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The induced path number of the complements of some graphs

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Abstract

The induced path number $\rho(G)$ of a graph G is defined as the minimum number of subsets into which the vertex set $V(G)$ of G can be partitioned such that each subset induces a path. In this paper we investigate the induced path number of the complement of graphs. We also look at Nordhaus-Gaddum type results for trees, bipartite graphs and graphs in general.

1 Introduction

We generally use the notation and terminology of [6]. Let $S \subseteq V(G)$. The subgraph of G *induced by* S, denoted $\langle S \rangle$, is the graph having vertex set S and edge set those edges of G having both endpoints in S. For a graph G, the *induced path number* $\rho(G)$ is defined by Chartrand et al. in [5] as the minimum number of subsets into which the vertex set $V(G)$ of G can be partitioned such that each subset induces a path. They

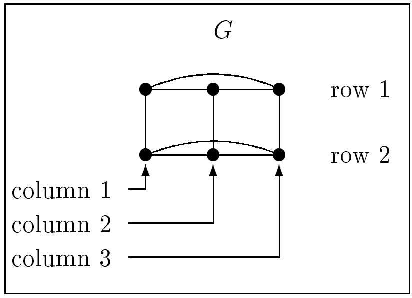

Figure 1: Rows and columns of G

investigated the induced path number for bipartite graphs and presented formulas for the induced path number of complete bipartite graphs and complete binary trees. They also determined bounds on the induced path number of trees and considered the induced path numbers of meshes, hypercubes and butterflies. Broere, Jonck and Voigt in [4] and Broere and Jonck in [2] further studied the induced path number of graphs.

In [1] an encompassing theory of partitions of the vertex set $V(G)$ of a graph G is discussed. The contents of this paper does not fit into the framework given in [1] since the property "to be an induced path" is not hereditary. Nevertheless, the topic studied in this paper has given rise to interesting results on notions that are typical in [1], viz. uniquely partitionable graphs and critical graphs.

The following results for paths, cycles, empty graphs and complete graphs are immediate and take little or no explanation.

Observation 1 *For the path on n vertices, $\rho(P_n) = 1$.*

Observation 2 *For the cycle on n vertices, $\rho(C_n) = 2$.*

Observation 3 *For the complete graph on n vertices, $\rho(K_n) = \lceil \frac{n}{2} \rceil$. In other words, for any positive integer k, $\rho(K_{2k}) = k$ and $\rho(K_{2k+1}) = k + 1$.*

The *cartesian product* of two graphs G_1 and G_2, denoted $G_1 \times G_2$, has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(u_1, u_2)(v_1, v_2) \mid u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)\}$. In Figure 1 we indicate, with an example, what we mean by the rows and columns of a graph of the form $G = G_1 \times G_2$. Here, $G = K_2 \times K_3$ and in this case there is a complete graph K_3 in every row and a complete graph K_2 in every column.

The following results are known for the cartesian product of paths, complete graphs and cycles.

Theorem 1 (Chartrand et al. in [5]) *For $m \geq 2$, $n \geq 2$, $\rho(P_m \times P_n) = 2$.*

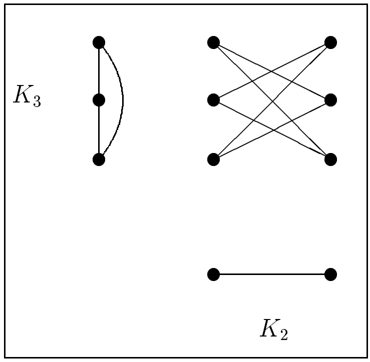

Figure 2: $\overline{K_3 \times K_2}$

Theorem 2 (Broere et al. in [3]) *Suppose $n \geq m$. Then*

$$\rho(K_m \times K_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ even and } n > m \\ \frac{n}{2} + \lceil \frac{m}{4} \rceil & \text{if } n \text{ is even and } n = m \\ \frac{n-1}{2} + \lceil \frac{m}{2} \rceil & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 3 (Broere et al. in [3]) *Suppose m and n are positive integers. Then*

$$\rho(C_m \times C_n) \leq 3.$$

Furthermore, if m and n are both odd, then $\rho(C_m \times C_n) = 3$.

A special case of the above theorem is the following.

Theorem 4 (Broere et al. in [3]) *Suppose a, k are positive integers, $n = 4a$ and $m = 2a(2k-1)+1$. Then*

$$\rho(C_m \times C_n) = 2.$$

2 The induced path number of the complement of certain classes of graphs

An example of a graph of the form $\overline{K_m \times K_n}$ is shown in Figure 2. This complement can also be described, as a product of graphs, by the following:

For the graphs K_m and K_n the graph $\overline{K_m \times K_n}$ has vertex set $V(K_m) \times V(K_n)$ and the vertices (a, b) and (c, d) are adjacent if and only if

a is adjacent to c in K_m
and
b is adjacent to d in K_n ,
that is, $a \neq c$ and $b \neq d$.

Lemma 1 *If $G = K_m \times K_n$, then $\overline{G}$ does not have a path of order six as induced subgraph.*

Proof: Suppose $\overline{G}$ has an induced path of order six. Let this path be $v_1v_2v_3v_4v_5v_6$. Since the three vertices v_1, v_3 and v_5 are mutually independent, they must all three be in the same row or in the same column. Without loss of generality we may assume they are in the same row. Since v_6 is nonadjacent to v_1 as well as to v_3, it must then also be in the same row as v_1 and v_3. But then v_6 is in the same row as v_5, which is not possible, since v_5 is adjacent to v_6. ■

Theorem 5 *If $m = 3^a \cdot 2^b$ and $n = 2^a \cdot 3^b$ where a, b are nonnegative integers and $G = K_m \times K_n$, then*

$$\rho(\overline{G}) = \left\lceil \frac{mn}{5} \right\rceil.$$

Proof:

(i) Since by Lemma 1, no six or more vertices of $\overline{G}$ induce a path, we have that

$$\rho(\overline{G}) \geq \left\lceil \frac{mn}{5} \right\rceil.$$

(ii) By using induction on $a + b$, we now prove that

$$\rho(\overline{G}) \leq \left\lceil \frac{mn}{5} \right\rceil.$$

If $a = b = 0$, then $K_m \times K_n = K_1 \times K_1 = K_1$ and

$$\rho(\overline{G}) = 1 \leq \left\lceil \frac{1}{5} \right\rceil = \left\lceil \frac{mn}{5} \right\rceil.$$

Now suppose that

$$\rho(\overline{G}) \leq \left\lceil \frac{mn}{5} \right\rceil$$

for all values $a + b \leq k$. Let $m = 3^a \cdot 2^b$ and $n = 2^a \cdot 3^b$ with $a + b = k + 1$. Consider the graph $\overline{G}$ and partition the vertices of $\overline{G}$ in $\frac{m}{2} \times \frac{n}{3} = \frac{mn}{6}$ (or $\frac{m}{3} \times \frac{n}{2} = \frac{mn}{6}$) blocks of 2 by 3 vertices (or 3 by 2 vertices) each.

In every block of six vertices, form an induced path of order five in such a manner that the vertices not used form a graph $\overline{G_1}$ where $G_1 = K_{\frac{m}{2}} \times K_{\frac{n}{3}}$ (or $G_1 = K_{\frac{m}{3}} \times K_{\frac{n}{2}}$).

This is always possible since $m = 3^a \cdot 2^b$ and $n = 2^a \cdot 3^b$. Then we have

$$\begin{aligned}
\rho(\overline{G}) &\leq \rho(\overline{G_1}) + \frac{mn}{6} \\
&\leq \left\lceil \frac{\frac{m}{2} \cdot \frac{n}{3}}{5} \right\rceil + \frac{mn}{6}, \text{ by the induction hypothesis} \\
&= \left\lceil \frac{mn + 5mn}{30} \right\rceil \\
&= \left\lceil \frac{mn}{5} \right\rceil.
\end{aligned}$$

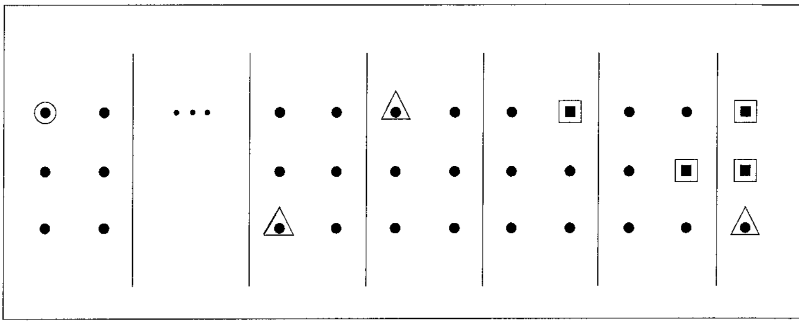

Figure 3: $\overline{K_3 \times K_n}$, n odd

We conclude that $\rho(\overline{G}) = \lceil \frac{mn}{5} \rceil$. ■

Note that $\rho(\overline{K_2 \times K_2}) = 2 = \lceil \frac{mn}{5} \rceil + 1$.

Conjecture 1 $\rho(\overline{K_m \times K_n}) = \lceil \frac{mn}{5} \rceil$ *except when*

1. $m = 3$ and $n \in \{3, 5, 8, 10, 11, 13, 14, 15, \ldots\}$
2. m and n odd, $m, n \geq 5$ and $mn \equiv 0 \pmod 5$.

In cases (1) and (2) we conjecture that

$$\rho(\overline{K_m \times K_n}) = \left\lceil \frac{mn}{5} \right\rceil + 1.$$

It is easy to check that $\rho(\overline{K_m \times K_n}) = \lceil \frac{mn}{5} \rceil$ for the values of $(3, n)$ excluded by (1): $(3,2), (3,4), (3,6), (3,7), (3,9)$ and $(3,12)$. For all other values of $(3, n)$ we can support this conjecture by showing that $\rho(\overline{K_m \times K_n}) \leq \lceil \frac{3n}{5} \rceil + 1$:

Clearly $\rho(\overline{K_3 \times K_3}) = 3 = \lceil \frac{9}{5} \rceil + 1$.

A partition of the $3n$ vertices of $\overline{K_3 \times K_n}$, n odd, in $\lceil \frac{3n}{5} \rceil + 1$ induced paths where $a = \lfloor \frac{n}{2} \rfloor$ paths are of order five, one path is of order four, $b = \lfloor \frac{3n-5a-4}{3} \rfloor$ if $n \geq 11$ ($b = \lfloor \frac{3n-5a}{3} \rfloor$ if $n < 11$) paths are of order three and at most one path is of order one or two (if $3n - 5a - 4(1) - 3b = 1$ or 2) is shown in Figure 3.

A partition of the $3n$ vertices of $\overline{K_3 \times K_n}$, n even, in $\lceil \frac{3n}{5} \rceil + 1$ induced paths where $a = \frac{n}{2}$ paths are of order five, $b = \lfloor \frac{3n-5a}{3} \rfloor$ paths are of order three and at most one path is of order one or two (if $3n - 5a - 3b = 1$ or 2) is shown in Figure 4.

Thus $\rho(K_3 \times K_n) \leq \lceil \frac{3n}{5} \rceil + 1$.

We can further support this conjecture by showing that $\rho(\overline{K_m \times K_n}) \leq \lceil \frac{mn}{5} \rceil + 1$ for the pairs (m, n) of case (2) too. For this we let $m = 5a$. A partition of the $5an$

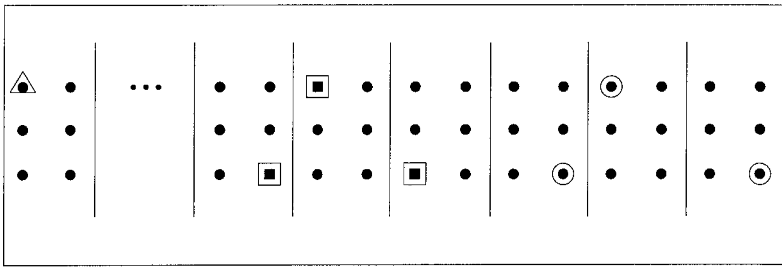

Figure 4: $\overline{K_3 \times K_n}$, n even

vertices of $\overline{K_{5a} \times K_n}$, $a = 1, 3, 5, \ldots$, in $\lceil \frac{5an}{5} \rceil + 1 = an + 1$ induced paths can be found as follows:

First partition the vertices in two classes V_1 and V_2 such that $\langle V_1 \rangle = \overline{K_{5a} \times K_{n-3}}$ and $\langle V_2 \rangle = \overline{K_{5a} \times K_3}$. A partition of the $5a(n-3)$ vertices of $\langle V_1 \rangle$into $a(n-3)$ induced paths of order five is shown in Figure 5—note that this partition is based on the fact that $n - 3$ is even.

In case (2) we showed that

$$\rho(\overline{K_{5a} \times K_3}) \leq \left\lceil \frac{15a}{5} \right\rceil + 1 = 3a + 1.$$

Thus we have

$$\begin{aligned} \rho(\overline{K_{5a} \times K_n}) &\leq a(n-3) + 3a + 1 \\ &= an + 1 \\ &= \left\lceil \frac{5an}{5} \right\rceil + 1 \end{aligned}$$

■

For use in some of our following results, we prove the following.

Theorem 6 $\rho(\overline{P_n}) = \lceil \frac{n}{4} \rceil$ *if* $n \geq 4$.

Proof: The result is clear if $n = 4$.

Suppose there are five vertices in $\overline{P_n}$, $n \geq 5$, that induce a path. Then P_n contains a C_3, which is impossible. Thus we have

$$\rho(\overline{P_n}) \geq \left\lceil \frac{n}{4} \right\rceil, \ n \geq 5.$$

A partition of the n vertices of $\overline{P_n}$ in $\lceil \frac{n}{4} \rceil$ induced paths is now described: Suppose the vertices of P_n are $\{v_1, v_2, \ldots, v_n\}$ with each v_i adjacent to v_{i+1}, $i < n$.

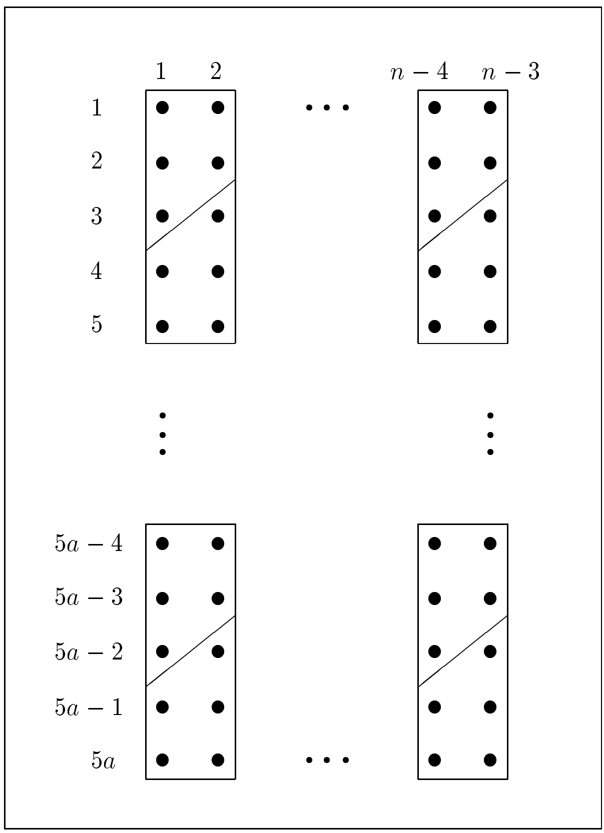

Figure 5: $< V_1 >$

Then there are integers p and r such that $n = 4p + r$ and $0 \leq r < 4$. If $r = 0$, the p sets of vertices of the form $\{v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}\}, i = 0, 1, \ldots, p-1$ suffice.

If $r > 0$, the p sets $\{v_{4i+2}, v_{4i+3}, v_{4i+4}, v_{4i+5}\}$ $(i = 0, 1, \ldots, p-1)$ and the set $\{v_1, v_{4p+2}, \ldots, v_{4p+r}\}$ (of $1, 2$ or 3 vertices, depending on the value of r) each induces a path.

Thus $\rho(\overline{P_n}) \leq \lceil \frac{n}{4} \rceil$, $n \geq 5$. ■

Next we find the induced path number of the complement of a cycle.

Theorem 7 *If $n \neq 3, 4, 7$, then $\rho(\overline{C_n}) = \lceil \frac{n}{4} \rceil$. Furthermore, $\rho(\overline{C_3}) = 3$, $\rho(\overline{C_4}) = 2$ and $\rho(\overline{C_7}) = 3$.*

Proof: It is straightforward to verify that $\rho(\overline{C_3}) = \rho(\overline{K_3}) = 3$, and $\rho(\overline{C_4}) = \rho(2K_2) = 2$.

Let the vertices of C_n be $v_1, v_2, v_3, \ldots, v_n$ with $v_i v_{i+1} \in E(C_n)$ for $1 \leq i < n$ and $v_n v_1 \in E(C_n)$. Since C_n is triangle-free, $\overline{C_n}$ cannot contain an induced P_5, so $\rho(\overline{C_n}) \geq \lceil \frac{n}{4} \rceil$. We consider five cases:

Case 1: $n = 4k$ where $k \geq 2$. In $\overline{C_n}$, the k sets $\{v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}\}$, for $0 \leq i \leq k-1$, each induce a P_4. Thus $\rho(\overline{C_n}) = k = \lceil \frac{n}{4} \rceil$.

Case 2: $n = 4k + 1$ where $k \geq 1$. In $\overline{C_n}$, the k sets $\{v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}\}$, for $0 \leq i \leq k-1$, each induce a P_4, and the set $\{v_n\}$ induces a P_1. Thus $\rho(\overline{C_n}) = k+1 = \lceil \frac{n}{4} \rceil$.

Case 3: $n = 4k + 2$ where $k \geq 1$. If $k = 1$, then in $\overline{C_6}$ the two sets $\{v_1, v_3, v_4\}$ and $\{v_2, v_5, v_6\}$ each induce a P_3 so that $\rho(\overline{C_n}) = 2 = \lceil \frac{n}{4} \rceil$. For $k \geq 2$, in $(\overline{C_n})$, the $k-1$ sets $\{v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}\}$, for $0 \leq i \leq k-2$, each induce a P_4. The two sets $\{v_{4k-3}, v_{4k-2}, v_{4k+1}\}$ and $\{v_{4k-1}, v_{4k}, v_{4k+2}\}$ each induce a P_3. Thus $\rho(\overline{C_n}) = (k-1) + 2 = \lceil \frac{n}{4} \rceil$.

Case 4: $n = 4k + 3$ where $k \geq 2$ In $\overline{C_n}$, the $k-1$ sets $\{v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}\}$, for $0 \leq i \leq k-2$, each induce a P_4. The set $\{v_{4k-2}, v_{4k-1}, v_{4k}, v_{4k+1}\}$ induces a P_4 and the set $\{v_{4k-3}, v_{4k+2}, v_{4k+3}\}$ induces a P_3. Thus $\rho(\overline{C_n}) = (k-1) + 2 = \lceil \frac{n}{4} \rceil$.

Case 5: $n = 7$. Clearly, $\rho(\overline{C_7}) \geq 2$. Suppose $\rho(\overline{C_7}) = 2$. Then one induced path in $\overline{C_7}$ contains four vertices that induce a P_4, and furthermore, these vertices must be consecutive in C_7. But then the three remaining vertices do not induce a P_3 in $\overline{C_7}$. Thus, $\rho(\overline{C_7}) \geq 3$. The three sets $\{v_1\}, \{v_2, v_3, v_6\}, \{v_4, v_5, v_7\}$ form a path partition of $\overline{C_7}$. Thus $\rho(\overline{C_7}) = 3$. ■

We complete this section by considering the induced path number of the complement of products of paths and cycles. Note that $\overline{C_3 \times C_3} = \overline{K_3 \times K_3}$, so that $\rho(\overline{C_3 \times C_3}) = 3$.

Theorem 8 *Let $m, n \neq 3, 4$. Then $\rho\left(\overline{C_m \times C_n}\right) = \left\lceil \frac{mn}{4} \right\rceil$.*

Proof: Let $G = C_m \times C_n$ where $m, n \neq 3, 4$.

Note that there is no induced P_5 in $\overline{G}$, since otherwise G contains a triangle. Therefore,

$$\rho\left(\overline{C_m \times C_n}\right) \geq \left\lceil \frac{mn}{4} \right\rceil .$$

Now we prove that $\rho\left(\overline{C_m \times C_n}\right) \leq \left\lceil \frac{mn}{4} \right\rceil$:

Case 1: $m \equiv 0 \pmod 4$ or $n \equiv 0 \pmod 4$

Suppose that $n \equiv 0 \pmod 4$. Note that $\overline{C_n}$ is an induced subgraph of $\overline{G}$. To find an induced path partition for $\overline{G}$, partition each of the m rows into $\rho\left(\overline{C_n}\right)$ partition sets. Then

$$\rho\left(\overline{C_m \times C_n}\right) \leq m \left\lceil \frac{n}{4} \right\rceil = \left\lceil \frac{mn}{4} \right\rceil .$$

If $n \not\equiv 0 \pmod 4$, then $m \equiv 0 \pmod 4$ and the result follows similarly.

Case 2: $m \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$

Note that mn is also $1 \pmod 4$ and that $\left\lceil \frac{mn}{4} \right\rceil = \frac{mn+3}{4}$. We first form the following $m\left(\frac{n-1}{4}\right)$ sets in the rows: $\{v_{i1}, \ldots, v_{i4}\}, \ldots, \{v_{i(n-4)}, \ldots, v_{i(n-1)}\}$, $1 \leq i \leq m$. Then we form the following $\frac{m-1}{4}+1$ sets in column n: $\{v_{1n}, \ldots, v_{4n}\}, \ldots, \{v_{(m-4)n}, \ldots, v_{(m-1)n}\}$, $\{v_{mn}\}$. Each of these $\frac{mn+3}{4}$ sets induces a path.

Case 3: $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$

Note that mn is also $2 \pmod 4$ and that $\left\lceil \frac{mn}{4} \right\rceil = \frac{mn+2}{4}$. We first form the following $m\left(\frac{n-2}{4}\right)$ sets in the rows: $\{v_{i1}, \ldots, v_{i4}\}$, $\{v_{i6}, \ldots, v_{i9}\}$, $\ldots$, $\{v_{i(n-4)}, \ldots, v_{i(n-1)}\}$, $1 \leq i \leq m$. Then we form the following $\frac{m-1}{4}$ sets in column 5: $\{v_{15}, \ldots, v_{45}\}$, $\ldots$, $\{v_{(m-4)5}, \ldots, v_{(m-1)5}\}$. We also form the $\frac{m-1}{4}$ sets in column n: $\{v_{2n}, \ldots, v_{5n}\}, \ldots$, $\{v_{(m-3)n}, \ldots, v_{mn}\}$. Lastly, we form the set $\{v_{m5}, v_{1n}\}$. Each of these $\frac{mn+2}{4}$ sets induces a path.

Case 4: $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$

Note that mn is $3 \pmod 4$ and that $\left\lceil \frac{mn}{4} \right\rceil = \frac{mn+1}{4}$. We first form the following $(m-2)\left(\frac{n-3}{4}\right)$ sets in the rows: $\{v_{i1}, \ldots, v_{i4}\}$, $\{v_{i5}, \ldots, v_{i8}\}$, $\ldots$, $\{v_{i(n-6)}, \ldots, v_{i(n-3)}\}$, $1 \leq i \leq m-2$. Then we form the following $3\left(\frac{m-1}{4}\right)$ sets in column j, $j \in \{n-2, n-1, n\}$: $\{v_{1j}, \ldots, v_{4j}\}$, $\ldots$, $\{v_{(m-4)j}, \ldots, v_{(m-1)j}\}$. Next we form the following $2\left(\frac{n-7}{4}\right)$ sets in row i, $i \in \{m-1, m\}$: $\{v_{i1}, \ldots, v_{i4}\}$, $\ldots$, $\{v_{i(n-10)}, \ldots, v_{i(n-7)}\}$. Lastly we form three sets: $\{v_{m(n-4)}, v_{(m-1)(n-6)}, v_{m(n-6)}\}$, $\{v_{(m-1)(n-5)}, v_{(m-1)(n-4)}, v_{(m-1)(n-3)}, v_{m(n-5)}\}$ and $\{v_{m(n-3)}, v_{m(n-2)}, v_{m(n-1)}, v_{mn}\}$, Each of these $\frac{mn+1}{4}$ sets induces a path.

Case 5: $m \equiv 2 \pmod 4$ and $n \equiv 2 \pmod 4$

Note that mn is $0 \pmod 4$ and that $\left\lceil \frac{mn}{4} \right\rceil = \frac{mn}{4}$. We first form the following $m\left(\frac{n-2}{4}\right)$ sets in the rows: $\{v_{i1}, \ldots, v_{i4}\}, \ldots, \{v_{i(n-5)}, \ldots, v_{i(n-2)}\}$, $1 \leq i \leq m$. Then we form the following $2\left(\frac{m-2}{4}\right)$ sets in column $n-1$ and column n: $\{v_{1(n-1)}, \ldots, v_{4(n-1)}\}$, $\ldots$, $\{v_{(m-5)(n-1)}, \ldots, v_{(m-2)(n-1)}\}, \{v_{2n}, \ldots, v_{5n}\}$, $\ldots$, $\{v_{(m-4)n}, \ldots, v_{(m-1)n}\}$. Lastly we form the set $\{v_{(m-1)(n-1)}, v_{m(n-1)}, v_{1n}, v_{mn}\}$. Each of these $\frac{mn}{4}$ sets induces a path.

Case 6: $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$

Note that mn is $2 \pmod 4$ and that $\left\lceil \frac{mn}{4} \right\rceil = \frac{mn+2}{4}$. We first form the following $m\left(\frac{n-3}{4}\right)$ sets in the rows: $\{v_{i1}, \ldots, v_{i4}\}$, $\ldots$, $\{v_{i(n-6)}, \ldots, v_{i(n-3)}\}$, $1 \leq i \leq m$. Then we form the following $3\left(\frac{m-2}{4}\right)$ sets in column j, $j \in \{n-2, n-1, n\}$: $\{v_{1j}, \ldots, v_{4j}\}$,

..., $\{v_{(m-5)j}, \ldots, v_{(m-2)j}\}$. Lastly, we form the two sets $\{v_{(m-1)(n-2)}, v_{m(n-1)}, v_{mn}\}$ and $\{v_{m(n-2)}, v_{(m-1)(n-1)}, v_{(m-1)n}\}$. Each of these $\frac{mn+2}{4}$ sets induces a path.

Case 7: $m \equiv 3 \pmod 4$ and $n \equiv 3 \pmod 4$

Note that mn is 1 (mod 4) and that $\lceil \frac{mn}{4} \rceil = \frac{mn+3}{4}$. We first form the following $m\left(\frac{n-3}{4}\right)$ sets in the rows: $\{v_{i1}, \ldots, v_{i4}\}$, ..., $\{v_{i(n-6)}, \ldots, v_{i(n-3)}\}$, $1 \leq i \leq m$. Then we form the following $3\left(\frac{m-3}{4}\right)$ sets in column j, $j \in \{n-2, n-1, n\}$: $\{v_{1j}, \ldots, v_{4j}\}$, ..., $\{v_{(m-6)j}, \ldots, v_{(m-3)j}\}$. Lastly, we form the two sets $\{v_{i(n-2)}, v_{(i-1)(n-1)}, v_{(i-1)n}\}$, $i \in \{m-1, m\}$ and the set $\{v_{(m-2)(n-2)}, v_{m(n-1)}, v_{mn}\}$. Each of these $\frac{mn+3}{4}$ sets induces a path. ■

The following observation and lemma will be useful in proving some later results.

Observation 4 *If H is an induced subgraph of G, then*

1. *$G - H$ is an induced subgraph of G,*
2. *$\overline{H}$ is an induced subgraph of $\overline{G}$ and*
3. *$\overline{G - H} = \overline{G} - \overline{H}$ is an induced subgraph of $\overline{G}$.*

Lemma 2 *If H is any induced subgraph of a graph G, then*

$$\rho(G) \leq \rho(H) + \rho(G - H).$$

Proof: Suppose H is an induced subgraph of G, $V_1, \ldots, V_k$ is an induced path partition of H where $k = \rho(H)$ and $U_1, \ldots, U_j$ is an induced path partition of $G - H$ where $j = \rho(G - H)$. Clearly, $V_1, \ldots, V_k, U_1, \ldots, U_j$ is a partition of $V(G)$. Also, each of the sets $V_1, \ldots, V_k, U_1, \ldots, U_j$ induce a path in G since H and $G - H$ are induced subgraphs of G. Hence we have an induced path partition of G using $k + j = \rho(H) + \rho(G - H)$ sets. Therefore, $\rho(G) \leq \rho(H) + \rho(G - H)$. ■

For any $1 \leq j < m$, we have $\overline{C_k \times P_j}$ is an induced subgraph of $\overline{C_k \times C_m}$, so we have $\rho(\overline{C_k \times C_m}) \leq \rho(\overline{C_k \times P_j}) + \rho(\overline{C_k \times P_{m-j}})$.

The graph $C_3 \times P_m$ can be thought of as a graph where the vertices are arranged into 3 rows and m columns. Each column induces a C_3 and each row induces a P_m. We use a_{ij} to denote the vertex in row i, column j.

Lemma 3 *If $G = C_3 \times P_m$ or $G = C_3 \times C_m$, $m \geq 4$, then $\overline{G}$ does not have a path of order six as an induced subgraph.*

Proof: Suppose $\overline{G}$ has an induced path of order six. Let this path be $v_1v_2v_3v_4v_5v_6$. Then in G the vertices v_1, v_3, v_4, v_6 induced two cycles of length 3 which share an edge. But this cannot happen. Thus $\overline{G}$ does not contain any induced paths of order six. ■

Thus, each induced path in $\overline{C_3 \times P_m}$ and $\overline{C_3 \times C_m}$ has at most 5 vertices.

Lemma 4 *If $G = C_3 \times P_m$ or $G = C_3 \times C_m$, $m \geq 4$, then any induced path partition of $\overline{G}$ contains at most $\lfloor \frac{m}{2} \rfloor$ induced paths of order five.*

Proof: Let $v_1v_2v_3v_4v_5$ be an induced path of order five in $\overline{G}$. In G, the vertices v_1, v_3, v_5 induce a C_3. Therefore, these three vertices come from the same C_3 in G. We have $v_1v_4, v_2v_5, v_2v_4 \in E(G)$, so v_2, v_4 are in the same C_3 in G. Thus, each induced P_5 in $\overline{G}$ uses two C_3's in G. Therefore, any induced path partition of $\overline{G}$ contains at most $\lfloor \frac{m}{2} \rfloor$ induced paths of order five. ■

We now give a lower bound on the induced path partition number of $\overline{C_3 \times C_m}$ and $\overline{C_3 \times P_m}$ for $m \geq 4$.

Theorem 9 *For $m \geq 4$, if $G = C_3 \times P_m$ or $G = C_3 \times C_m$ then $\rho(\overline{G}) \geq \lceil \frac{5m}{8} \rceil$.*

Proof: Let l_i be the number of induced paths of order i in a minimum induced path partition of $\overline{G}$. Counting the number of vertices, $3m = 5l_5 + 4l_4 + 3l_3 + 2l_2 + 1l_1 \leq 5l_5 + 4(l_4 + l_3 + l_2 + l_1) \leq 5l_5 + 4(\rho(\overline{G}) - l_5)$. Rearranging, $\rho(\overline{G}) \geq \frac{3m - l_5}{4}$. By Lemma 2, $l_5 \leq \lceil \frac{m}{2} \rceil$ and therefore $\rho(\overline{G}) \geq \lceil \frac{5m}{8} \rceil$. ■

Next, we give an upper bound on $\rho(\overline{C_3 \times P_m})$ and $\rho(\overline{C_3 \times C_m})$ for $m \geq 4$.

Theorem 10 *For $m \geq 4$, if $G = C_3 \times P_m$ or $G = C_3 \times C_m$ then $\rho(\overline{G}) \leq \lceil \frac{2m}{3} \rceil$ for $m \geq 4$.*

Proof: Let $m \geq 4$. For $m = 4$, $\rho(\overline{G}) \leq 3$ since the three sets $\{v_{11}, v_{21}, v_{31}, v_{22}, v_{32}\}$, $\{v_{13}, v_{23}, v_{33}, v_{24}, v_{34}\}$, $\{v_{12}, v_{14}\}$ each induce a path in $\overline{G}$.

For $m = 5$, $\rho(\overline{G}) \leq 4$ since the four sets $\{v_{11}, v_{21}, v_{31}, v_{22}, v_{32}\}$, $\{v_{23}, v_{33}, v_{14}, v_{24}, v_{34}\}$, $\{v_{12}, v_{15}\}$, $\{v_{13}, v_{25}, v_{35}\}$ each induce a path in $\overline{G}$.

For $m = 6$, $\rho(\overline{G}) \leq 4$ since the four sets $\{v_{11}, v_{21}, v_{31}, v_{22}, v_{32}\}$, $\{v_{23}, v_{33}, v_{14}, v_{24}, v_{34}\}$, $\{v_{12}, v_{13}, v_{15}\}$, $\{v_{25}, v_{35}, v_{16}, v_{26}, v_{36}\}$ each induce a path in $\overline{G}$.

For $m = 7$, $\rho(\overline{G}) \leq 5$ since the five sets $\{v_{11}, v_{21}, v_{31}, v_{22}, v_{32}\}$, $\{v_{23}, v_{33}, v_{14}, v_{24}, v_{34}\}$, $\{v_{12}, v_{13}, v_{15}\}$, $\{v_{25}, v_{35}, v_{16}\}$, $\{v_{26}, v_{36}, v_{17}, v_{27}, v_{37}\}$ each induce a path in $\overline{G}$.

For $m = 8$, $\rho(\overline{G}) \leq 6$ since the six sets $\{v_{11}, v_{21}, v_{31}, v_{22}, v_{32}\}$, $\{v_{23}, v_{33}, v_{14}, v_{24}, v_{34}\}$, $\{v_{12}, v_{17}\}$, $\{v_{13}, v_{16}\}$, $\{v_{15}, v_{25}, v_{35}, v_{26}, v_{36}\}$, $\{v_{27}, v_{37}, v_{18}, v_{28}, v_{38}\}$ each induce a path in $\overline{G}$.

For $m = 9$, $\rho(\overline{G}) \leq 6$ since the six sets $\{v_{11}, v_{21}, v_{31}, v_{22}, v_{32}\}$, $\{v_{23}, v_{33}, v_{14}, v_{24}, v_{34}\}$, $\{v_{12}, v_{13}, v_{37}\}$, $\{v_{15}, v_{25}, v_{35}, v_{16}, v_{36}\}$, $\{v_{26}, v_{17}, v_{27}, v_{18}\}$, $\{v_{28}, v_{38}, v_{19}, v_{29}, v_{39}\}$ each induce a path in $\overline{G}$.

We proceed by induction on m. The result holds for $4 \leq m \leq 9$. So suppose $m \geq 10$ and for all integers k with $4 \leq k < m$ we have $\rho(\overline{C_3 \times P_k}) \leq \lceil \frac{2k}{3} \rceil$ and $\rho(\overline{C_3 \times C_k}) \leq \lceil \frac{2k}{3} \rceil$. Then

$$\rho(\overline{C_3 \times C_m}) \leq \rho(\overline{C_3 \times P_{m-6}}) + \rho(\overline{C_3 \times P_6})$$

$$\leq \lceil \frac{2(m-6)}{3} \rceil + 4 = \lceil \frac{2m}{3} \rceil$$

and similarly, $\rho(\overline{C_3 \times P_m}) \leq \lceil \frac{2m}{3} \rceil$. ■

We summarize with the following:

Theorem 11 *For $m \geq 4$, $\lceil \frac{5m}{8} \rceil \leq \rho(\overline{C_3 \times P_m}) \leq \lceil \frac{2m}{3} \rceil$ and $\lceil \frac{5m}{8} \rceil \leq \rho(\overline{C_3 \times C_m}) \leq \lceil \frac{2m}{3} \rceil$.*

We finish this section by considering the complements of products of paths and cycles with C_4.

Theorem 12 *For $m \geq 2$, $\rho(\overline{C_4 \times P_m}) = m$.*

Proof: The graph $\overline{C_4 \times P_m}$ contains no induced paths of order five, since $C_4 \times P_m$ does not contain an induced C_3. Thus $\rho(\overline{C_4 \times P_m}) \geq \frac{4m}{4} = m$.

Now $\rho(\overline{C_4 \times P_2}) = 2$ since the two sets $\{v_{11}, v_{21}, v_{12}, v_{42}\}$, $\{v_{22}, v_{31}, v_{32}, v_{41}\}$ each induce a P_4 in $\overline{C_4 \times P_2}$. Next, $\rho(\overline{C_4 \times P_3}) = 3$ since the three sets $\{v_{11}, v_{21}, v_{12}, v_{13}\}$, $\{v_{22}, v_{31}, v_{32}, v_{23}\}$, $\{v_{41}, v_{42}, v_{43}, v_{33}\}$ each induce a P_4 in $\overline{C_4 \times P_m}$.

Suppose $m \geq 4$ and for all integers k with $3 \leq k < m$ we have $\rho(\overline{C_4 \times P_k}) \leq k$. Then $m \leq \rho(\overline{C_4 \times P_m}) \leq \rho(\overline{C_4 \times P_{m-2}}) + \rho(\overline{C_4 \times P_2}) \leq (m-2) + 2 = m$. ■

Theorem 13 *For $m \geq 4$, $\rho(\overline{C_4 \times C_m}) = m$.*

Proof: Suppose $m \geq 4$. The graph $\overline{C_4 \times C_m}$ contains no induced paths of order five, since $C_4 \times C_m$ does not contain an induced C_3. Thus $\rho(\overline{C_4 \times C_m}) \geq \frac{4m}{4} = m$. Then $m \leq \rho(\overline{C_4 \times C_m}) \leq \rho(\overline{C_4 \times P_{m-2}}) + \rho(\overline{C_4 \times P_2}) = (m-2) + 2 = m$. ■

3 Nordhaus-Gaddum type results

It is interesting what relationships hold with a given parameter and its complement. One of the most famous results is due to Nordhaus and Gaddum [7] who in 1956 gave bounds on the chromatic number of a graph and its complement. Here the chromatic number of a graph G, denoted $\chi(G)$, is the minimum number of subsets into which the vertex set of G can be partitioned such that each induced subset contains no edges.

Theorem 14 (Nordhaus and Gaddum [7]) *For any graph G with n vertices,*

1. $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$ *and*
2. $n \leq \chi(G) \cdot \chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2$.

In this section we investigate bounds on the sum of the induced path number of a graph and its complement.

Theorem 15 *For any graph G of order n,*

$$\sqrt{n} \leq \rho(G) + \rho(\overline{G}) \leq \left\lceil \frac{3n}{2} \right\rceil.$$

Proof: The proof of the upper bound is done by induction on n, the number of vertices in G.

Suppose $n = 1$. Then $G = \overline{G} = K_1$, $\rho(G) = \rho(\overline{G}) = 1$ and the result holds.

Suppose $n = 2$. Then $\{G, \overline{G}\} = \{K_2, \overline{K_2}\}$, $\rho(G) + \rho(\overline{G}) = 1 + 2 = 3$ and the result holds.

Suppose $n = 3$. Then $\{G, \overline{G}\} \subseteq \{K_3, \overline{K_3}, P_3, \overline{P_3}\}$. If $\{G, \overline{G}\} = \{K_3, \overline{K_3}\}$, $\rho(G) + \rho(\overline{G}) = 3+2 = 5$ and the result holds. If $\{G, \overline{G}\} = \{P_3, \overline{P_3}\}$, $\rho(G)+\rho(\overline{G}) = 1+2 = 3$ and the result holds.

Now, suppose that any graph H with m vertices where $1 \leq m < n$ satisfies $\rho(H) + \rho(\overline{H}) \leq \lceil \frac{3m}{2} \rceil$ and suppose G is any graph with $n \geq 4$ vertices. Let P_k be a longest induced path in G.

Case 1: $k = n$. Then $\rho(\overline{G}) = \rho(\overline{P_p}) = \lceil \frac{n}{4} \rceil$ and $\rho(G) + \rho(\overline{G}) = 1 + \lceil \frac{n}{4} \rceil < \lceil \frac{3n}{2} \rceil$.

Case 2: $k < n$. Let $H = G - P_k$. Now H has $1 \leq n - k < n$ vertices, and by the inductive hypothesis, $\rho(H) + \rho(\overline{H}) \leq \lceil \frac{3(n-k)}{2} \rceil$.

Case 2.1: $k = 1$. Then $G = \overline{K_n}$ and $\overline{G} = K_n$. Hence, $\rho(G) + \rho(\overline{G}) = n + \lceil \frac{n}{2} \rceil = \lceil \frac{3n}{2} \rceil$.

Case 2.2: $k = 2$. Then $G - H = P_2$ and $\rho(\overline{G - H}) = 2$. By Lemma 2, $\rho(G) \leq \rho(H) + 1$ and $\rho(\overline{G}) \leq \rho(\overline{H}) + 2$. Hence, $\rho(G) + \rho(\overline{G}) \leq \lceil \frac{3(n-2)}{2} \rceil + 3 = \lceil \frac{3n}{2} \rceil$.

Case 2.3: $k = 3$. Then $G - H = P_3$ and $\rho(\overline{G - H}) = 2$. By Lemma 2, $\rho(G) \leq \rho(H) + 1$ and $\rho(\overline{G}) \leq \rho(\overline{H}) + 2$. Hence, $\rho(G) + \rho(\overline{G}) \leq \lceil \frac{3(n-3)}{2} \rceil + 3 < \lceil \frac{3n}{2} \rceil$.

Case 2.4: $k \geq 4$. Then $G - H = P_k$ where $k \geq 4$ and by Theorem 6, $\rho(\overline{G - H}) = \lceil \frac{k}{4} \rceil$. By Lemma 2, $\rho(G) \leq \rho(H) + 1$ and $\rho(\overline{G}) \leq \rho(\overline{H}) + \lceil \frac{k}{4} \rceil$. Hence, $\rho(G) + \rho(\overline{G}) \leq \lceil \frac{3(n-k)}{2} \rceil + \lceil \frac{k}{4} \rceil + 1 \leq \frac{3n-3k+1}{2} + \frac{k+3}{4} + 1 = \frac{6n-6k+3+k+3+4}{4} = \frac{6n-5k+9}{4} < \frac{6n}{4} \leq \lceil \frac{3n}{2} \rceil$. Therefore, in every case, $\rho(G) + \rho(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$.

Note that the only time it is possible for $\rho(G) + \rho(\overline{G}) = \lceil \frac{3n}{2} \rceil$ is when the order of the longest path is 1 or 2.

Now for the lower bound. Let $x = \rho(G)$. Then by the pigeonhole principle, G contains a partition class with an induced path of length at least $\lceil \frac{n}{x} \rceil$. Thus, $\overline{G}$ contains a complete graph of order $\left\lceil \frac{\lceil \frac{n}{x} \rceil}{2} \right\rceil$. And $\overline{G}$ will require at least $\left\lceil \frac{\lceil \frac{n}{x} \rceil}{4} \right\rceil$ partition classes. Hence, $\rho(\overline{G}) \geq \left\lceil \frac{\lceil \frac{n}{x} \rceil}{4} \right\rceil \geq \frac{n}{4x}$. So, $\rho(G) + \rho(\overline{G}) = x + \rho(\overline{G}) \geq x + \frac{n}{4x}$.

Now, $f(x) = x + \frac{n}{4x}$ has a minimum when $f'(x) = 1 - \frac{n}{4x^2} = 0$ since $f''(x) = \frac{n}{2x^3} > 0$. This happens when $x = \frac{\sqrt{n}}{2}$. Hence, $f(x) \geq f(\frac{\sqrt{n}}{2}) = \frac{\sqrt{n}}{2} + \frac{n}{4\frac{\sqrt{n}}{2}} = \sqrt{n}$. Therefore, $\rho(G) + \rho(\overline{G}) \geq \sqrt{n}$. ■

The upper bound in this theorem is achieved by the complete graph K_n.

The lower bound is achieved by the following in general described graph G :

Recall that in a complete p-partite graph H, it is possible to partition $V(H)$ into p subsets $V_1, \ldots, V_p$ such that $E(H) = \{uv \ / \ u \in V_i, v \in V_j, 1 \leq i \neq j \leq p\}$. Note that each V_i is an independent set. Now, by replacing each independent set V_i in H by a path P_{4p}, the *complete p-path-partite* graph $K_{P_{4p}, \ldots, P_{4p}}$ is formed. Let

$v_{i1}, v_{i2}, \ldots, v_{i(4p)}$ be the vertices of the i^{th} P_{4p} where $p = 2, 3, \ldots$ and $i = 1, 2, \ldots, p$. Let G be $K_{P_{4p}, \ldots, P_{4p}}$ except for the edges $\{v_{12}v_{22},\ v_{17}v_{27},\ v_{1(10)}v_{2(10)},\ v_{1(15)}v_{2(15)},\ v_{1(18)}v_{2(18)},\ \ldots,\ v_{1(4p-1)}v_{2(4p-1)},\ v_{23}v_{33},\ v_{26}v_{36},\ v_{2(11)}v_{3(11)},\ v_{2(14)}v_{3(14)},\ v_{2(19)}v_{3(19)},\ \ldots,\ v_{2(4p-2)}v_{3(4p-2)},\ \ldots\}$

$$\cup\{v_{(p-1)2}v_{p2}, v_{(p-1)7}v_{p7}, v_{(p-1)10}v_{p(10)}, v_{(p-1)15}v_{p(15)}, v_{(p-1)18}v_{p(18)}, \ldots, v_{(p-1)(4p-1)}v_{p(4p-1)}\} \text{ when } p \text{ is even or}$$

$$\cup\{v_{(p-1)3}v_{p3}, v_{(p-1)6}v_{p6}, v_{(p-1)11}v_{p(11)}, v_{(p-1)14}v_{p(14)}, v_{(p-1)19}v_{p(19)} \ldots, v_{(p-1)(4p-2)}v_{p(4p-2)}\} \text{ when } p \text{ is odd.}$$

See Figure 6 for the case $p = 2$ and $i = 2$.

Note that G has $4p^2$ vertices. In G, the p P_{4p}'s form p induced paths so that $\rho(G) \le p$. In $\overline{G}$, there are the following p induced paths: $\{v_{13}, v_{11}, v_{14}, v_{12}, v_{22}, v_{24}, v_{21}, v_{23}, v_{33}, \ldots, v_{p3}$ when p is even or v_{p2} when p is odd$\}$ on the first four vertices of each row,

$\{v_{16}, v_{18}, v_{15}, v_{17}, v_{27}, v_{25}, v_{28}, v_{26}, v_{36}, \ldots, v_{p6}$ when p is even or v_{p7} when p is odd$\}$ on the next four vertices of each row, $\ldots$,

$\{v_{1(4p-2)}, v_{1(4p)}, v_{1(4p-3)}, v_{1(4p-1)}, v_{2(4p-1)}, v_{2(4p-3)}, v_{2(4p)}, v_{2(4p-2)}, v_{3(4p-2)}, \ldots, v_{p(4p-2)}$ when p is even or $v_{p(4p-1)}$ when p is odd$\}$ on the last four vertices of each row. Again see Figure 6 for the case $p = 2$ and $i = 2$.

Hence $\rho(\overline{G}) \le p$ and therefore $\rho(G) + \rho(\overline{G}) \le 2p$. By the above theorem, $\rho(G) + \rho(\overline{G}) \ge 2p$. Therefore $\rho(G) + \rho(\overline{G}) = 2p = \sqrt{4p^2}$.

Theorem 16 *For any bipartite graph G of order n*

$$1 + \left\lceil \frac{n}{4} \right\rceil \le \rho(G) + \rho(\overline{G}) \le \left\lceil \frac{3n}{2} \right\rceil.$$

Proof: Let G be a bipartite graph with partite sets of order a and b where $a \le b$.

The upper bound follows from Theorem 15.

Now $\overline{G}$ is spanned by the two disjoint complete graphs K_a and K_b. A set in the vertex partition of $\overline{G}$ contains at most two vertices from the K_a and at most two vertices from the K_b. Thus $\overline{G}$ contains no induced P_5. Thus, $\rho(\overline{G}) \ge \lceil \frac{n}{4} \rceil$. Since $\rho(G) \ge 1$, the result holds. ■

The bounds given by this theorem are sharp: The graph $\overline{K_n}$, the complement of the complete graph, is bipartite and achieves the upper bound. For the lower bound, we have by Theorem 6 that

$$\rho(P_n) + \rho(\overline{P_n}) = 1 + \left\lceil \frac{n}{4} \right\rceil, \ n \ge 4.$$

4 Nordhaus-Gaddum Results for Trees

In this section, we give a lower and upper bound on the sum of the path partition number of a tree and its complement. Furthermore, we give a construction of all the trees which achieve this lower bound.

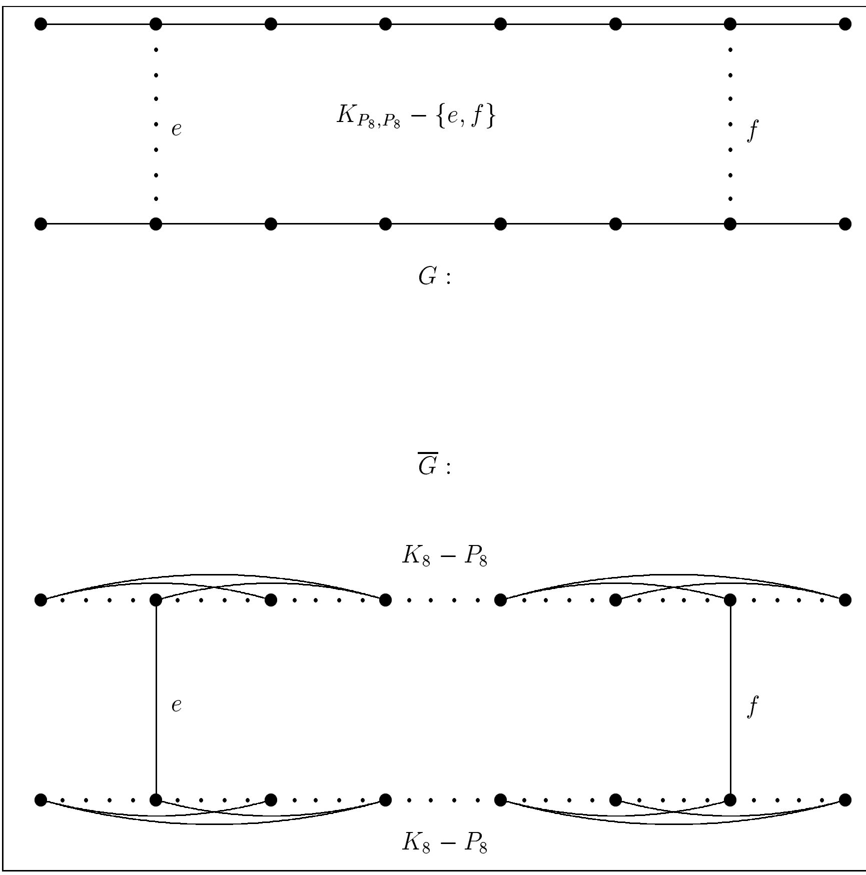

Figure 6: $\rho(G) + \rho(\overline{G}) = \sqrt{n}$

Theorem 17 *Let T be a tree and let L be the set of all leaves of T. Then $\rho(T) + \rho(\overline{T}) \geq |L|$.*

Proof: Each set in any induced path partition of T contains at most 2 leaves of T, so that $\rho(T) \geq |L|/2$. In $\overline{T}$, the vertices of L form a clique. Each set in any induced path partition of $\overline{T}$ contains at most 2 vertices of L. Thus, $\rho(\overline{T}) \geq |L|/2$. ■

Let $\mathcal{T}$ be the family of trees T such that $\rho(T)+\rho(\overline{T}) = |L|$. Consider the following two operations on any tree T:

Type 1: Attach a P_3 to T by adding an edge between a non-leaf of the P_3 and any non-leaf of T.

Type 2: Attach a P_4 to T by adding an edge between a non-leaf of the P_4 and any non-leaf of T.

We define two classes of trees:

$\mathcal{C}_3 = \{T \mid T$ is a tree that can be obtained from a P_3 by a finite non-empty sequence of at least one operation of Type 1 and Type 2$\}$

$\mathcal{C}_4 = \{T \mid T$ is a tree that can be obtained from a P_4 by a finite sequence of operations of Type 1 and Type 2$\}$.

We will show that $\mathcal{T} = \mathcal{C}_3 \cup \mathcal{C}_4$.

Theorem 18 *$\mathcal{C}_3 \cup \mathcal{C}_4 \subseteq \mathcal{T}$.*

Proof: Let $T \in \mathcal{C}_3 \cup \mathcal{C}_4$. Then T is constructed using k P_3's and m P_4's, so that $n = 3k + 4m$ and $|L| = 2k + 2m$. By the definition of $\mathcal{C}_3 \cup \mathcal{C}_4$, if $m = 0$ then $k \geq 2$. Now $\rho(T) = |L|/2$ because we can form an induced path partition of T by taking each of the k P_3's and m P_4's.

Now consider $\overline{T}$. Let each P_3 in T have vertices x_i, y_i, z_i, where $x_i, z_i \in L$ and $0 \leq i \leq k$. Let each P_4 in T have vertices a_j, b_j, c_j, d_j where $a_j, d_j \in L$ and $0 \leq j \leq m$. If $k = 0$ or $k \geq 2$, in $\overline{T}$ we form an induced path partition as follows: first form the m sets $\{a_j, b_j, c_j, d_j\}$. If $k = 0$, this forms an induced path partition of $\overline{T}$ with $|L|/2$ sets. If $k \geq 2$, take also the $k - 1$ sets $\{x_i, y_i, z_{i+1}\}$ for $1 \leq i \leq k - 1$ and the set $\{x_n, y_n, z_1\}$.

If $k = 1$, let a_m, b_m, c_m, d_m be a P_4 in T such that $b_m y_1 \in E(T)$. We form an induced path partition with $|L|/2$ sets as follows: take the $m - 1$ sets $\{a_j, b_j, c_j, d_j\}$ for $1 \leq j \leq m - 1$ and the two sets $\{x_1, y_1, a_m, b_m\}$, $\{z_1, c_m, d_m\}$.

Thus $T \in \mathcal{T}$. ■

Now we prove that $\mathcal{T} \subseteq \mathcal{C}_3 \cup \mathcal{C}_4$.

Lemma 5 *If $T \in \mathcal{T}$ then every vertex in T is either a leaf or adjacent to a leaf.*

Proof: Suppose that T is a tree where $\rho(T) + \rho(\overline{T}) = |L|$ and $x \in V$ is not a leaf or adjacent to a leaf. Then $L \cup \{x\}$ forms a clique in $\overline{T}$. So, $\rho(\overline{T}) \geq \frac{|L|+1}{2}$ and $\rho(T) \geq \frac{|L|}{2}$. Hence, $\rho(T) + \rho(\overline{T}) > |L|$, a contradiction. ■

Theorem 19 $\mathcal{T} \subseteq \mathcal{C}_3 \cup \mathcal{C}_4$.

Proof: Let $T \in \mathcal{T}$. That means $\rho(T) + \rho(\overline{T}) = |L|$, and $\rho(T) = \rho(\overline{T}) = |L|/2$. Let $V_1, V_2, \ldots, V_{\rho(T)}$ be an induced path partition for T. We know that each V_i contains exactly two leaves of T.

Claim: $|V_i| = k \leq 4$. Suppose $V_i = \{x_1, x_2, x_3, ..., x_k\}$ where $k \geq 5$ and $x_j x_{j+1} \in E(T)$ for $1 \leq j \leq k-1$. Then by Lemma 5, x_3 is adjacent to a leaf in T, say z, and $z \in V_t$ where $i \neq t$. But then V_t cannot contain two leaves. Thus, $|V_i| \leq 4$.

Claim: $|V_i| \geq 3$. Suppose $|V_i| = 2$. Then V_i consists of just two leaves, which cannot induce a path in T, unless $T = P_2$. But if $T = P_2$, then $T \notin \mathcal{T}$. Thus $|V_i| \geq 3$.

Thus, each V_i induces a P_3 or P_4 in T, each contains two leaves, so that T is constructed by joining up with edges the non-leaves of the P_3's and P_4's. Thus, $T \in \mathcal{C}_3 \cup \mathcal{C}_4$. ■

We conclude with an upper bound on the sum of the path partition number of a tree and its complement.

Theorem 20 *For any tree T of order $n \geq 3$,*

$$\rho(T) + \rho(\overline{T}) \leq \left\lceil \frac{3\,(n-1)}{2} \right\rceil .$$

Proof:

Suppose $n = 3$. Then $\{T, \overline{T}\} = \{P_3, \overline{P_3}\}$ and $\rho\,(T) + \rho\left(\overline{T}\right) = 1 + 2 = 3$ and the result holds.

Now suppose $n \geq 3$. Since T is connected, T must contain an induced P_3. Then $\rho(T) + \rho(\overline{T}) \leq \rho(P_3) + \rho(\overline{P_3}) + \rho(T - P_3) + \rho(\overline{T - P_3})$.

Note that $T - P_3$ may not be connected. By Theorem 15, $\rho(T - P_3) + \rho(\overline{T - P_3}) \leq \left\lceil \frac{3(n-3)}{2} \right\rceil$, and clearly $\rho(P_3) + \rho(\overline{P_3}) = 3$.

Case 1: n is odd. Then $\rho(T) + \rho(\overline{T}) \leq 1 + 2 + \frac{3(n-3)}{2} = \frac{3(n-1)}{2}$.

Case 2: n is even. Note that $\left\lceil \frac{3(n-1)}{2} \right\rceil = \frac{3n-2}{2}$. Then $\rho(T) + \rho(\overline{T}) \leq 1 + 2 + \left\lceil \frac{3(n-3)}{2} \right\rceil = 3 + \frac{3(n-3)+1}{2} = \frac{3n-2}{2}$. ■

This bound is sharp. If $T = K_{1,n-1}$, then $\overline{T} = K_1 \cup K_{n-1}$ and $\rho\,(T) + \rho\left(\overline{T}\right) = (n-2) + \left(1 + \lceil \frac{n-1}{2} \rceil\right) = n - 1 + \lceil \frac{n-1}{2} \rceil = \left\lceil \frac{3(n-1)}{2} \right\rceil$.

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