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On the largest minimal blocking set in $\mathbf{P}^2(\mathbb{F}_8)$

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Abstract

A new description of the unique minimal 23–blocking set of $\mathbf{P}^2(\mathbb{F}_8)$ is given.

1 Introduction

A *blocking set* in a projective plane is a set of points intersecting every line, but containing no line entirely. A blocking set is said to be *minimal* if it is minimal with respect to set–theoretic inclusion. Generally, one is interested in the existence of blocking sets in finite projective planes and perhaps proving their uniqueness.

In a recent paper Barát and Innamorati [1] studied the largest minimal blocking sets of the projective plane $\mathbf{P}^2(\mathbb{F}_8)$. They proved that the Bruen-Thas bound for the size of a minimal blocking set, that is $q\sqrt{q}+1$, is sharp for $q = 8$. Further, they exhibited an interesting example and proved its uniqueness from a combinatorial point of view.

In this paper, we give a construction of a minimal blocking set B of $\mathbf{P}^2(\mathbb{F}_8)$ of size 23 based on the geometry of the Klein quartic. By construction, the linear automorphism group of B has order 7 (in the paper [1], the authors claim that the automorphism group of their blocking set has order 21, but we think this is supposed to be the automorphism group of B as a subgroup of the group $\mathrm{P\Gamma L}(3,\mathbb{F}_8)$.) By the combinatorial uniqueness of minimal 23-blocking sets of $\mathbf{P}^2(\mathbb{F}_8)$ proved in [1], our blocking set is isomorphic to the Barát-Innamorati blocking set.

2 Singer cycles and the Klein quartic

Let $\mathbb{F}_8$ be a cubic extension of $\mathbb{F}_2$. Let ω be a primitive element of $\mathbb{F}_8$ and $m(x) = x^3 + a_2x^2 + a_1x + a_0$ its minimal polynomial over $\mathbb{F}_2$. The companion matrix $C(m)$

of $m(x)$ given by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{pmatrix}$$

induces a linear collineation ϕ of $\mathbf{P}^2(\mathbb{F}_2)$ of order $q^2+q+1=7$ called a *Singer cycle* of $\mathrm{PGL}(3,\mathbb{F}_2)$.

All Singer cycles of $\mathrm{PGL}(3,\mathbb{F}_2)$ form a single conjugacy class and the matrix $C(m)$ is conjugate in $\mathrm{GL}(3,\mathbb{F}_8)$ to the diagonal matrix

$$D = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^4 \end{pmatrix}$$

by the matrix

$$E = \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & \omega^4 \\ \omega^2 & \omega^4 & \omega \end{pmatrix}$$

Let σ denote the linear collineation of $\mathbf{P}^2(\mathbb{F}_8)$ induced by D. It fixes the points $E_0=(1,0,0)$, $E_1=(0,1,0)$, $E_2=(0,0,1)$.

The linear collineation T of $\mathbf{P}^2(\mathbb{F}_8)$ given by

$$(X_0,X_1,X_2)\longmapsto(X_2,X_0,X_1)$$

has order three and acts on the points E_0,E_1,E_2 as the cycle $(E_0E_1E_2)$. The group $\langle T\rangle$ normalizes $S=\langle\sigma\rangle$ and $N=\langle T,\sigma\rangle$ is the normalizer of S in a $\mathrm{PGL}(3,\mathbb{F}_2)$ embedded in $\mathrm{PGL}(3,\mathbb{F}_8)$.

The orbit of the point $U=(1,1,1)$ under the action of S is given by

$$\Pi_2=\{\sigma^i(U) : i=0,\ldots,6\}=\{(1,\omega^i,\omega^{3i})\}.$$

Π_2 may be viewed as a subgeometry of $\mathbf{P}^2(\mathbb{F}_8)$ which turns out to be a projective plane of order 2. More precisely, Π_2 is a projective subplane of $\mathbf{P}^2(\mathbb{F}_8)$ (lying in a non-canonical position) isomorphic to $\mathbf{P}^2(\mathbb{F}_2)$.

Let $\mathcal{X}$ denote a projective, non–singular, algebraic plane curve of degree d over $GF(2)$ which is invariant under the Singer cycle ϕ of $\mathrm{PGL}(\mathbb{F}_2)$.

The main result in [2] states that either $\deg(\mathcal{X})=4$ or $\deg(\mathcal{X})\geq 7$. In the former case $\mathcal{X}$ is projectively equivalent to the famous *Klein curve* $\mathcal{X}_2$ with equation

$$XY^3+X^3Z+YZ^3=0.$$

The curve $\mathcal{X}_2$ has genus three and $\mathrm{Aut}(\mathcal{X}_2)$ is the linear group $\mathrm{PSL}(2,\mathbb{F}_7)\simeq\mathrm{PSL}(3,\mathbb{F}_2)$ which has 168 elements.

From [4], the Klein quartic over $\mathbb{F}_8$ has 24 rational points (Weierstrass points of weight 1) on which $\mathrm{PSL}(3,\mathbb{F}_2)$ acts transitively. In $\mathbf{P}^2(\mathbb{F}_8)\setminus\{\mathcal{X}_2\}$ the group $\mathrm{PSL}(3,\mathbb{F}_2)$

has two orbits, namely, the Baer subplane Π_2 and one orbit of size 42 covering the remaining points of $\mathbf{P}^2(\mathbb{F}_8)$.

A line of $\mathbf{P}^2(\mathbb{F}_8)$ meets Π_2 in either 0, or 1 or 3 points. The 73 lines of $\mathbf{P}^2(\mathbb{F}_8)$ are partitioned as follows. There are 7 lines meeting Π_2 in 3 points (yielding all lines of Π_2), 42 lines meet Π_2 in exactly one point and 24 lines are external to Π_2. Simple calculations show that the 7 lines are external to $\mathcal{X}_2$, the 42 lines are 4–secants of $\mathcal{X}_2$ and the remaining 24 lines are 2–secants of $\mathcal{X}_2$. In particular, it turns out that $\mathcal{X}_2$ is a 24–arc of type $(0, 2, 4)$.

The line–sets described above are all complete orbits under $\mathrm{PSL}(3, \mathbb{F}_2)$.

In particular, each 2–secant of $\mathcal{X}_2$ is obtained by joining pairs of fixed points of the 7–Sylow subgroups of $\mathrm{PSL}(3, \mathbb{F}_2)$; each 4–secant of $\mathcal{X}_2$ is stabilized by a subgroup $C_2 \times C_2$.

The group $\langle \sigma \rangle$ is conjugate in $\mathrm{PSL}(3, \mathbb{F}_2)$ to a 7–Sylow of $\mathrm{PSL}(3, \mathbb{F}_2)$ and its normalizer N has order 21. The group N has five orbits on the pointset of $\mathbf{P}^2(\mathbb{F}_8)$, namely, the sets $\{E_0, E_1, E_2\}$ and $\mathcal{X}_2 \setminus \{E_0, E_1, E_2\}$, one orbit of size 21 consisting of the non–vertex points of the triangle $E_0E_1E_2$ and one orbit, say O, of size 21, covering the remaining points of $\mathbf{P}^2(\mathbb{F}_8)$.

Our purpose is to prove that the set $B = \mathcal{O} \cup \{E_i\} \cup \{E_j\}$, for any two distinct indices $i, j \in \{0, 1, 2\}$, is a minimal blocking set of size 23.

3 The proof

First of all note that the lines E_iE_j, $i, j = 0, 1, 2$, are 2–secants of $\mathcal{X}_2$.

If ℓ_3 is a 3–secant of Π_2 (arising from a line of Π_2) then it is disjoint from $\mathcal{X}_2$. Since ℓ_3 meets each line E_iE_j, $i, j = 0, 1, 2$ in one point, it follows that $|B \cap \ell_3| = 3$.

If ℓ_1 is a 1–secant of Π_2, then ℓ_1 is a 4–secant of $\mathcal{X}_2$. It may happen that at most one point E_i, $i = 0, 1, 2$, lies on ℓ_1. If E_i lies on ℓ_1 and $E_i \notin B$ then ℓ_1 meets E_jE_k, $j, k \neq i$ and we have $|B \cap \ell_1| = 3$. If E_i lies on ℓ_1 and $E_i \in B$, then $|B \cap \ell_1| = 4$. If E_i does not lie on ℓ_1 then ℓ_1 meets each line E_iE_j and so $|B \cap \ell_1| = 1$.

A simple calculation shows that any line of the pencil with centre E_i, apart from E_iE_j and E_iE_k, meets Π_2 in one point. This means that there exist exactly 21 lines of $\mathbf{P}^2(\mathbb{F}_8)$ that are 1–secant to Π_2 and that do contain no point E_i. These lines meet $\mathcal{O}$ in only one point.

If ℓ_0 is an external line to Π_2, then ℓ_0 is a 2–secant of $\mathcal{X}_2$. If ℓ_0 is not the line E_iE_j, then ℓ_0 meets each line E_iE_j, $i < j$, $i, j = 0, 1, 2$ and so $|B \cap \ell_0| = 4$. If $\ell_0 = E_iE_j$, then $|B \cap \ell_0| = 2$.

Of course, the lines E_iE_k and E_jE_k meet B in exactly one point (the points E_i and E_j, respectively).

We have proved that B is a blocking set of $\mathbf{P}^2(\mathbb{F}_8)$ of size 23. Since $\mathcal{O}$ is a full orbit of N, for each point of $\mathcal{O}$ there exists exactly one 1–secant. Then B admits exactly 23 1–secants and thus it is minimal. Of course, B contains the union of three Fano subplanes.

The proof is now complete.

Remark 1 An alternative description of the minimal 23–blocking set given above is the following. Consider the three Klein quartics $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of $\mathbf{P}^2(\mathbb{F}_8)$ with equations:

$$\omega XY^3 + \omega^2 X^3 Z + \omega^4 YZ^3 = 0,$$
$$\omega^2 XY^3 + \omega^4 X^3 Z + \omega YZ^3 = 0,$$
$$\omega^4 XY^3 + \omega X^3 Z + \omega^2 YZ^3 = 0,$$

respectively.

It is easy to show that these three curves share the points E_0, E_1 and E_2 and the subplane Π_2. Now, it is possible to select a subplane, say π, of order two on one of the three curves, say $\mathcal{C}_1$, then apply the Frobenius automorphism of order three of $\mathbb{F}_8$, and obtain three disjoint subplanes lying on $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, respectively. Adding to the union of these three subplanes any two of the points E_0, E_1, E_2, the minimal 23–blocking set is obtained.

Remark 2 In [3] we proved that the automorphism group of the Pellikaan's curve $X_1^4X_2 + X_2^4X_3 + X_3^4X_1 = 0$ defined over the field $\mathbb{F}_{27}$ is the normalizer N of a Singer cycle S of $\mathbf{P}^2(\mathbb{F}_3)$ of order 13. Looking at the orbits of N on the pointset of $\mathbf{P}^2(\mathbb{F}_{27})$ we found a minimal blocking set B of size 80 with arrow $(80_1, 287_2, 195_3, 91_4, 65_5, 39_8)$. The blocking set B is obtained by gluing two orbits of N of size 39 and any two of the points $E_0 = (1,0,0)$, $E_1 = (0,1,0)$, $E_2 = (0,0,1)$. Again, B contains the union of six subplanes of order three. Its automorphism group is S. Notice that we also found other two minimal 80–blocking sets B' and B'' of $\mathbf{P}^2(\mathbb{F}_{27})$ with arrow $(80_1, 326_2, 156_3, 52_4, 65_5, 39_6, 39_7)$ and $(80_1, 287_2, 182_3, 130_4, 26_5, 13_6, 39_8)$ having the same automorphism group of B.

With the same technique, in $\mathbf{P}^2(\mathbb{F}_{64})$ we found a minimal blocking set of size 254 with arrow $(254_1, 631_2, 097_3, 1008_4, 504_5, 504_6, 63_8, 147_9, 63_{10})$ admitting a cyclic group of order 21.

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