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Completely separating systems of k-sets for $7 \leq k \leq 10$

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Abstract

An (n)**Completely Separating System** $\mathcal{C}$ is a collection of subsets of $[n] = \{1, \ldots, n\}$ such that for all distinct $a, b \in [n]$ there are subsets $A, B \in \mathcal{C}$ with $a \in A - B$ and $b \in B - A$. $R(n, k)$ denotes the minimum possible size of a completely separating system $\mathcal{C}$ on $[n]$ with $|A| = k$ for each $A \in \mathcal{C}$. Exact values of $R(n, k)$ are known for $k \leq 6$ and for some other values of n and k. Upper and lower bounds are known in all other cases. $R(n, k)$ is fully determined here for $7 \leq k \leq 10$ and general results are provided to aid in determining $R(n, k)$ for larger k in certain cases.

1 Introduction and Basic Results

This paper extends previous work in [6] and [9] in determining the minimum size of completely separating systems (CSSs) with a single block size. This section contains necessary definitions concerning CSSs and antichains, as well as an important theorem concerning antichains.

Let $k < n$. An (n, k)**Completely Separating System (or** (n, k)**CSS)** $\mathcal{C}$ is an (n)CSS in which each block (or set) is of size k. Given $x, y \in [n]$, x is said to **dominate** y in a collection of blocks $\mathcal{C}$ if for each $A \in \mathcal{C}, x \in A$ whenever $y \in A$. In any CSS, no element can dominate another element. The size of $\mathcal{C}$ is the number of blocks in $\mathcal{C}$, and it is denoted by $|\mathcal{C}|$. The integers $\mathbf{R(n)}$ and $\mathbf{R(n,k)}$ are defined by: $R(n) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is an } (n)CSS\}$ and $R(n, k) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is an } (n, k)CSS\}$. In what follows R is often written instead of $R(n, k)$. An (n)CSS for which $|\mathcal{C}| = R(n)$ is a **minimal** (n)CSS and an (n, k)CSS $\mathcal{C}$ for which $|\mathcal{C}| = R(n, k)$ is a **minimal** (n, k)CSS. The **volume** of a collection of sets $\mathcal{B}$ is $V(\mathcal{B}) = \sum_{A \in \mathcal{B}} |A|$. A CSS is said to be **fair** if each element occurs in either r or $r + 1$ blocks for some integer r. For an (n, k)CSS $\mathcal{C}$, $V(\mathcal{C}) = k|\mathcal{C}|$. Two CSSs are said to be **isomorphic** if one can be obtained from the other by a permutation of $1, \ldots, n$. A CSS is said to be **unique** if there is no non-isomorphic CSS of the same size. Given $\mathcal{C}$, the **complementary collection $\mathcal{C}'$** is $\mathcal{C}' = \{A' = [n] - A, A \in \mathcal{C}\}$.

CSSs have a dual formulation as antichains, and this has been important in the development of CSSs. An **antichain** on $[r]$ is a collection $\mathcal{A}$ of distinct subsets of $[r]$ such that for any distinct $A, B \in \mathcal{A}$, $A \not\subseteq B$. Let $\mathcal{A} = \{A_1, \ldots, A_r\}$ be a collection of subsets of $[n]$. Cai [1] defined the **dual** $\mathcal{A}^*$ of $\mathcal{A}$ to be the collection $\mathcal{A}^* = \{X_1, \ldots, X_n\}$ of subsets of $[r]$ given by $X_i = \{k : i \in A_k\}$. Antichains are the duals of CSSs (Spencer [10]): If $\mathcal{A}$ is a CSS then its dual $\mathcal{A}^*$ is an antichain and vice versa. A **flat antichain** is an antichain $\mathcal{A}$ such that $||A| - |B|| \leq 1$ for each $A, B \in \mathcal{A}$. A flat antichain is the dual of a fair CSS. The following theorem is due to Lieby [4] and Kisvölcsey [2].

Theorem 1.1 (FLAT - The Flat Antichain Theorem). *Given any antichain $\mathcal{A}$ on $[n]$, there is a flat antichain $\mathcal{A}'$ on $[n]$ with $|\mathcal{A}| = |\mathcal{A}'|$ and $V(\mathcal{A}) = V(\mathcal{A}')$.*

Some necessary results on $R(n)$ include Theorem 3.1 of [7] and Lemma 2 of [6]. Here they are partially restated in the following lemma, using the convention that each element must occur at least once.

Lemma 1.1.

1. *If $n \leq 4$ then $R(n) = n$.*
2. *If $n = 5$ or 6 then $R(n) = 4$ and there is a unique way of achieving $R(n)$ in each case.*
3. *$R(n)$ is a non-decreasing function of n.*
4. *$R(7) = 5$.*
5. *For $1 \leq k < n$, $R(n, k) = R(n, n - k)$.*

By Theorem 3.1 of [7] there is more than one minimal (n)CSS for $n = 3$ or 4. The unique minimal (n)CSSs for $n = 5$ or 6 are shown in Note 2.2.

In [6] and [9] the values of $R(n, k)$ for $n \geq k^2/2$, for $k \leq 6$, and for many values of n with $7 \leq k \leq 10$ are determined. The recent paper [3] has an asymptotic emphasis, and it determines upper and lower bounds on $R(n, k)$ for $n \geq 2k$. Its Corollary 1 states that $R(\binom{rm}{r}, \binom{rm-1}{r-1}) = rm$ for $r \geq 1$ and $m \geq 2$. This confirms some of the values of $R(n, k)$ stated below, such as $R(20, 10) = 6$.

In this paper the remaining unknown values for $R(n, k)$ for $7 \leq k \leq 10$ are determined. These values are: $k = 7$ for $n = 14$ to 18; $k = 8$ for $n = 15$ to 25; $k = 9$ for $n = 16$ to 33 and $k = 10$ for $n = 17$ to 41. The determination of these values of $R(n, k)$ involves several approaches which are described below. The derived values of $R(n, k)$ are included in Table 1. It has often been the case that an (n, k)CSS which achieves a known lower bound on $R(n, k)$ has been constructed, but they are not

n	k=1	2	3	4	5	6	7	8	9	10
2	2									
3	3									
4	4	4								
5	5	5								
6	6	6	4							
7	7	7	5							
8	8	8	6	5						
9	9	9	6	6						
10	10	10	7	5	6					
11	11	11	8	6	6					
12	12	12	8	6	6	6				
13	13	13	9	7	6	7				
14	14	14	10	7	7	7	6			
15	15	15	10	8	6	7	7			
16	16	16	11	8	7	7	7	6		
17	17	17	12	9	7	7	7	7		
18	18	18	12	9	8	7	8	7	6	
19	19	19	13	10	8	7	8	7	7	
20	20	20	14	10	8	8	8	8	7	6
21	21	21	14	11	9	7	8	8	7	7
22	22	22	15	11	9	8	8	8	8	7
23	23	23	16	12	10	8	8	8	8	7
24	24	24	16	12	10	8	8	8	8	8
25	25	25	17	13	10	9	8	9	8	8
26	26	26	18	13	11	9	8	9	8	8
27	27	27	18	14	11	9	9	9	9	8
28	28	28	19	14	12	10	8	9	9	8
29	29	29	20	15	12	10	9	9	9	9
30	30	30	20	15	12	10	9	9	9	9
31	31	31	21	16	13	11	9	9	9	9
32	32	32	22	16	13	11	10	9	9	9
33	33	33	22	17	14	11	10	9	10	9
34	34	34	23	17	14	12	10	9	10	10
35	35	35	24	18	14	12	10	10	10	10
36	36	36	24	18	15	12	11	9	10	10
37	37	37	25	19	15	13	11	10	10	10
38	38	38	26	19	16	13	11	10	10	10
39	39	39	26	20	16	13	12	10	10	10
40	40	40	27	20	16	14	12	10	10	10
41	41	41	28	21	17	14	12	11	10	10

Table 1: Values of $R(n,k)$ for $2 \leq n \leq 41$ and $k \leq 10$.

included here for reasons of space. Many of them can be found in the appendix at http://adt.curtin.edu.au/theses/available/adt-WCU20020429.124825/. The author should be contacted directly for other desired constructions.

In the first approach, CSSs were constructed which achieve the lower bound in [6] or which achieve bounds derived in this paper. These were for: $k = 7$ and $n = 14$; $k = 8$ and $n = 16, 23, 24$; $k = 9$ and $n = 18, 20, 21, 26, 30, 31, 32$; $k = 10$ and $n = 20, 21, 22, 23, 38, 39, 40, 41$.

The next approach involved computer-based searches. The searches were feasible because of FLAT. They were carried out by B. McKay [5] or C. Ramsay [8] effectively relying upon the following. If a flat antichain of size n and volume kR does not exist on $[R]$, then there is no antichain of size n and volume kR on $[R]$ by FLAT. Hence $R(n,k) > R$. Further, the antichain must have the correct configuration to be the dual of an (n,k)CSS in R blocks and this is another factor that may increase the lower bound on $R(n,k)$ given in [6]. If the dual of an antichain happens to have the same size as the lower bound on $R(n,k)$ for a particular value of n and k then $R(n,k)$ has been determined. The derived bounds were:

$k = 7$: $R(n, 7) \geq 7$ for $n = 15, 16$.
$k = 8$: $R(n, 8) \geq 7$ for $n = 17, 18, 19$ and $R(n, 8) \geq 8$ for $n = 21, 22$.
$k = 9$: $R(n, 9) \geq 7$ for $n = 19$; $R(n, 9) \geq 8$ for $n = 22, 23, 24, 25$ and $R(29, 9) \geq 9$.
$k = 10$: $R(n, 10) \geq 8$ for $n = 26, 27, 28$; $R(n, 10) \geq 9$ for $n = 29, 32, 33$ and $R(n, 10) \geq 10$ for $n = 34, 37$.

Given the values obtained by the first two approaches, Lemma 1.1 can be applied to determine $R(n, k)$ for: $k = 8$ and $n = 15$; $k = 9$ and $n = 16, 17$; $k = 10$ and $n = 18, 19$.

The remainder of this paper is devoted to structural properties of minimal (n, k)CSSs (Section 2) which are then applied to determine the remaining unknown values of $R(n, k)$ for $7 \leq k \leq 10$ (Section 3). The derivations in each case are from the combinatorial design perspective. There is no apparent gain in considering these from the antichain perspective. Some subsequent work in progress by the author relies upon the structures derived here.

The remaining unknown values of $R(n, k)$ for $k \leq 10$ are: $k = 7$ for $n = 17, 18$; $k = 8$ for $n = 20, 25$; $k = 9$ for $n = 27, 28, 33$; $k = 10$ for $n = 17, 24, 25, 30, 31, 35, 36$.

2 Structural Results

For $k > 1$, no singleton block can occur in any (n, k)CSS. This means that every element of $[n]$ must occur in at least 2 blocks in any minimal (n, k)CSS and hence $V(\mathcal{C}) \geq 2n$ and then $kR(n, k) \geq 2n$. A p-**element** in an (n, k)CSS $\mathcal{C}$ is an element of $[n]$ which occurs in p blocks of $\mathcal{C}$. The notation $V(X)$ is used in this section to denote the number of terms in an array X.

In what follows the blocks of a CSS are represented as rows in an array. In some of these representations extra spaces may be included to highlight some underlying structure in the CSS represented. It is convenient to represent the elements of a completely separating system on an n-block by elements of $[n]$ for the 2-elements and by alphabetic characters for the 3-elements.

Suppose that $\mathcal{C}$ is a minimal (n, k)CSS. Then the **excess** of $\mathcal{C}$ is $E = kR - 2n$. Note that when $kR \leq 3n$, E is the maximum number of elements of $[n]$ that can occur in more than two blocks in $\mathcal{C}$. Define $t = t(n, k) = \max\{0, \lceil 2(3n - kR)/R \rceil\} = \lceil 2(n - E)/R \rceil$ when $t > 0$. For the cases considered here t represents the minimum number of 2-elements that must occur in at least one block in $\mathcal{C}$. This is stated in Lemma 2.1. Let $d = n - E$.

Note 2.1. 1. d is the minimum number of 2-elements that must occur in a minimal (n, k)CSS with $kR \leq 3n$.
2. A CSS $\mathcal{C}$ which achieves $R(n, k)$ has at least $2d/R$ 2-elements in one of its blocks. Note that this provides an alternate definition of t as $t = \lceil \frac{2d}{R} \rceil$.

A common approach to proving the results included here is to determine d for a minimal $(n, k)CSS$ $\mathcal{C}$, yielding bounds on the number of 2-elements and 3-elements in $\mathcal{C}$. This imposes constraints on the structure of a CSS, which then either improves the lower bound on $R(n,k)$ or guides in the construction of a CSS with a given value of R.

2.1 Basic constraints

Henceforth it is assumed that $k \geq 7$.

Lemma 2.1. *Let $\mathcal{C}$ be a minimal $(n,k)CSS$ with $R \leq k$. Then there is a block in $\mathcal{C}$ which contains at least $t(n,k)$ 2-elements.*

Proof. There are at least $3n - kR$ distinct 2-elements in $\mathcal{C}$, which therefore occupy $2(3n-kR)$ places in the R blocks of $\mathcal{C}$. Thus some block contains at least $\lceil \frac{2(3n-kR)}{R} \rceil = t(n,k)$ 2-elements. □

Theorem 2.1. *Let $\mathcal{C}$ be a $(n,k)CSS$ with $|\mathcal{C}| \leq k$. Then each block in $\mathcal{C}$ contains at most $|\mathcal{C}| - 5$ 2-elements.*

Proof. Suppose that $\mathcal{A} = \{A_1, \ldots, A_{|\mathcal{C}|}\}$ is a $(n,k)CSS$ with $|\mathcal{C}| \leq k$ in which one block, say A_1, contains $s \geq |\mathcal{C}| - 4$ 2-elements, say $1, \ldots, s$. A_1 will contain $k - s \leq 4$ other elements, say $x_1, \ldots, x_{k-s}$. The elements $1, \ldots, s$ must occur in s distinct blocks other than A_1 to be completely separated from one another, and none of the x_i can occur in these blocks. Thus the $k - s \leq 4$ elements $x_1, \ldots, x_{k-s}$ need to be completely separated by the $|\mathcal{C}| - s - 1 \leq k - s - 1$ blocks $A_{s+2}, \ldots, A_{|\mathcal{C}|}$. This is not possible by Lemma 1.1. □

Corollary 2.1. *If $\mathcal{C}$ is a minimal $(n,k)CSS$ with $R(n,k) \leq k$, then $R(n,k) - t(n,k) \geq 5$. Moreover, equality only holds if the exact number of 2-elements is exactly $t(n,k)$.*

Proof. Immediate from the theorem and Lemma 2.1. □

Corollary 2.1 asserts that if $\mathcal{C}$ is minimal then the number of 2-elements in each block of $\mathcal{C}$ is at most $R - 5$. Corollary 2.1 is often useful when trying to construct a minimal (n,k)CSS, or to show that a given lower bound on $R(n,k)$ cannot be the actual value of $R(n,k)$. The following corollary provides another useful lower bound on R.

Corollary 2.2.

$$R(n,k) \geq \lceil \frac{5 - 2k + \sqrt{(2k-5)^2 + 24n}}{2} \rceil. \tag{1}$$

Proof. Corollary 2.1 can be applied to obtain

$$R \geq 5 + 2\frac{3n - kR}{R}.$$

This easily leads to (1). □

2.2 $R(n,k)$ for $t = k - 5$ or $t = k - 6$

Consider $(n,k)CSS$s for $2n < kR \leq 3n$, with $R \leq k$ and $t = R - 5$. This case is common enough to specify the following general structure.

Theorem 2.2. *Assume $\mathcal{C}$ is a $(n,k)CSS$ which achieves $R = R(n,k)$, $2n < kR \leq 3n$, with $R = k$ and with $t = R-5$. Then, without loss of generality, $\mathcal{C}$ has the array form M shown below.*

$$\begin{array}{cccccccccc}
1 & 2 & \dots & \dots & R-5 & a & b & c & d & e \\
1 & & & & & & & & & \\
2 & & & & W & & & X & & \\
\vdots & & & & & & & & & \\
R-5 & & & & & & & & & \\
a & b & c & & & & & & & \\
a & d & e & & Y & & & Z & & \\
b & d & & & & & & & & \\
c & e & & & & & & & &
\end{array}$$

Here, excluding the elements $1, \dots, R-5, a, \dots, e$ placed as shown above,
W is the subarray of 2-elements in rows 2 to $R-4$ of M;
X is the subarray of 3-elements in rows 2 to $R-4$ of M;
Y is the subarray of 2-elements in rows $R-3$ to R of M;
Z is the subarray of 3-elements in rows $R-3$ to R of M;
and no row of M contains more than $R-5$ 2-elements.

Proof. By Lemma 2.1, Theorem 2.1 and Corollary 2.1 it can be asserted that:
(i) t is the greatest number of 2-elements appearing in any block in $\mathcal{C}$
(ii) $\mathcal{C}$ must contain a block A with exactly t 2-elements
(iii) There are t other blocks each containing one of the 2-elements of A. There are $k - t = 5$ elements in A other than the 2-elements, say a, b, c, d, e, and these cannot appear with the 2-elements of A elsewhere in $\mathcal{C}$. As $R = k$ they must be completely separated in the remaining $R - t - 1 = 4$ blocks. By Lemma 1.1 the elements a, b, c, d, e can be uniquely separated in four blocks as shown in M. Hence the structure of M is determined as shown. ☐

Note 2.2. 1. Swapping rows R and $R-1$ does not change the underlying structure of the array. The same can be said for rows $R-2$ and $R-3$.
2. In Theorem 2.2, each row of W contains at most $R-6$ 2-elements else there are more than t 2-elements in a row of M. Similarly each row of Y contains at most $k-5$ 2-elements.
3. Whenever the subarray

$$\begin{array}{ccc}
a & b & c \\
a & d & e \\
b & d & \\
c & e &
\end{array}$$

occurs in a CSS then it is the unique way to completely separate 5 elements in 4 sets. The same 2-element may occur in each of the last two rows, but no 2-element

may occur in each of any other pair of rows of the subarray to which this subarray belongs.
4. Whenever the subarray

$$\begin{array}{ccc} a & b & c \\ a & d & e \\ b & d & f \\ c & e & f \end{array}$$

occurs in a CSS then it is the unique way to completely separate 6 elements in 4 sets. No 2-element can occur twice in the same rows of the subarray to which this subarray belongs.

Corollary 2.3. *Assume $\mathcal{C}$ is a minimal $(n,k)CSS$ with $R = k$ and $t = R - 5$. Assume M, W, Y are as defined in Theorem 2.2.*
Then
(i) at most one 2-element can occur twice in Y.
(ii) $V(W) \le (R-5)(R-6)$.
(iii) $V(W) \ge d - t - 1 = d - R + 4 = 3n - R^2 - R + 4$.
(iv) $R \ge \frac{5+\sqrt{6n-27}}{2}$.

Proof. (i) If a 2-element occurs twice in Y then to have it completely separated from each element of $\{a,b,c,d,e\}$, it must occur in rows $R-1$ and R of M.
Hence there is only one 2-element that can occur twice in Y.
(ii) W contains $R-5$ rows and cannot contain more than $R-6$ 2-elements in each row by Note 2.2.
(iii) By Note 2.1 there are at least d 2-elements in $\mathcal{C}$. Thus there are at least $d-(R-5)$ 2-elements in $\mathcal{C}$ other than $1,\ldots,R-5$. At most one 2-element occurs only in Y so $d-(R-5)-1$ occur at least once in W. Hence the result.
(iv) This follows from (ii) and (iii). □

Theorem 2.3. *Assume $\mathcal{C}$ is a minimal $(n,k)CSS$ with $R = k-1$ and $t = R-5$. Then $\mathcal{C}$ has the form of the array M shown below.*

$$\begin{array}{ccccccccccc} 1 & 2 & \ldots & \ldots & R-5 & a & b & c & d & e & f \\ 1 & & & & & & & & & & \\ 2 & & & W & & & & X & & & \\ \vdots & & & & & & & & & & \\ R-5 & & & & & & & & & & \\ a & b & c & Y & & & & Z & & & \\ a & d & e & & & & & & & & \\ b & d & f & & & & & & & & \\ c & e & f & & & & & & & & \end{array}$$

Here W, X, Y, Z have the same meaning as in Theorem 2.2.

Proof. Lemma 1.1 asserts that there is a unique way of completely separating 6 elements in 4 blocks. This unique way is as shown for the elements a,b,c,d,e,f in the last four rows of M. The rest of the proof mimics the proof of Theorem 2.2. □

Note 2.3. 1. In Theorem 2.3 a permutation of the last four rows is an isomorphic CSS.
2. In Theorem 2.3, each row of W contains at most $R-6$ 2-elements and each row of Y contains at most $R-5$ 2-elements.

Corollary 2.4. *Assume $\mathcal{C}$ is a minimal $(n,k)CSS$ with $R=k-1$ and $t=R-5$. Then*
(i) No 2-element occurs twice in Y.
(ii) $V(W) \le (R-5)(R-6) = (k-6)(k-7)$.
(iii) $V(W) \ge d-t = d-R+5 = 3n-kR-R+5$.
(iv) W contains at least $d-5R+25$ distinct 2-elements which occur in W only.
(v) $R \ge \frac{9+\sqrt{24n-119}}{4}$.

Proof. The proof follows the same argument as in the proof of Corollary 2.3 but applied to Theorem 2.3. ☐

3 $R(n,k)$ for $k = 7, 8, 9, 10$

The remaining unknown values of $R(n,k)$ for $k \le 10$ are determined in this section. It should be noted that in each proof which relies upon Theorems 2.2 or 2.3, the collection of 2-elements in W must be distinct due to the limitations imposed by the constraints expressed in Note 2.2.

Lemma 3.1.
(i) $R(17,7) = 7$.
(ii) $R(18,7) = 8$.

Proof. For $n = 17, 18$ examples of $(n,7)$CSSs containing the appropriate numbers of blocks have been constructed. It is necessary to show that smaller $(n,7)$CSSs do not exist. In [6] it is shown that $R(17,7) \ge 6$ and $R(18,7) \ge 7$.
(i) This follows from Corollary 2.2 which implies that $R(17,7) \ge 7$.
(ii) If $R(18,7) = 7$, then $d = 5, t = 2$ and Theorem 2.2 applies with $V(W) = 2$ and $V(Y) \ge 4$. Thus the structure must be as follows.

1	2	a	b	c	d	e	
1	3						X
2	4						
a	b	c					
a	d	e					Z
b	d	5					
c	e	5					

The blocks of 3-elements in Z in rows 6 and 7 must be disjoint because of the 2-element 5 occurring in those rows.
If 3 occurs in row 4 or 5, say row 4, then the five 3-elements of row 2 must be completely separated in rows 3, 5, 6 and 7. By Lemma 1.1 there is a unique way

of completely separating 5 elements in 4 blocks and by Note 2.2 this allows only one other 2-element to occur twice in the same rows, if complete separation is to be maintained. Hence, if 4 does not also occur in row 4, then either 4 or 5 will not be completely separated from one of these 3-elements.

If 4 does occur in row 4, then the two 3-elements of row 4 in Z, say f and g, must occur in rows 5 to 7 in the configuration fg, f, g respectively so that the element 3,4 and 5 are not dominated. This leaves six 3-elements of which five occur in row 2. These need to be completely separated in rows 3, 5, 6 and 7. This cannot be done as at least four of these are needed to fill row 3, and this contradicts Note 2.2.3.

If 3 occurs in row 6 or 7, say row 7, then the three 3-elements of Z in row 7, say f, g and h, must be completely separated in rows 3 to 5 as pairs, say fg, fh, gh respectively. As $t = 2$ this forces 4 to be in row 6. This leaves the five 3-elements $i, \ldots, m$ to fill row 2 and these have to be completely separated in rows 3 to 6. An application of Note 2.2 shows that 4 will be dominated by one of these 3-elements. □

Lemma 3.2.
(i) $R(20, 8) = 8$.
(ii) $R(25, 8) = 9$.

Proof. For $n = 20, 25$, examples of $(n, 8)$CSSs containing the appropriate numbers of blocks have been constructed. In [6] it is shown that $R(20, 8) \geq 7$ and $R(25, 8) \geq 8$.

(i) Assume that $\mathcal{C}$ is a $(20, 8)$CSS in 7 blocks. Then $d = 4$ and $t = 2$ so any block in $\mathcal{C}$ contains at most two 2-elements. Then the structure of Theorem 2.3 must occur as shown.

$$
\begin{array}{cccccccc}
1 & 2 & a & b & c & d & e & f \\
1 & & & & W & & X & \\
2 & & & & & & & \\
a & b & c & & & & & \\
a & d & e & & & & & \\
b & d & f & & Y & & Z & \\
c & e & f & & & & &
\end{array}
$$

If $n = 20$ then, by Note 2.2.4, W must contain the 2-elements 3 and 4, say, with 3 in the first row and 4 in the second row of W. It may also be assumed that the first row of X consists of the 3-elements g,h,i,j,k,l and that 3 and 4 occur once each in Y as $t = 2$.

By Note 2.2.4, if 3 and 4 are not in the same row of Y then g, h, i, j, k, l cannot be completely separated without one of them dominating at least one of 3 or 4.

If 3 and 4 are in the same row of Y, say row 4, then g, h, i, j, k, l must be completely separated in rows 3, 5, 6 and 7 , by Note 2.2. The remaining three 3-elements of row 3 must be chosen from $\{m, n, o, p\}$ and these 3-elements cannot occur in row 4. This means that there are insufficient elements left to fill row 4 without 4 being dominated by some 3-element.

(ii) In [6] it was shown that $R(25, 8) \geq 8$. If $R(25, 8) = 8$ then there is a contradiction as $V(W) \leq 6$ by part (ii) of Corollary 2.3 and $V(W) \geq 7$ by part (iii). Thus $R(25, 8) > 8$. □

Lemma 3.3.
(i) $R(27,9) = 9$.
(ii) $R(28,9) = 9$.
(iii) $R(33,9) = 10$.

Proof. For $n = 27, 28, 33$ examples of $(n,9)$CSSs containing the appropriate numbers of blocks have been constructed. In [6] it is shown that $R(n,9) \geq 8$ for $n = 27$ or 28 and $R(33,9) \geq 9$.

(i) Assume that $\mathcal{C}$ is a $(27,9)$CSS in 8 blocks. Then $d = 9$ and $t = 3$. $\mathcal{C}$ has the structure of Theorem 2.3 with some positions filled as shown.

1	2	3	a	b	c	d	e	f
1	4	5	g	h	i	j	k	l
2	6	7						
3	8	9						
a	b	c	4					
a	d	e	5					
b	d	f				Z		
c	e	f						

The 3-elements of row 2, g, h, i, j, k, l must be completely separated in rows 3,4,7 and 8 in the unique way shown in Note 2.2. Consequently none of the 2-elements 6,7,8,9 can occur in rows 7 or 8. Thus row 5 must contain the elements 6 and 8 and row 6 must contain the elements 7 and 9. Then the 3-elements of row 3 or row 4 must occur only in rows 3,4,7 and 8. It is not hard to check that there must be 4 distinct 3-elements m, n, o, p which fill the remaining places in the rows 3,4,7, and 8. This leaves the 3-elements q, r which cannot be completely separated in rows 5 and 6 only. Thus $R(27,9) > 8$.
(ii) $R(28,9) > 8$ by Corollary 2.4.
(iii) In [6] it is shown that $R(33,9) \geq 9$. $R(33,9) > 9$ by Corollary 2.3. □

Lemma 3.4.
(i) $R(17,10) = 7$.
(ii) $R(n,10) = 8$ *for* $n = 24, 25$.
(iii) $R(n,10) = 9$ *for* $n = 30, 31$.
(iv) $R(n,10) = 10$ *for* $n = 35, 36$.

Proof. (i) $R(17,10) = R(17,7) = 7$ by Lemma 3.1 and Lemma 1.1.

For $n = 24, 25, 30, 31, 35, 36$ examples of (n,k)CSSs containing the appropriate numbers of blocks have been constructed.
(ii) Let $n = 24$ or 25. In [6] it is shown that $R(n,10) \geq 7$. Assume $R(n,10) = 7$. Assume $\mathcal{C}$ is a $(24,10)$CSS in 7 blocks. Then $d = 2$ and $t = 1$. If a block in $\mathcal{C}$, say A, contains more than one 2-element, then each of these 2-elements must occur in different blocks of $\mathcal{C}$ other than A. Then the remaining eight elements of A cannot be completely separated in the four blocks of $\mathcal{C}$ which do not contain these 2-elements of A by Lemma 1.1. Therefore, it can be assumed that $\mathcal{C}$ has at most one 2-element

in each block and thus $\mathcal{C}$ has the following partial form.

1	a	b	c	d	e	f	g	h	i
1	j	k	l	m	n	o	p	q	r
2	.	.	.						
2	.	.	.						
.	.	.							
.	.	.							
.	.	.							

Assume four or more elements of $\{a,\ldots,i\}$ or 4 or more elements of $\{j,\ldots,r\}$ occur in row 3 only or row 4 only. Then these 3-elements must be completely separated in the last 3 rows of $\mathcal{C}$. This is impossible by Lemma 1.1.

Therefore each of rows 3 and 4 contain at most 6 elements of $\{a,\ldots,r\}$. Therefore, at least three elements of $\{s,t,u,v\}$ occur in each of row 3 and 4 to fill those rows. Then at least one of these elements, say s, occurs in both rows 3 and 4. This means that the element 2 is not completely separated from s. This completes the proof for $n=24$.

Assume that $\mathcal{C}$ is a $(25,10)$CSS in 7 blocks. Then $d=5$ and $t=2$. Therefore a block in $\mathcal{C}$, say A, contains at least two 2-elements which occur again in distinct blocks in $\mathcal{C}$. By Lemma 1.1 it is impossible to completely separate the remaining eight elements of A in the four blocks which do not contain these 2-elements of A. This completes the proof of this part.

(iii) Let $n=30$ or 31. In [6] it is shown that $R(n,10)\geq 8$. Assume $\mathcal{C}$ is a $(n,10)$ CSS in 8 blocks. Then $d\geq 10$ and $t\geq 3$. For each value of n, each block $A\in\mathcal{C}$ can contain at most two 2-elements by an application of Lemma 1.1 to the remaining seven elements of A. This contradicts $t\geq 3$ so $R(n,10)>8$.

(iv) In [6] it is shown that $R(35,10)\geq 9$. Assume $\mathcal{C}$ is a $(35,10)$CSS in 9 blocks. Then $d=15$ and $t=4$. Thus $\mathcal{C}$ must have the form of Theorem 2.3 with the meaning of W as defined in that theorem. This is expressed in the array representation of $\mathcal{C}$ shown below in the form of row 1 and the reoccurrence of the elements of row 1 in other rows. Other constraints can be imposed on $\mathcal{C}$. These are shown in the same array and are explained below in order.

1	2	3	4	a	b	c	d	e	f		
1	5	6	7		g	h	i	j	k	l	
2	8	9	10		g	h	i		m	n	o
3	11	12	13		g	j	k		m	n	p
4	14	15		h	j	l		m	o		p
a	b	c		5	8	11	14				
a	d	e		6	9	12	15				
b	d	f		7	10	13					
c	e	f		i	k	l		n	o		p

1. $11\leq V(W)\leq 12$ by Theorem 2.3 and Note 2.2.

2. Hence the 3-elements $g,\ldots,l$ must occur as shown in row 2. The 2-elements 5,6,7 in row 2 must reoccur in three other rows of the array with at least two of these in rows 6-9. Each of the 2-elements $8,\ldots,15$ must reoccur in rows 6-9. Then the elements $g,\ldots,l$ must be completely separated in the other four rows excluding

row 1. By Lemma 1.1 and Note 2.2 this can be done in a unique way in four rows with 3 elements in each row. If a 2-element of row 2, say 5, reoccurs in row 5 then the elements $g \ldots, l$ cannot be completely separated without dominating one fo the elements $8, \ldots, 15$. Therefore the occurrences of the elements $5, \ldots, 15$ and $g, \ldots, l$ must occur exactly as shown in the array.
3. The four 3-elements m, n, o, p must occur as shown. For $V(W) = 11$ or 12 this means that at least three of the remaining four or three p-elements respectively, $p \geq 3$, occur in rows 6-9 only. Then not all of them can be completely separated from one another. Therefore $R(35, 10) > 9$.

In [6] it is shown that $R(36, 10) \geq 9$. $R(36, 10) > 9$ by Corollary 2.4(v). □

4 Final Comments

Table 1 and the results in [6] provides a complete set of values of $R(n, k)$ for $k \leq 10$. The table only shows values of $R(n, k)$ for $k \leq n/2$. The missing values can be determined by applying Lemma 1.1. The author would like to thank P. Lieby, B. McKay, L. Rylands, J. Simpson and the referee for their contributions to this paper. This paper was partially supported by a CDU Project Grant.

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