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P_7-factorization of complete bipartite graphs

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Abstract

In this paper, it is shown that necessary and sufficient conditions for the existence of a P_7-factorization of the complete bipartite graph $K_{m,n}$ are
(1) $4n \geq 3m$,
(2) $4m \geq 3n$,
(3) $m + n \equiv 0 \pmod 7$, and
(4) $7mn/[6(m+n)]$ is an integer.

1 Introduction

Let P_k be the path on k vertices and $K_{m,n}$ be the complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$. A subgraph F of $K_{m,n}$ is called a spanning subgraph of $K_{m,n}$ if F contains all the vertices of $K_{m,n}$. A P_k-factor of $K_{m,n}$ is a spanning subgraph F of $K_{m,n}$ such that every component of F is a P_k and every pair of P_k's have no vertex in common. A P_k-factorization of $K_{m,n}$ is a set of edge-disjoint P_k-factors of $K_{m,n}$ which is a partition of the set of edges of $K_{m,n}$. In paper [6], the P_k-factorization of $K_{m,n}$ is defined as a resolvable $(m, n, k, 1)$ bipartite P_k-design. The graph $K_{m,n}$ is called P_k-factorizable whenever it has a P_k-factorization. For graph theoretical terms, see [4].

When k is an even number, the spectrum problem for a P_k-factorization of $K_{m,n}$ has been completely solved (see [3], [6] and [8]). When k is an odd number, the spectrum problem for a P_k-factorization of $K_{m,n}$ seems to be much less tractable. In the early paper [5], Ushio gave a necessary and sufficient condition for the existence of P_3-factorization of $K_{m,n}$. Some further work was done by Ushio and Tsuruno in [7] and Du in [1], [2] and the author and B. Du in [9]. In paper [6], Ushio gave the following conjecture (Conjecture 5.3 in [6]).

Conjecture 1.1 Let m and n be positive integers and k be odd. Then $K_{m,n}$ has a P_k-factorization if and only if (1) $(k+1)n \geq (k-1)m$, (2) $(k+1)m \geq (k-1)n$, (3) $m + n \equiv 0$ $(mod\ k)$, and (4) $kmn/[(k-1)(m+n)]$ is an integer.

Very recently, Du and the author [10] have shown that Ushio Conjecture is true for $k = 5$. In this paper we will show that Ushio Conjecture is true when $k = 7$. That is, we shall prove

Theorem 1.2 *Let m and n be positive integers. Then $K_{m,n}$ has a P_7-factorization if and only if*
(1) $4n \geq 3m$,
(2) $4m \geq 3n$,
(3) $m + n \equiv 0 \ (mod\ 7)$, *and*
(4) $7mn/[6(m + n)]$ *is an integer.*

2 Main result

Using simple computation, we have the following necessary condition for the existence of a P_7-factorization of the complete bipartite graph $K_{m,n}$.

Theorem 2.1 *If $K_{m,n}$ has a P_7-factorization, then* (1) $4n \geq 3m$, (2) $4m \geq 3n$, (3) $m + n \equiv 0 \ (mod\ 7)$, *and* (4) $7mn/[6(m + n)]$ *is an integer.*

In the remainder of the paper we prove the sufficiency of Theorem 1.2. For any two integers x and y, we use $\gcd(x, y)$ to denote the greatest common divisor of x and y. The following lemma is obvious.

Lemma 2.2 *Let a, b, p and q be positive integers, if* $\gcd(ap, bq) = 1$, *then*

$$\gcd(ap + bq, pq) = 1.$$

We first prove the following result, which is used later in this paper.

Theorem 2.3 *If $K_{m,n}$ has a P_7-factorization, then $K_{sm,sn}$ has a P_7-factorization for every positive integer s.*

Proof Let $\{F_i : 1 \leq i \leq s\}$ be a 1-factorization of $K_{s,s}$ (which exists by [4]). For each $i \in \{1, 2, \cdots, s\}$, replace every edge of F_i by a $K_{m,n}$ to get a factor G_i of $K_{sm,sn}$ such that the graph G_i are pairwise edge-disjoint and their union is $K_{sm,sn}$. Since $K_{m,n}$ has a P_7-factorization, it is clear that the graph G_i, too, has a P_7-factorization. Consequently, $K_{sm,sn}$ has a P_7-factorization. This proves the theorem.

Now we start to prove our main result Theorem 1.2. There are three cases to consider.

Case $4m = 3n$: In this case, from Theorem 2.3, $K_{m,n}$ has a P_7-factorization, since $K_{3,4}$ has a P_7-factorization:

$$y_1x_1y_2x_2y_3x_3y_4, \quad y_3x_1y_4x_2y_1x_3y_2.$$

Case $3m = 4n$: Obviously, $K_{m,n}$ has a P_7-factorization.

Case $4m > 3n$ and $4n > 3m$: In this case, let $a = (4n - 3m)/7$, $b = (4m - 3n)/7$, $t = (m+n)/7$, and $r = 7mn/[6(m+n)]$. Then from Conditions (1)–(4) in Theorem 1.2, a, b, t, r are integers and $0 < a < m$ and $0 < b < n$. We have $3a+4b = m$ and $4a + 3b = n$. Hence $r = 2(a+b) + ab/[6(a+b)]$. Let $z = ab/[6(a+b)]$, which is a positive integer. And let $\gcd(3a, 4b) = d$, $3a = dp$, $4b = dq$, where $\gcd(p, q) = 1$. Then $z = dpq/[6(4p+3q)]$. These equalities imply the following equalities:

$$d = \frac{6(4p+3q)z}{pq},$$
$$m = \frac{6(p+q)(4p+3q)z}{pq},$$
$$n = \frac{(16p+9q)(4p+3q)z}{2pq},$$
$$r = \frac{(p+q)(16p+9q)z}{pq},$$
$$a = \frac{2p(4p+3q)z}{pq},$$
$$b = \frac{3q(4p+3q)z}{2pq}.$$

Now we can establish the following lemma.

Lemma 2.4
(1) If $\gcd(p, 9) = 1$ *and* $\gcd(q, 16) = 1$*, then*

$$m = 12(p+q)(4p+3q)s, \qquad n = (16p+9q)(4p+3q)s,$$
$$a = 4p(4p+3q)s, \quad b = 3q(4p+3q)s, \quad r = 2(p+q)(16p+9q)s,$$

for some positive integer s.

(2) If $\gcd(p, 9) = 1$ *and* $\gcd(q, 16) = 2$*, let* $q = 2q_1$*. Then*

$$m = 6(p+2q_1)(2p+3q_1)s, \qquad n = (8p+9q_1)(2p+3q_1)s,$$
$$a = 2p(2p+3q_1)s, \quad b = 3q_1(2p+3q_1)s, \quad r = (p+2q_1)(8p+9q_1)s,$$

for some positive integer s.

(3) If $\gcd(p, 9) = 1$ *and* $\gcd(q, 16) = 4$*, let* $q = 4q_2$*. Then*

$$m = 6(p+4q_2)(p+3q_2)s, \qquad n = 2(4p+9q_2)(p+3q_2)s,$$
$$a = 2p(p+3q_2)s, \quad b = 6q_2(p+3q_2)s, \quad r = (p+4q_2)(4p+9q_2)s,$$

for some positive integer s.

(4) If $\gcd(p, 9) = 1$ *and* $\gcd(q, 16) = 8$*, let* $q = 8q_3$*. Then*

$$m=3(p+8q_3)(p+6q_3)s,\quad n=2(2p+9q_3)(p+6q_3)s,$$
$$a=p(p+6q_3)s,\quad b=6q_3(p+6q_3)s,\quad r=(p+8q_3)(2p+9q_3)s,$$

for some positive integer s.

(5) If $\gcd(p,9)=1$ *and* $\gcd(q,16)=16$*, let* $q=16q_4$*. Then*

$$m=3(p+16q_4)(p+12q_4)s,\quad n=4(p+9q_4)(p+12q_4)s,$$
$$a=p(p+12q_4)s,\quad b=12q_4(p+12q_4)s,\quad r=2(p+16q_4)(p+9q_4)s,$$

for some positive integer s.

(6) If $\gcd(p,9)=3$ *and* $\gcd(q,16)=1$*, let* $p=3p_1$*. Then*

$$m=12(3p_1+q)(4p_1+q)s,\quad n=3(16p_1+3q)(4p_1+q)s,$$
$$a=12p_1(4p_1+q)s,\quad b=3q(4p_1+q)s,\quad r=2(3p_1+q)(16p_1+3q)s,$$

for some positive integer s.

(7) If $\gcd(p,9)=3$ *and* $\gcd(q,16)=2$*, let* $p=3p_1$ *and* $q=2q_1$*. Then*

$$m=6(3p_1+2q_1)(2p_1+q_1)s,\quad n=3(8p_1+3q_1)(2p_1+q_1)s,$$
$$a=6p_1(2p_1+q_1)s,\quad b=3q_1(2p_1+q_1)s,\quad r=(3p_1+2q_1)(8p_1+3q_1)s,$$

for some positive integer s.

(8) If $\gcd(p,9)=3$ *and* $\gcd(q,16)=4$*, let* $p=3p_1$ *and* $q=4q_2$*. Then*

$$m=6(3p_1+4q_2)(p_1+q_2)s,\quad n=6(4p_1+3q_2)(p_1+q_2)s,$$
$$a=6p_1(p_1+q_2)s,\quad b=6q_2(p_1+q_2)s,\quad r=(3p_1+4q_2)(4p_1+3q_2)s,$$

for some positive integer s.

(9) If $\gcd(p,9)=3$ *and* $\gcd(q,16)=8$*, let* $p=3p_1$ *and* $q=8q_3$*. Then*

$$m=3(3p_1+8q_3)(p_1+2q_3)s,\quad n=6(2p_1+3q_3)(p_1+2q_3)s,$$
$$a=3p_1(p_1+2q_3)s,\quad b=6q_3(p_1+2q_3)s,\quad r=(3p_1+8q_3)(2p_1+3q_3)s,$$

for some positive integer s.

(10) If $\gcd(p,9)=3$ *and* $\gcd(q,16)=16$*, let* $p=3p_1$ *and* $q=16q_4$*. Then*

$$m=3(3p_1+16q_4)(p_1+4q_4)s,\quad n=12(p_1+3q_4)(p_1+4q_4)s,$$
$$a=3p_1(p_1+4q_4)s,\quad b=12q_4(p_1+4q_4)s,\quad r=2(3p_1+16q_4)(p_1+3q_4)s,$$

for some positive integer s.

(11) If $\gcd(p, 9) = 9$ *and* $\gcd(q, 16) = 1$*, let* $p = 9p_2$*. Then*

$$m = 4(9p_2 + q)(12p_2 + q)s, \quad n = 3(16p_2 + q)(12p_2 + q)s,$$
$$a = 12p_2(12p_2 + q)s, \quad b = q(12p_2 + q)s, \quad r = 2(9p_2 + q)(16p_2 + q)s,$$

for some positive integer s.

(12) If $\gcd(p, 9) = 9$ *and* $\gcd(q, 16) = 2$*, let* $p = 9p_2$ *and* $q = 2q_1$*. Then*

$$m = 2(9p_2 + 2q_1)(6p_2 + q_1)s, \quad n = 3(8p_2 + q_1)(6p_2 + q_1)s,$$
$$a = 6p_2(6p_2 + q_1)s, \quad b = q_1(6p_2 + q_1)s, \quad r = (9p_2 + 2q_1)(8p_2 + 3q_1)s,$$

for some positive integer s.

(13) If $\gcd(p, 9) = 9$ *and* $\gcd(q, 16) = 4$*, let* $p = 9p_2$ *and* $q = 4q_2$*. Then*

$$m = 2(9p_2 + 4q_2)(3p_2 + q_2)s, \quad n = 6(4p_2 + q_2)(3p_2 + q_2)s,$$
$$a = 6p_2(3p_2 + q_2)s, \quad b = 2q_2(3p_2 + q_2)s, \quad r = (9p_2 + 4q_2)(4p_2 + q_2)s,$$

for some positive integer s.

(14) If $\gcd(p, 9) = 9$ *and* $\gcd(q, 16) = 8$*, let* $p = 9p_2$ *and* $q = 8q_3$*. Then*

$$m = (9p_2 + 8q_3)(3p_2 + 2q_3)s, \quad n = 6(2p_2 + q_3)(3p_2 + 2q_3)s,$$
$$a = 3p_2(3p_2 + 2q_3)s, \quad b = 2q_3(3p_2 + 2q_3)s, \quad r = (9p_2 + 8q_3)(2p_2 + q_3)s,$$

for some positive integer s.

(15) If $\gcd(p, 9) = 9$ *and* $\gcd(q, 16) = 16$*, let* $p = 9p_2$ *and* $q = 16q_4$*. Then*

$$m = (9p_2 + 16q_4)(3p_2 + 4q_4)s, \quad n = 12(p_2 + q_4)(3p_2 + 4q_4)s,$$
$$a = 3p_2(3p_2 + 4q_4)s, \quad b = 4q_4(3p_2 + 4q_4)s, \quad r = 2(9p_2 + 16q_4)(p_2 + q_4)s,$$

for some positive integer s.

Proof We assume that $\gcd(p, q) = 1$, $\gcd(p, 9) = 1$ and $\gcd(q, 16) = 1$ hold. Then $\gcd(16p + 9q, 2) = \gcd(4p + 3q, 2) = 1$ and $\gcd(16p, 9q) = \gcd(4p, 3q) = 1$ hold. It is easy to see that $n = (16p + 9q)(4p + 3q)z/(2pq)$. By Lemma 2.2, we see that $\gcd(16p+9q, pq) = \gcd(4p+3q, pq) = 1$. Therefore, $z/(2pq)$ must be an integer. Let $s = z/(2pq)$. Then the equalities in (1) hold.

The proof of the equalities in (2)–(4), (6)–(10) and (12)–(15) are similar to (1).

We assume that $\gcd(p, q) = 1$, $\gcd(p, 9) = 1$, $\gcd(q, 16) = 16$ and $q = 16q_4$ hold. Then $\gcd(p+16q_4, 2) = \gcd(p+12q_4, 2) = 1$ and $\gcd(p, 16q_4) = \gcd(p, 9q_4) = 1$ hold.

It is easy to see that $r = (p+16q_4)(p+9q_4)z/(pq_4)$. By Lemma 2.2, we see that $\gcd(p+16q_4, pq_4) = \gcd(p+9q_4, pq_4) = 1$. Therefore, $z/(pq_4)$ must be an integer. Let $z' = z/(pq_4)$. Then we have $m = 3(p+16q_4)(p+12q_4)z'/2$ is an integer. We see $z'/2$ must be an integer. Let $s = z'/2$. Then the equalities in (5) hold.

The proof of the equalities in (11) are similar to (5).

This proves the lemma.

For our main result, we need the following direct constructions. We use $\lceil x \rceil$ to denote the least integer not less than x and $\lfloor x \rfloor$ the largest integer not exceeding x.

Lemma 2.5 *For any positive integers p and q, let $m = 6(p+2q)(2p+3q)$ and $n = (8p+9q)(2p+3q)$. Then $K_{m,n}$ has a P_7-factorization.*

Proof Let $a = 2p(2p+3q)$, $b = 3q(2p+3q)$, $r = (p+2q)(8p+9q)$, $r_1 = p+2q$ and $r_2 = 8p+9q$. Let X and Y be two partite sets of $K_{m,n}$ and set

$$X = \{x_{i,j} : 1 \le i \le r_1,\ 1 \le j \le 6(2p+3q)\},$$
$$Y = \{y_{i,j} : 1 \le i \le r_2,\ 1 \le j \le 2p+3q\}.$$

We will construct a P_7-factorization of $K_{m,n}$. We remark in advance that the additions in the first subscripts of $x_{i,j}$'s and $y_{i,j}$'s are taken modulo r_1 and r_2 in $\{1, 2, \cdots, r_1\}$ and $\{1, 2, \cdots, r_2\}$, respectively, and the additions in the second subscripts of $x_{i,j}$'s and $y_{i,j}$'s are taken modulo $6(2p+3q)$ and $2p+3q$ in $\{1, 2, \cdots, 6(2p+3q)\}$ and $\{1, 2, \cdots, 2p+3q\}$, respectively.

For each $1 \le i \le p$, let

$$E_i = \{x_{i,j+(2p+3q)(u-1)+3(2p+3q)v}\, y_{8(i-1)+4v+u+w,j+2i-1+w} : 1 \le j \le 2p+3q,\ 1 \le u \le 3,\ 0 \le v \le 1,\ 0 \le w \le 1\}.$$

For each $1 \le i \le q$, let $E_{p+i} =$

$$\{x_{p+2(i-1)+\lceil (u+w)/2 \rceil, j+(2p+3q)(v-1)+3(2p+3q)[1-(\lceil u/2 \rceil - \lfloor u/2 \rfloor)](1-w)+3(2p+3q)(\lceil u/2 \rceil - \lfloor u/2 \rfloor)w}\, y_{8p+9(i-1)+3(v-1)+u, 2p+j+3(i-1)+u} : 1 \le j \le 2p+3q,\ 1 \le u \le 3,\ 1 \le v \le 3,\ 0 \le w \le 1\}.$$

Let $F = \bigcup_{1 \le i \le p+q} E_i$. Then the graph F is a P_7-factor of $K_{m,n}$. Define a bijection σ from $X \cup Y$ onto $X \cup Y$ in such a way that $\sigma(x_{i,j}) = x_{i+1,j}$ and $\sigma(y_{i,j}) = y_{i+1,j}$. For each $i \in \{1, 2, \cdots, r_1\}$ and each $j \in \{1, 2, \cdots, r_2\}$, let

$$F_{i,j} = \{\sigma^i(x)\sigma^j(y) : x \in X, y \in Y, xy \in F\}.$$

It is easy to show that the graphs $F_{i,j}$ $(1 \le i \le r_1,\ 1 \le j \le r_2)$ are P_7-factors of $K_{m,n}$ and their union is $K_{m,n}$. Thus, $\{F_{i,j} : 1 \le i \le r_1,\ 1 \le j \le r_2\}$ is a P_7-factorization of $K_{m,n}$. This proves the lemma.

The proof of the following lemma is similar to Lemma 2.5, so we only give the represention of X, Y, E_i and E_{p+i}.

Lemma 2.6 *For any positive integers p and q, let $m = 6(p+4q)(p+3q)$ and $n = 2(4p+9q)(p+3q)$. Then $K_{m,n}$ has a P_7-factorization.*

Proof Let $a = 2p(p+3q)$, $b = 6q(p+3q)$, $r = (p+4q)(4p+9q)$, $r_1 = p+4q$, $r_2 = 4p+9q$ and

$$\begin{aligned} X &= \{x_{i,j} : 1 \le i \le r_1,\ 1 \le j \le 6(p+3q)\}, \\ Y &= \{y_{i,j} : 1 \le i \le r_2,\ 1 \le j \le 2(p+3q)\}. \end{aligned}$$

For each $1 \le i \le p$, let

$$\begin{aligned} E_i = \{&x_{i,j+2(p+3q)u}y_{4(i-1)+u+v+1,j+2i-1+v} \\ &: 1 \le j \le 2(p+3q),\ 0 \le u \le 2,\ 0 \le v \le 1\}. \end{aligned}$$

For each $1 \le i \le q$, let

$$\begin{aligned} E_{p+i} = \{&x_{p+4(i-1)+u+w,j+2(p+3q)v}y_{4p+9(i-1)+3v+u,2p+j+6(i-1)+2u+w-1} \\ &: 1 \le j \le 2(p+3q),\ 1 \le u \le 3,\ 0 \le v \le 2,\ 0 \le w \le 1\}. \end{aligned}$$

Lemma 2.7 *For any positive integers p and q, let $m = 3(p+8q)(p+6q)$ and $n = 2(2p+9q)(p+6q)$. Then $K_{m,n}$ has a P_7-factorization.*

Proof Let $a = p(p+6q)$, $b = 6q(p+6q)$, $r = (p+8q)(2p+9q)$, $r_1 = p+8q$, $r_2 = 2p+9q$ and

$$\begin{aligned} X &= \{x_{i,j} : 1 \le i \le r_1,\ 1 \le j \le 3(p+6q)\}, \\ Y &= \{y_{i,j} : 1 \le i \le r_2,\ 1 \le j \le 2(p+6q)\}. \end{aligned}$$

For each $1 \le i \le p$, let

$$\begin{aligned} E_i = \{&x_{i,j+(p+6q)(u-1)}y_{2(i-1)+\lceil (u+v)/2\rceil,j+i+[1-(\lceil u/2\rceil-\lfloor u/2\rfloor)](p+6q)(1-v)+(\lceil u/2\rceil-\lfloor u/2\rfloor)(p+6q)v} \\ &: 1 \le j \le p+6q,\ 1 \le u \le 3,\ 0 \le v \le 1\}. \end{aligned}$$

For each $1 \le i \le q$, let

$$\begin{aligned} E_{p+i} = \{&x_{p+4u+8(i-1)+w+v,j+(p+6q)(h-1)}y_{2p+9(i-1)+3(h-1)+w,j+(p+6q)u+p+w+3v+6(i-1)} \\ &: 1 \le j \le p+6q,\ 0 \le u \le 1,\ 0 \le v \le 1,\ 1 \le w \le 3,\ 1 \le h \le 3\}. \end{aligned}$$

Lemma 2.8 *For any positive integers p and q, let $m = 6(3p+2q)(2p+q)$ and $n = 3(8p+3q)(2p+q)$. Then $K_{m,n}$ has a P_7-factorization.*

Proof Let $a = 6p(2p+q)$, $b = 3q(2p+q)$, $r = (3p+2q)(8p+3q)$, $r_1 = 3p+2q$, $r_2 = 8p+3q$ and

$$X = \{x_{i,j} : 1 \le i \le r_1,\ 1 \le j \le 6(2p+q)\},$$
$$Y = \{y_{i,j} : 1 \le i \le r_2,\ 1 \le j \le 3(2p+q)\}.$$

For each $1 \le i \le p$, let

$$E_i = \{x_{3(i-1)+u,j+3(2p+q)v}\, y_{8(i-1)+4v+u+w,j+6(i-1)+2u+w-1} : 1 \le j \le 3(2p+q),\ 1 \le u \le 3,\ 0 \le v \le 1,\ 0 \le w \le 1\}.$$

For each $1 \le i \le q$, let

$$E_{p+i} = \{x_{3p+2(i-1)+\lceil (u+v)/2\rceil,j+3(2p+q)[1-(\lceil u/2\rceil-\lfloor u/2\rfloor)](1-v)+3(2p+q)(\lceil u/2\rceil-\lfloor u/2\rfloor)v}\, y_{8p+3(i-1)+u,6p+j+u+3(i-1)} : 1 \le j \le 3(2p+q),\ 1 \le u \le 3,\ 0 \le v \le 1\}.$$

Lemma 2.9 *For any positive integers p and q, let $m = 6(3p+4q)(p+q)$ and $n = 6(4p+3q)(p+q)$. Then $K_{m,n}$ has a P_7-factorization.*

Proof Let $a = 6p(p+q)$, $b = 6q(p+q)$, $r = (3p+4q)(4p+3q)$, $r_1 = 3p+4q$, $r_2 = 4p+3q$ and

$$X = \{x_{i,j} : 1 \le i \le r_1,\ 1 \le j \le 6(p+q)\},$$
$$Y = \{y_{i,j} : 1 \le i \le r_2,\ 1 \le j \le 6(p+q)\}.$$

For each $1 \le i \le p$, let

$$E_i = \{x_{3(i-1)+u,j}\, y_{4(i-1)+u+v,j+6(i-1)+2(u-1)+v} : 1 \le j \le 6(p+q),\ 1 \le u \le 3,\ 0 \le v \le 1\}.$$

For each $1 \le i \le q$, let

$$E_{p+i} = \{x_{3p+4(i-1)+u+v,j}\, y_{4p+3(i-1)+u,6p+j+6(i-1)+2(u-1)+v} : 1 \le j \le 6(p+q),\ 1 \le u \le 3,\ 0 \le v \le 1\}.$$

The proof of Theorem 1.2: By applying Theorem 2.3 with Lemmas 2.4 to 2.9, it can be seen that when the parameters m and n satisfy conditions (1)–(4) in Theorem 1.2, the graph $K_{m,n}$ has a P_7-factorization. This completes the proof of Theorem 1.2.

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