assume that $|N(w) \cap N(A)| = 1$. Then $|N(w) \cap \overline{A}| = 2$. In this case, by using the same reasoning as above we can deduce that $|N(x) \cap A| \geq 2$, and hence $|N(x) \cap A| = 2$. So, $N(x) \cap N(A) = \{y, w\}$. Now we claim that $N(x) \cap A \subseteq T \cap A$. For otherwise, let $z' \in N(x) \cap A - T \cap A$. Let T' be a smallest separating set such that $x, z' \in T$ and let F' be a T'-fragment. Then, the same discussion for T, F still hold for T', F', so we have $z' \in N(w)$, which contradicts the assumption that $|N(w) \cap A| = 1$. Hence, we have $N(x) \cap A \subseteq$

$T \cap A \subseteq N(w) \cap A$, and thus $N(x) \cap A = N(w) \cap A$. Clearly, in this case we have $|N(x) \cap \overline{A}| = 1$. Still by Lemma3, $N(x) \cap \overline{A} \subsetneq N(w) \cap \overline{A}$. Let $N(x) \cap \overline{A} = \{u'\}$. Then $A' = \overline{A} - \{u'\}$ is a fragment. As $u' \in N(w)$, we have $|N(w) \cap A'| = 1$. Note that $N(A') = N(A) \cup \{u'\} - \{x\}$, we have $N(w) \cap N(A') = \{u'\}$. We claim that $|A'| \geq 3$. Assume that $|A'| \leq 2$. Then $|\overline{A}| \leq 3$, and thus $|\overline{A}| = 3$, which means that $\overline{A}$ is also an $\mathcal{S}$-atom. By Assertion 1, we have $\overline{A} \cap V_5 = \emptyset$. As $|A'| = 2$, we have $|N(w) \cap A'| \geq 2$, which contradicts the fact that $|N(w) \cap A'| = 1$. So, $|A'| \geq 3$. By Lemma3, $N(w) \cap A' \subsetneq N(u') \cap A'$. In this situation we observe that each edge incident to w is contained in some triangles, a contradiction. So, $|N(w) \cap N(A)| \geq 2$, and thus $|N(w) \cap N(A)| = 2$ and $|N(w) \cap \overline{A}| = 1$. ■

We are ready to complete the proof of theorem 2. Let $N(w) \cap A = \{z, z_1\}$, $N(w) \cap N(A) = \{x, w_1\}$ and $N(w) \cap \overline{A} = \{z_2\}$. If $|N(x) \cap A| = 1$, then $N(x) \cap A = \{z\}$. Let $A_1 = A - \{z\}$. Then, A_1 is a fragment such that $N(A_1) = N(A) \cup \{z\} - \{x\}$. Clearly, $w \in N(A_1)$ and $N(w) \cap N(A_1) = \{z, w_1\}$. If $|A_1| = 2$, as $A_1 \cap V_5 = \emptyset$, then $|N(w) \cap A_1| = 2$, a contradiction. So, $|A_1| \geq 3$. Then, by Lemma3, $z_1 \in N(w_1)$. As $N(w) \cap \overline{A} = \{z_2\}$, still by Lemma3, we have $z_2 \in N(w_1) \cup N(x)$. As $zx \in E(G)$, it follows that each edge incident to w is in some triangle, a contradiction. Hence, $|N(x) \cap A| \geq 2$. So, $N(x) \cap N(A) = \{w, y\}$ and $|N(x) \cap \overline{A}| = 1$. By Lemma3, we have $N(x) \cap \overline{A} \subseteq N(w) \cap \overline{A}$, and thus $N(x) \cap \overline{A} = N(w) \cap \overline{A} = \{z_2\}$, implying that $N(A) \cup \{z_2\} - \{x, w\}$ is a smallest separating set of cardinality 4, a contradiction. This proves Theorem 2. ■

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Appendix

We give an outline showing that G_1, G_2 are contraction critical 5-connected.

1. We first show that G_1, G_2 are 5-connected by the following three steps.

(i) For $i = 1, 2$, $G_i - \{x, y\}$ is 3-connected.

(ii) In $G_i - x$ there are four internally vertex disjoint paths from y to the other vertices, and in $G_i - y$ there are also four internally vertex disjoint paths from x to the other vertices.

(ii) There is no such separating set T of cardinality 4 in G_i that satisfies $T \cap \{x, y\} = \emptyset$.

2. In G_i, we can observe that the two end vertices of many edges have a common neighbor of degree 5, and the two end vertices of the remaining edges are also contained in a separating set of cardinality 5.

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