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# Product constructions for large sets of resolvable MTSs and DTSs

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**Abstract**

An $LRMTS(v)$ (respectively, $LRDTS(v)$) is a large set consisting of $v-2$ (respectively, $3(v-2)$) disjoint resolvable Mendelsohn (respectively, directed) triple systems of order $v$. We have presented the tripling constructions for LRMTSs (see Chang, *Discr. Math.*, to appear) and LRDTSs (see Zhou and Chang, *Acta Mathematica Sinica*, to appear), with a newly defined structure TRIQ or DTRIQ being used. Lei (*Discr. Math.* 257 (2002), 63–81) introduced a concept called LR-design in order to obtain the product construction for large sets of Kirkman triple systems (KTSs). In this paper, we utilize both TRIQ (or DTRIQ) and LR-design to present the product constructions for LRMTSs and LRDTSs, which generalize the tripling constructions mentioned above. Applying the product constructions with the known LRMTSs, LRDTSs, TRIQs (or DTRIQs) and LR-designs, we obtain the existence of an $LRMTS(v)$ and an $LRDTS(v)$ for $v = 3^n m(2\cdot k_1^{n_1}+1)(2\cdot k_2^{n_2}+1)\cdots(2\cdot k_t^{n_t}+1)$ where $n\geq 1$, $t\geq 0$, $n_i\geq 1$, $k_i\in\{7,\ 13\}$ $(i=1,2,\cdots,t)$ and $m\in\{1,\ 4,\ 5,\ 7,\ 11,\ 13,\ 17,\ 23,\ 25,\ 35,\ 37,\ 41,\ 43,\ 47,\ 53,\ 55,\ 57,\ 61,\ 65,\ 67,\ 91,\ 123\}\cup\{(7^k+2)/3,\ (13^k+2)/3,\ (25^k+2)/3,\ 2^{2k+1}25^j+1:\ k\geq 0 \text{ and } j\geq 0\}$.

## 1 Introduction

Let $X$ be a finite set. In what follows an *ordered pair* of $X$ is always an ordered pair $(x,y)$ where $x\neq y\in X$. A *cyclic triple* on $X$ is a set of three ordered pairs $(x,y)$, $(y,z)$ and $(z,x)$ of $X$, which is denoted by $\langle x,y,z\rangle$ (or $\langle y,z,x\rangle$, or $\langle z,x,y\rangle$). A *transitive triple* on $X$ is a set of three ordered pairs $(x,y)$, $(y,z)$ and $(x,z)$ of $X$, which is denoted by $(x,y,z)$.

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An *oriented triple system* of order $v$ is a pair $(X, \mathcal{B})$ where $X$ is a $v$-set and $\mathcal{B}$ is a collection of cyclic or transitive triples on $X$, called *blocks*, such that every ordered pair of $X$ belongs to exactly one block of $\mathcal{B}$. In particular, if the triples in $\mathcal{B}$ are all cyclic (respectively, transitive), then $(X, \mathcal{B})$ is called a *Mendelsohn* (respectively, *directed*) *triple system* and denoted by $MTS(v)$ (respectively, $DTS(v)$).

An $MTS(v)$ (or $DTS(v)$) $(X, \mathcal{B})$ is called *resolvable* if its block set $\mathcal{B}$ can be partitioned into subsets (called *parallel class*es), each containing every element of $X$ exactly once. A resolvable $MTS(v)$ (respectively, $DTS(v)$), denoted by $RMTS(v)$ (respectively, $RDTS(v)$), is easily seen to contain $v-1$ parallel classes.

A *large set* of $MTS(v)$ (respectively, $DTS(v)$), denoted by $LMTS(v)$ (respectively, $LDTS(v)$), is a collection $\{(X, \mathcal{B}_i)\}$, where every $(X, \mathcal{B}_i)$ is an $MTS(v)$ (respectively, $DTS(v)$) and all $\mathcal{B}_i$'s form a partition of all cyclic (respectively, transitive) triples on $X$. It is easy to see that an $LMTS(v)$ consists of $v-2$ disjoint $MTS(v)$s and an $LDTS(v)$ consists of $3(v-2)$ disjoint $DTS(v)$s. An $LRMTS(v)$ (respectively, $LRDTS(v)$) denotes an $LMTS(v)$ (respectively, $LDTS(v)$) in which each $MTS(v)$ (respectively, $DTS(v)$) is resolvable.

We summarize the known existence results on LRMTSs and LRDTSs as follows.

**Theorem 1.1** ([11, 4, 6, 1, 12]) *There exist an $LRMTS(v)$ and an $LRDTS(v)$ for $v = 69$, 123, 141, 159, $7^k+2$, $13^k+2$, $25^k+2$, $3^n m$, where $k \geq 0$, $n \geq 1$, $m \in \{1, 4, 5, 7, 11, 13, 17, 25, 35, 37, 43, 55, 57, 61, 65, 67, 91, 123\} \cup \{2^{2r+1}25^s + 1 : r \geq 0, s \geq 0\}$.*

The orders $4 \cdot 3^n$ for $n \geq 1$ in Theorem 1.1 come from the tripling constructions given in [1, 12] as follows.

**Theorem 1.2** (Tripling Constructions, [1, 12]) *If there exist both an $LRMTS(v)$ (respectively, $LRDTS(v)$) and a $TRIQ(v)$ (respectively, $DTRIQ(v)$), then there exists an $LRMTS(3v)$ (respectively, $LRDTS(3v)$).*

In the following product constructions, we will also use the structures TRIQ and DTRIQ. So, we recall their definitions and existence results.

A *quasigroup* of order $v$ is a pair $(X, \circ)$, where $X$ is a $v$-set and $\circ$ is a binary operation on $X$ such that equations $a \circ x = b$ and $y \circ a = b$ are uniquely solvable for every pair of elements $a, b$ in $X$. A quasigroup $(X, \circ)$ is called *idempotent* if the identity $x \circ x = x$ holds for all $x$ in $X$. An idempotent quasigroup of order $v$ is denoted by $IQ(v)$. Furthermore, an idempotent quasigroup $(X, \circ)$ is called *resolvable* if all $v(v-1)$ pairs of distinct elements of $X$ can be partitioned into subsets $T_i$, $1 \leq i \leq 3(v-1)$, such that every $\Gamma_i = \{(x, y, x \circ y) : (x, y) \in T_i\}$ (called *parallel class*) is a partition of $X$. A resolvable idempotent quasigroup of order $v$ is denoted by $RIQ(v)$.

An $IQ(v)$ is called *first-transitive*, if there exists a group $G$ of order $v$ acting transitively on $X$ which forms an automorphism group of $(X, \circ)$. A first-transitive $RIQ(v)$ is briefly denoted by $TRIQ(v)$.

Take any fixed ordered pair $(i,j)$ $(i \neq j)$. For an $IQ$ $(X,\circ)$ and the given ordered pair $(i,j)$, define a set $T^X(i,j)$ of transitive triples of $X \times \{i,j\}$ as follows: for each ordered pair $(x,y)$, $x \neq y \in X$, let $t(x,y,x\circ y)$ be the three transitive triples of $X \times \{i,j\}$ defined by

$$t(x,y,x\circ y) = \{((x,i),(y,i),(x\circ y,j)),((x,i),(x\circ y,j),(y,i)),((x\circ y,j),(x,i),(y,i))\}. \tag{1.1}$$

Set

$$T^X(i,j) = \bigcup_{x\neq y\in X} t(x,y,x\circ y). \tag{1.2}$$

The $IQ$ $(X,\circ)$ is called *second-transitive* provided that $T^X(i,j)$ can be partitioned into three sets $T_0^X(i,j)$, $T_1^X(i,j)$ and $T_2^X(i,j)$ such that

(a) the three transitive triples in $t(x,y,x\circ y)$ belong to different $T_k^X(i,j)$s $(k=0,$ $1,\ 2)$;

(b) if $a \neq b \in X$, each of the ordered pairs $((a,i),(b,j))$ and $((b,j),(a,i))$ belongs to exactly one transitive triple in each of $T_0^X(i,j)$, $T_1^X(i,j)$ and $T_2^X(i,j)$.

It is worth noting that we make a slight modification to the definition of second-transitivity in [12] (where the values of $i$ and $j$ are restricted to $\{0,1,2\}$). It is obvious that the two definitions are equivalent because the second-transitivity does not depend on the choice of the ordered pair $(i,j)$.

An $IQ(v)$ $(X,\circ)$ with both first- and second-transitivity is called *doubly transitive*. A doubly transitive $RIQ(v)$ is denoted by $DTRIQ(v)$.

The existence of $TRIQ(v)$ and $DTRIQ(v)$ is known as follows.

**Theorem 1.3** ([1, 12]) *A $TRIQ(v)$ exists if and only if $v$ is a positive integer such that $3|v$ and $v \not\equiv 2 \pmod 4$; A $DTRIQ(v)$ exists if and only if $v$ is a positive integer such that $3|v$ and $v \not\equiv 2 \pmod 4$.*

Another important concept is LR-design, which was introduced by Lei in [7].

Let $X$ be a $v$-set. An LR-design of order $v$ (briefly $LR(v)$) is a collection $\{(X,\mathcal{A}_k^j): 1 \leq k \leq \frac{v-1}{2}, j=0,1\}$ of $v-1$ $KTS(v)$s with following properties:

(i) Let the resolution of $\mathcal{A}_k^j$ be $\Gamma_k^j = \{A_k^j(h) : 1 \leq h \leq \frac{v-1}{2}\}$. There is an element in each $\Gamma_k^j$, say, $A_k^j(1)$, such that

$$\bigcup_{k=1}^{\frac{v-1}{2}} A_k^0(1) = \bigcup_{k=1}^{\frac{v-1}{2}} A_k^1(1) = \mathcal{A}$$

and $(X,\mathcal{A})$ is a $KTS(v)$.

(ii) For any triple $T=\{x,y,z\} \subseteq X$, $x \neq y \neq z \neq x$, there exist $k$, $j$ such that $T \in \mathcal{A}_k^j$.

The known existence results on LR-design are as follows.

**Theorem 1.4** ([7]) *There exists an* $LR(2 \cdot 13^n + 1)$ *for* $n \geq 0$.

**Theorem 1.5** ([3]) *There exists an* $LR(2 \cdot 7^n + 1)$ *for* $n \geq 0$.

In Section 2, we will present the product constructions for LRMTSs and LRDTSs. In Section 3, we apply the constructions to update the existence results on LRMTSs and LRDTSs.

## 2 Product constructions for LRMTSs and LRDTSs

In the following constructions, we need the concept of complete mapping in a finite group.

A *complete mapping* of a group $(G, \cdot)$ is a bijection mapping $x \to \theta(x)$ of $G$ upon $G$ such that the mapping $\eta(x) = x \cdot \theta(x)$ is again a bijection mapping of $G$ upon $G$.

The following existence result has been stated in [1].

**Lemma 2.1** ([1, Lemma 2.7]) *If there exists an* $IQ(v)$ $(X, \circ)$ *with a sharply transitive automorphism group* $G$, *then* $G$ *has a complete mapping.*

**Remark 2.2** *Suppose that* $X_1$ *is a* $u$*-set and* $(X_1, \circ)$ *is a* $TRIQ(u)$. *By the definition of* $TRIQ$, *we have:*

(A) *There is a sharply transitive automorphism group* $G = \{\sigma_0, \sigma_1, \cdots, \sigma_{u-1}\}$ *on* $(X_1, \circ)$. *By Lemma 2.1,* $G$ *has a complete mapping, say,* $\phi$, *and let* $\sigma^* = [\phi(\sigma)]^{-1}$ *for* $\sigma \in G$. *Then by the definition of complete mapping, we have*

$$\{\sigma(\sigma^*)^{-1} : \sigma \in G\} = G. \tag{2.3}$$

(B) *All* $u(u-1)$ *pairs of distinct elements of* $X_1$ *can be partitioned into subsets* $S_i$ $(1 \leq i \leq 3(u-1))$, *such that every* $\Gamma_i = \{(x, y, x \circ y) : (x, y) \in S_i\}$ *is a partition of* $X_1$.

*If* $(X_1, \circ)$ *is also second-transitive, i.e.,* $(X_1, \circ)$ *is a* $DTRIQ(u)$, *then another property should hold:*

(C) *For any fixed ordered pair* $(i, j)$ $(i \neq j)$, $T^{X_1}(i, j) = \bigcup_{x \neq y \in X_1} t(x, y, x \circ y)$, *where* $t(x, y, x \circ y)$ *is defined in* (1.1). *By the property of second-transitivity,* $T^{X_1}(i, j)$ *can be partitioned into 3 sets* $T_0^{X_1}(i, j)$, $T_1^{X_1}(i, j)$ *and* $T_2^{X_1}(i, j)$ *satisfying:*

  (a) *the three transitive triples in* $t(x, y, x \circ y)$ *belong to different* $T_l^{X_1}(i, j)$*s* $(l = 0, 1, 2)$;

  (b) *if* $a \neq b \in X_1$, *each of the ordered pairs* $((a, i), (b, j))$ *and* $((b, j), (a, i))$ *belongs to exactly one transitive triple in each of* $T_0^{X_1}(i, j)$, $T_1^{X_1}(i, j)$ *and* $T_2^{X_1}(i, j)$.

*Furthermore, suppose that* $X_2$ *is a* $v$*-set with a linear order "*$<$*" (i.e., for any* $x \neq y$, $x, y \in X_2$, *either* $x < y$ *or* $y < x$*). And suppose that* $\{(X_2, \mathcal{A}_k^j) : 1 \leq k \leq \frac{v-1}{2}, j = 0, 1\}$ *is an* $LR(v)$ *satisfying condition* (D)*:*

(D) (i) *Let the resolution of* $\mathcal{A}_k^j$ *be* $\Gamma_k^j = \{A_k^j(h) : 1 \leq h \leq \frac{v-1}{2}\}$*. There is an element in each* $\Gamma_k^j$*, say,* $A_k^j(1)$*, such that*

$$\bigcup_{k=1}^{\frac{v-1}{2}} A_k^0(1) = \bigcup_{k=1}^{\frac{v-1}{2}} A_k^1(1) = \mathcal{A}$$

*and* $(X_2, \mathcal{A})$ *is a* $KTS(v)$*.*

(ii) *For any triple* $T = \{x, y, z\} \subseteq X_2$, $x \neq y \neq z \neq x$*, there exist* $k$*,* $j$ *such that* $T \in \mathcal{A}_k^j$*.*

The symbols and properties in Remark 2.2 will be used in the proofs of both Theorem 2.3 and Theorem 2.4. In addition, we stipulate some notations for the use in the following proofs.

In (1.2), we give a symbol $T^X(i, j)$. For the comparing proofs of the following theorems, we introduce an analogous symbol $C^X(i, j)$ in an $IQ$ $(X, \circ)$. For a fixed ordered pair $(i, j)$, define

$$C^X(i, j) = \bigcup_{x \neq y \in X} \{\langle (x, i), (y, i), (x \circ y, j) \rangle\}.$$

Moreover, if $\pi$ is a permutation of $X$, we denote by $\pi C^X(i, j)$ (resp., $\pi T_l^X(i, j)$, $0 \leq l \leq 2$) the set of the cyclic (resp., transitive) triples in $C^X(i, j)$ (resp., $T_l^X(i, j)$, $0 \leq l \leq 2$) by replacing each occurrence of $(x, j)$ with $(\pi(x), j)$ but keeping those occurrences with the second component "$i$" unchanged, say,

$$\pi C^X(i, j) = \cup_{x \neq y \in X} \{\langle (x, i), (y, i), (\pi(x \circ y), j) \rangle\}.$$

**Theorem 2.3** *If there exist an* $LRMTS(u)$ *and a* $TRIQ(u)$*, and there exists an* $LR(v)$*, then there exists an* $LRMTS(uv)$*.*

**Proof** Suppose that $(X_1, \circ)$ is the $TRIQ(u)$ in Remark 2.2 with the properties (A) and (B). Let $\{(X_2, \mathcal{A}_k^j) : 1 \leq k \leq \frac{v-1}{2}, j = 0, 1\}$ be the $LR(v)$ satisfying condition (D). Let $\{(X_1, \mathcal{B}_j) : 1 \leq j \leq u - 2\}$ be an $LRMTS(u)$. We will construct an $LRMTS(uv)$ on the set $Y = X_1 \times X_2$. The construction proceeds in 3 steps.

**Step 1**: For any $\{a, b, c\} \subseteq X_2$ with $a < b < c$, for $\sigma_i, \sigma_j \in G$ and $x \in X_1$, define

$$B_{ijx}^{(a,b,c)} = \{\{(x, a), (\sigma_j(x), b), (\sigma_i \sigma_j^*(x), c)\}\},$$

$$P_{ij}^{(a,b,c)} = \bigcup_{x \in X_1} \{\langle u, v, w \rangle, \langle w, v, u \rangle : \{u, v, w\} \in B_{ijx}^{(a,b,c)}\},$$

and

$$\mathcal{A}_i^{(a,b,c)} = \bigcup_{\sigma_j \in G} P_{ij}^{(a,b,c)}.$$

Noting the formula (2.3), we have: (1) For $m \neq n \in \{a,b,c\}$, $x,y \in X_1$, each of the ordered pairs $((x,m),(y,n))$ and $((y,n),(x,m))$ belongs to exactly one triple of $\mathcal{A}_i^{(a,b,c)}$; (2) $\mathcal{A}_i^{(a,b,c)}$ and $\mathcal{A}_{i'}^{(a,b,c)}$ are disjoint for $i \neq i'$.

For each $a' \in X_2$, we have $u-2$ disjoint $RMTS(u)$s $(X_1 \times \{a'\},\ \mathcal{B}_j^{(a')})$ for $j = 1, 2, \cdots, u-2$, where $\mathcal{B}_j^{(a')} = \{\langle (x,a'),(y,a'),(z,a')\rangle : \langle x,y,z\rangle \in \mathcal{B}_j\}$.

For a given $j$, $1 \leq j \leq u-2$, take $\{a,b,c\} \in \mathcal{A}$ and $a<b<c$, define

$$\mathcal{C}_j = (\bigcup_{\{a,b,c\}\in\mathcal{A}} \mathcal{A}_j^{(a,b,c)}) \bigcup (\bigcup_{a' \in X_2} \mathcal{B}_j^{(a')}).$$

Then it is not difficult to check that each $(Y, \mathcal{C}_j)$ is an $MTS(uv)$ for $1 \leq j \leq u-2$.

$(Y,\mathcal{C}_j)$ is resolvable because $\mathcal{C}_j$ is the union of the $uv-1$ parallel classes in the following 2 parts.

Part I: For given $i$ and $k$, $0 \leq i \leq u-1$ and $1 \leq k \leq \frac{v-1}{2}$, $\bigcup_{\{a,b,c\}\in A_k^0(1)} P_{ij}^{(a,b,c)}$ consists of 2 parallel classes. So this part gives $u(v-1)$ parallel classes.

Part II: $\bigcup_{a'\in X_2} \mathcal{B}_j^{(a')}$ can be partitioned into $u-1$ parallel classes because of the resolvability of $\mathcal{B}_j$.

This step gives $u-2$ disjoint $RMTS(uv)$s on $Y$.

(The remaining $\mathcal{A}_0^{(a,b,c)}$ and $\mathcal{A}_{u-1}^{(a,b,c)}$ ($\{a,b,c\} \in \mathcal{A},\ a<b<c$) are saved for the use in the following two steps.)

**Step 2**: (making use of the block set $\mathcal{A}_0^{(a,b,c)}$)

For a given $\sigma_j \in G$, $j = 0,\ 1,\ \cdots,\ u-1$, define 3 permutations on $X_1$, namely $\alpha_j^{(s)}(s \in Z_3)$ as follows:

$$\alpha_j^{(0)} = \sigma_j, \qquad \alpha_j^{(1)} = \sigma_0\sigma_j^*\sigma_j^{-1}, \qquad \alpha_j^{(2)} = (\sigma_0\sigma_j^*)^{-1} = (\alpha_j^{(1)}\alpha_j^{(0)})^{-1}.$$

For given $k$ and $j$, $1 \leq k \leq \frac{v-1}{2}$ and $0 \leq j \leq u-1$, take $\{a,b,c\} \in A_k^0(1)$, $a<b<c$. Define

$$\mathcal{C}_{0j}^{(a,b,c)} = \alpha_j^{(0)} C^{X_1}(a,b) \cup \alpha_j^{(1)} C^{X_1}(b,c) \cup \alpha_j^{(2)} C^{X_1}(c,a),$$

and

$$\mathcal{D}_{kj}^{(0)} = \Big[\bigcup_{\{a,b,c\}\in A_k^0(1)} (P_{0j}^{(a,b,c)} \cup \mathcal{C}_{0j}^{(a,b,c)})\Big] \bigcup \Big[\bigcup_{\{a,b,c\}\in \mathcal{A}_k^0 \setminus A_k^0(1)} \mathcal{A}_j^{(a,b,c)}\Big].$$

Then it can be checked that each $(Y, \mathcal{D}_{kj}^{(0)})$ is an $RMTS(uv)$ for $1 \leq k \leq \frac{v-1}{2}$ and $0 \leq j \leq u-1$. Now we explain its parallel classes in 2 parts:

Part I: $\bigcup_{\{a,b,c\}\in A_k^0(1)} P_{0j}^{(a,b,c)}$ consists of two parallel classes.

By the property (B) in Remark 2.2, for a given $i$, $1 \leq i \leq 3(u-1)$, $\Gamma_i = \{(x,y,x\circ y) :\ (x,y) \in S_i\}$ is a partition of $X_1$. Define $\pi(S_i) = \{(\pi(x),\pi(y)) : (x,y) \in S_i\}$ for some $\pi \in G$. Since $\alpha_j^{(s)} \in G$ $(s \in Z_3)$, $\alpha_j^{(2)} = (\alpha_j^{(1)}\alpha_j^{(0)})^{-1}$ and $A_k^0(1)$ is a parallel class of $X_2$, we can conclude that

$$\bigcup_{\{a,b,c\}\in A_k^0(1)} (\alpha_j^{(0)} C^{S_i}(a,b) \cup \alpha_j^{(1)} C^{\alpha_j^{(0)}(S_i)}(b,c) \cup \alpha_j^{(2)} C^{\alpha_j^{(1)}\alpha_j^{(0)}(S_i)}(c,a))$$

is a partition of $Y$, where $C^R(e,f)=\bigcup_{(x,y)\in R}\{\langle(x,e),(y,e),(x\circ y,f)\rangle\}$ for $R\in\{S_i,\ \alpha_j^{(0)}(S_i),\ \alpha_j^{(1)}\alpha_j^{(0)}(S_i)\}$ and $(e,f)\in\{(a,b),\ (b,c),\ (c,a)\}$. Note that

$$C^X(e,f)=\bigcup_{i=1}^{3(u-1)}C^{S_i}(e,f)=\bigcup_{i=1}^{3(u-1)}C^{\alpha_j^{(0)}(S_i)}(e,f)=\bigcup_{i=1}^{3(u-1)}C^{\alpha_j^{(1)}\alpha_j^{(0)}(S_i)}(e,f).$$

It is easy to see that $\cup_{\{a,b,c\}\in A_k^0(1)}\mathcal{C}_{0j}^{(a,b,c)}$ can be partitioned into $3(u-1)$ parallel classes.

We have $3(u-1)+2$ parallel classes in this part.

Part II: For given $m$ and $i$, $2\le m\le\frac{v-1}{2}$ and $0\le i\le u-1$, $\bigcup_{\{a,b,c\}\in A_k^0(m)}P_{ji}^{(a,b,c)}$ provides 2 parallel classes. So, we get $u(v-3)$ parallel classes in this part.

Obviously, $\mathcal{D}_{kj}^{(0)}$ is the union of all the $uv-1$ parallel classes in Part I and II.

By formula (2.3), we have $\{\alpha_j^{(s)}:0\le j\le u-1\}=G$ $(s\in Z_3)$. With this fact, we can check that these $\frac{u(v-1)}{2}$ $RMTS(uv)$s are pairwise disjoint and they are obviously disjoint with those obtained in Step 1.

**Step 3**. (making use of the block set $\mathcal{A}_{u-1}^{(a,b,c)}$)

For a given $\sigma_j\in G$, $j=0,\ 1,\ \cdots,\ u-1$, define 3 permutations on $X_1$, namely $\beta_j^{(s)}(s\in Z_3)$ as follows:

$$\beta_j^{(0)}=\sigma_{u-1}\sigma_j^*,\qquad \beta_j^{(1)}=\sigma_j(\sigma_{u-1}\sigma_j^*)^{-1},\qquad \beta_j^{(2)}=(\sigma_j)^{-1}=(\beta_j^{(1)}\beta_j^{(0)})^{-1}.$$

For given $k$ and $j$, $1\le k\le\frac{v-1}{2}$ and $0\le j\le u-1$, take $\{a,b,c\}\in A_k^1(1)$, $a<b<c$. Define

$$\mathcal{C}_{u-1,j}^{(a,b,c)}=\beta_j^{(0)}C^{X_1}(a,c)\cup\beta_j^{(1)}C^{X_1}(c,b)\cup\beta_j^{(2)}C^{X_1}(b,a),$$

and

$$\mathcal{D}_{kj}^{(u-1)}=\Big[\bigcup_{\{a,b,c\}\in A_k^1(1)}(P_{u-1,j}^{(a,b,c)}\cup\mathcal{C}_{u-1,j}^{(a,b,c)})\Big]\cup\Big[\bigcup_{\{a,b,c\}\in\mathcal{A}_k^1\setminus A_k^1(1)}\mathcal{A}_j^{(a,b,c)}\Big].$$

The similar arguments as in Step 2 give $\frac{u(v-1)}{2}$ $RMTS(uv)$s $(Y,\mathcal{D}_{kj}^{(u-1)})$ for $1\le k\le\frac{v-1}{2}$ and $0\le j\le u-1$. Furthermore, these $RMTS(uv)$s are disjoint and also disjoint with those obtained in Steps 1 and 2.

We obtain a total of $uv-2$ disjoint $RMTS(uv)$s, a large set. This completes the proof. □

As we know, a large set of $RDTS(v)$ contains three times the number of "small" sets that a large set of $RMTS(v)$ does. We will see that the property of second-transitivity is just what we need for the product construction for LRDTSs.

**Theorem 2.4** *If there exist an $LRDTS(u)$ and a $DTRIQ(u)$, and there exists an $LR(v)$, then there exists an $LRDTS(uv)$.*

**Proof** It is similar to the proof of Theorem 2.3. Suppose that $(X_1,\circ)$ is the $DTRIQ(u)$ in Remark 2.2 with the properties (A), (B) and (C). Let $\{(X_2,\mathcal{A}_k^j)$ :

$1 \le k \le \frac{v-1}{2}, j = 0, 1\}$ be the $LR(v)$ satisfying condition (D). Let $\{(X_1, \mathcal{B}_j) : 1 \le j \le 3(u-2)\}$ be an $LRDTS(u)$. We will construct an $LRDTS(uv)$ on the set $Y = X_1 \times X_2$. The construction proceeds in 3 steps.

**Step 1**: For any $\{a, b, c\} \subseteq X_2$ with $a < b < c$, for $\sigma_i, \sigma_j \in G$ and $x \in X_1$, define

$$B_{ijx}^{(a,b,c)} = \{\{(x, a), (\sigma_j(x), b), (\sigma_i\sigma_j^*(x), c)\}\},$$

$$P_{0ij}^{(a,b,c)} = \bigcup_{x \in X_1} \{(u, v, w), (w, v, u) : \{u, v, w\} \in B_{ijx}^{(a,b,c)}\},$$

$$P_{1ij}^{(a,b,c)} = \bigcup_{x \in X_1} \{(u, w, v), (v, w, u) : \{u, v, w\} \in B_{ijx}^{(a,b,c)}\},$$

$$P_{2ij}^{(a,b,c)} = \bigcup_{x \in X_1} \{(w, u, v), (v, u, w) : \{u, v, w\} \in B_{ijx}^{(a,b,c)}\},$$

and

$$\mathcal{A}_{li}^{(a,b,c)} = \bigcup_{\sigma_j \in G} P_{lij}^{(a,b,c)} \ (0 \le l \le 2).$$

Noting the formula (2.3), we have: (1) For $m \neq n \in \{a, b, c\}$, $x, y \in X_1$, each of the ordered pairs $((x, m), (y, n))$ and $((y, n), (x, m))$ belongs to exactly one triple of $\mathcal{A}_{li}^{(a,b,c)}$; (2) $\mathcal{A}_{li}^{(a,b,c)}$ and $\mathcal{A}_{l'i'}^{(a,b,c)}$ are disjoint for $(i, l) \neq (i', l')$.

For each $a' \in X_2$, we have $3(u-2)$ disjoint $RDTS(u)$s $(X_1 \times \{a'\},\ \mathcal{B}_j^{(a')})$ for $1 \le j \le 3(u-2)$, where $\mathcal{B}_j^{(a')} = \{((x, a'), (y, a'), (z, a')) : (x, y, z) \in \mathcal{B}_j\}$.

For a given $j$, $1 \le j \le u-2$, take $\{a, b, c\} \in \mathcal{A}$ and $a < b < c$, define

$$\mathcal{C}_{lj} = (\bigcup_{\{a,b,c\} \in \mathcal{A}} \mathcal{A}_{lj}^{(a,b,c)}) \cup (\bigcup_{a' \in X_2} \mathcal{B}_{3j-2+l}^{(a')}).$$

Then each $(Y, \mathcal{C}_{lj})$ is an $RDTS(uv)$ for $1 \le j \le u-2$ and $0 \le l \le 2$. Furthermore, the $3(u-1)$ $RDTS$s in this step are disjoint.

(The remaining $\mathcal{A}_{l0}^{(a,b,c)}$ and $\mathcal{A}_{l,u-1}^{(a,b,c)}$ ($\{a, b, c\} \in \mathcal{A},\ a < b < c,\ 0 \le l \le 2$) are saved for the use in the following two steps.)

**Step 2**: (making use of the block set $\mathcal{A}_{l0}^{(a,b,c)}$ for $\{a, b, c\} \in \mathcal{A}$ and $0 \le l \le 2$.)

For a given $\sigma_j \in G$, $j = 0, 1, \cdots, u-1$, define 3 permutations on $X_1$, namely $\alpha_j^{(s)} (s \in Z_3)$ as follows:

$$\alpha_j^{(0)} = \sigma_j, \quad \alpha_j^{(1)} = \sigma_0\sigma_j^*\sigma_j^{-1}, \quad \alpha_j^{(2)} = (\sigma_0\sigma_j^*)^{-1} = (\alpha_j^{(1)}\alpha_j^{(0)})^{-1}.$$

For given $k$, $j$ and $l$, $1 \le k \le \frac{v-1}{2}$, $0 \le j \le u-1$ and $0 \le l \le 2$, take $\{a, b, c\} \in A_k^0(1)$, $a < b < c$. Define

$$\mathcal{C}_{l0j}^{(a,b,c)} = \alpha_j^{(0)} T_l^{X_1}(a, b) \cup \alpha_j^{(1)} T_l^{X_1}(b, c) \cup \alpha_j^{(2)} T_l^{X_1}(c, a),$$

and

$$\mathcal{D}_{lkj}^{(0)} = \Big[\bigcup_{\{a,b,c\} \in A_k^0(1)} (P_{l0j}^{(a,b,c)} \cup \mathcal{C}_{l0j}^{(a,b,c)})\Big] \cup \Big[\bigcup_{\{a,b,c\} \in \mathcal{A}_k^0 \setminus A_k^0(1)} \mathcal{A}_{lj}^{(a,b,c)}\Big].$$

Then each $(Y, \mathcal{D}_{lkj}^{(0)})$ is an $RDTS(uv)$ for $0 \leq l \leq 2$, $1 \leq k \leq \frac{v-1}{2}$ and $0 \leq j \leq u-1$. Furthermore, the $\frac{3u(v-1)}{2}$ $RDTS(uv)$s in this step are pairwise disjoint and they are also disjoint with those obtained in Step 1.

**Step 3**. (making use of the block set $\mathcal{A}_{l,u-1}^{(a,b,c)}$ for $\{a,b,c\} \in \mathcal{A}$ and $0 \leq l \leq 2$)

For a given $\sigma_j \in G$, $j = 0, 1, \cdots, u-1$, define 3 permutations on $X_1$, namely $\beta_j^{(s)} (s \in Z_3)$ as follows:

$$\beta_j^{(0)} = \sigma_{u-1}\sigma_j^*, \quad \beta_j^{(1)} = \sigma_j(\sigma_{u-1}\sigma_j^*)^{-1}, \quad \beta_j^{(2)} = (\sigma_j)^{-1} = (\beta_j^{(1)}\beta_j^{(0)})^{-1}.$$

For given $k$, $j$ and $l$, $1 \leq k \leq \frac{v-1}{2}$, $0 \leq j \leq u-1$ and $0 \leq l \leq 2$, take $\{a,b,c\} \in A_k^1(1)$, $a < b < c$. Define

$$\mathcal{C}_{l,u-1,j}^{(a,b,c)} = \beta_j^{(0)} T_l^{X_1}(a,c) \cup \beta_j^{(1)} T_l^{X_1}(c,b) \cup \beta_j^{(2)} T_l^{X_1}(b,a),$$

and

$$\mathcal{D}_{lkj}^{(u-1)} = \Big[ \bigcup_{\{a,b,c\} \in A_k^1(1)} (P_{l,u-1,j}^{(a,b,c)} \cup \mathcal{C}_{l,u-1,j}^{(a,b,c)}) \Big] \cup \Big[ \bigcup_{\{a,b,c\} \in \mathcal{A}_k^1 \setminus A_k^1(1)} \mathcal{A}_{lj}^{(a,b,c)} \Big].$$

Then each $(Y, \mathcal{D}_{lkj}^{(u-1)})$ is an $RDTS(uv)$ for $0 \leq l \leq 2$, $1 \leq k \leq \frac{v-1}{2}$ and $0 \leq j \leq u-1$. Furthermore, these $RDTS(uv)$s are disjoint and also disjoint with those obtained in Steps 1 and 2.

We obtain a total of $3(uv-2)$ disjoint $RDTS(uv)$s, a large set.

The details of the proof are omitted. But we should point out one thing. When considering the resolvability, there is no harm in disregarding the orientation of the triples. So the proof of the resolvability is similar to that in Theorem 2.3. Especially in Steps 2 and 3, if we disregard the orientation, $T_l^X(i,j)$ is actually the same as $C^X(i,j)$ for any $i \neq j$ and $0 \leq l \leq 2$. □

Note: There is an $LR(3)$ by Theorem 1.4. Take $LR(3)$ in Theorem 2.3 and Theorem 2.4, then we can obtain the tripling constructions in Theorem 1.2.

## 3 Updated results

By Theorem 1.3 and Theorem 2.3 (respectively, Theorem 2.4) we get the following result.

**Theorem 3.1** *Let $v$ be a positive integer such that $v \not\equiv 2 \pmod 4$. If there exist both an $LRMTS(v)$ (respectively, $LRDTS(v)$) and an $LR(u)$, then there exists an $LRMTS(uv)$ (respectively, $LRDTS(uv)$).*

Applying Theorem 3.1 recursively with the $LRMTS(v)$s and $LRDTS(v)$s from Theorem 1.1 and the $LR(u)$s from Theorem 1.4 and Theorem 1.5, we obtain the updated existence results on LRMTSs and LRDTSs.

**Theorem 3.2** *There exist an $LRMTS(v)$ and an $LRDTS(v)$ for $v = 3^n m(2\cdot k_1^{n_1}+1)(2\cdot k_2^{n_2}+1)\cdots(2\cdot k_t^{n_t}+1)$ where $n \geq 1$, $t \geq 0$, $n_i \geq 1$, $k_i \in \{7, 13\}$ $(i = 1, 2, \cdots, t)$ and $m \in \{1, 4, 5, 7, 11, 13, 17, 23, 25, 35, 37, 41, 43, 47, 53, 55, 57, 61, 65, 67, 91, 123\} \cup \{(7^k+2)/3, (13^k+2)/3, (25^k+2)/3, 2^{2k+1}25^j+1 : k \geq 0$ and $j \geq 0\}$.*

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