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On totally magic injections

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Abstract

A *totally magic injection* of a graph with v vertices and e edges is a one-to-one map taking the vertices and edges into the positive integers, such that the sum h of the label on a vertex and the labels on its incident edges is a constant independent of the choice of vertex, and the sum k of an edge label and the labels of the endpoints of the edge is constant. This is a generalization of a *totally magic labeling*, which maps onto the integers $1, 2, \cdots, v+e$, and arises in the attempt to find totally magic labelings. In this paper we explore the existence and properties of totally magic injections, and prove the existence of a new small totally magic injection.

1 Introduction

All graphs in this paper are finite, simple and undirected. Unless otherwise specified, the graph G has vertex set $V = V(G)$ and edge set $E = E(G)$ and we write e for $|E|$ and v for $|V|$.

For a given graph G, suppose λ is a map from $V(G) \cup E(G)$ to the integers. We define the weight of vertex x as

$$wt(x) = \lambda(x) + \sum_{y \sim x} \lambda(xy) \tag{1}$$

and define the weight of edge xy as

$$wt(xy) = \lambda(x) + \lambda(xy) + \lambda(y). \tag{2}$$

A *labeling* on a graph G is a one-to-one map from $V(G) \cup E(G)$ onto the integers $1, 2, \ldots, v+e$. A labeling is called *edge-magic* if all edges have the same weight, *vertex-magic* if all vertices have the same weight, and *totally magic* if it is both vertex-magic and edge-magic. In other words, a *totally magic labeling* λ on a graph G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \ldots, v+e$, with the property that, given any vertex x,

$$wt(x) = h$$

and given any edge xy,

$$wt(xy) = k$$

for some constants h and k. A graph having a totally magic labeling is called a *totally magic graph.*

Totally magic labelings have been discussed in [3]. It is shown in that paper that totally magic graphs are very rare. The only known infinite families consist of the unions of an odd number of triangles, mK_3, where m is odd, and the same graphs with precisely one edge deleted. (On the other hand, the members of these families with m even are never totally magic.) Only two other totally magic graphs are known: the single vertex K_1, and the graph obtained by appending an isolate to the two-edge path, $P_3 \cup K_1$. (We label paths by their number of vertices.)

2 Totally magic injections

A magic injection of G is a one-to-one mapping from the elements of G to the positive integers which has a magic property. In other words a magic injection is like a magic labeling, but the condition that the labels be consecutive has been removed. One can discuss edge-magic, vertex-magic or totally magic injections. It is easy to see that every graph has an edge-magic injection, and every graph has a vertex-magic injection except for those that have a component K_2 or two components K_1 (see, for example, [5]).

If the largest label used in an injection is m, we call m the *size* of the injection. The *deficiency* of an injection on a graph G is $m - v(G) - e(G)$, and the deficiency $def_e(G)$ of a graph G is the minimum value of $m - v(G) - e(G)$, such that there exists an injection on G of size m. We use the terms “edge-magic deficiency,” “vertex-magic deficiency” and “totally magic deficiency” in the obvious way.

If a disconnected graph is totally magic (or has a totally magic injection), the totally magic labeling induces a totally magic injection on its components. For example, the union of an *even* number of triangles has a totally magic injection, as does the graph obtained from it by deleting one edge. So we are interested in finding totally magic injections on graphs, even if they do not admit if totally magic labelings. For this reason totally magic injections have been studied by several authors (see [2, 4, 5, 6]). In what follows, it will be convenient to refer to a graph possessing a totally magic injection as a *TMI graph.*

3 Known TMI graphs

Theorem 1 *The star $K_{1,n}$ is TMI when $n > 2$.*

Proof. To label $K_{1,n}$, label the center 1 and the edges $2, 3, \ldots, n+1$. From consideration of the center vertex a totally magic injection will have $h = \frac{1}{2}(n+1)(n+2)$. Label the outer vertex attached to edge i with $h - i$; then $k = h + 1$.

To see that this is an injection, it is only necessary to check that

$$(n+1)(n+2)/2 - n - 1 > n + 1.$$

But this is obvious when $n > 2$. □

Of course, $K_{1,2} = P_3$ is totally magic, and $K_{1,1} = K_2$ has no totally magic injection.

Suppose α is a map from the vertices and edges of a graph G to the positive integers in which every vertex has weight h and every edge has weight k, where no two vertices and no two edges have the same label. Then α is a totally magic injection unless some vertex label and edge label are equal. If we add an integer constant t to each vertex label, the new map has the same properties, so long as the sum of t and the smallest vertex label is still positive. So by a suitable choice of t we obtain a totally magic injection with vertex constant $h+t$ and edge constant $k+2t$.

In particular, suppose G is a TMI graph with no isolate and α is a totally magic injection. Select t so that no vertex X and edge Y have $\alpha(X)+t = \alpha(Y)$, and no edge has $\alpha(Y) = h+t$. (This can always be done by choosing t sufficiently large.) Then append an isolated vertex, Z say, to G. The injection β, defined by $\beta(X) = \alpha(X)+t$ for vertices of G, $\beta(Y) = \alpha(Y)$ for edges, and $\beta(Z) = h+t$, is totally magic. We have

Theorem 2 *If G is a TMI graph with no isolate then $G \cup K_1$ is also TMI.*

Combining these theorems with the results stated earlier, the known TMI graphs with 7 or fewer vertices are as follows.

	TMGs	Others!
1 vertex:	K_1	none
2 vertices:	none	none
3 vertices:	K_3, P_3	none
4 vertices:	$P_3 \cup K_1$	$K_3 \cup K_1, K_{1,3}$
5 vertices:	none	$K_{1,3} \cup K_1, K_{1,4}$
6 vertices:	none	$K_3 \cup K_3, K_3 \cup P_3, K_{1,4} \cup K_1, K_{1,5}$
7 vertices:	none	$K_3 \cup K_3 \cup K_1, K_3 \cup P_3 \cup K_1,$ $K_{1,5} \cup K_1, K_{1,6}$

4 Known forbidden configuration theorems

The following theorems are proven in [3] as results about totally magic graphs, but all are essentially *forbidden configuration* theorems. If a graph G is in violation of one of them, then not only is G not totally magic, but G cannot be a TMI graph. So we restate them in the more general form:

Theorem 3 *A TMI graph cannot contain two isolated vertices.*

Theorem 4 *A TMI graph cannot contain a K_2 as component.*

Theorem 5 *If a TMI graph has a vertex x of degree 1, the component containing x is a star.*

Theorem 6 *If a TMI graph contains two adjacent vertices of degree 2, then the component containing them is a cycle of length 3.*

Theorem 7 *Suppose G contains two vertices, x_1 and x_2, that are each adjacent to precisely the same set $\{y_1, y_2, \ldots, y_d\}$ of other vertices. (It is not specified whether x_1 and x_2 are adjacent.) If $d > 1$ then G is not TMI.*

Theorem 8 *Suppose G contains two vertices, x and y, with a common neighbor. If x and y are nonadjacent and each has degree 2, or are adjacent and each has degree 3, then G is not TMI.*

Theorem 9 *Suppose a TMI graph contains a triangle. Then the sum of the labels of all edges outside the triangle and incident with any one vertex of the triangle is the same, whichever vertex is chosen.*

5 The totally magic equation matrix

Suppose λ is a totally magic labeling of a graph G with vertices $X_1, X_2, \ldots X_v$ and edges $Y_1, Y_2, \ldots Y_e$. If the wt functions are defined as in (1) and (2), then

$$\begin{aligned} wt(X_i) - h &= 0 \quad \text{for all } i \in \{1, 2, \ldots, v\}, \\ wt(Y_j) - k &= 0 \quad \text{for all } j \in \{1, 2, \ldots, e\}. \end{aligned} \tag{3}$$

Let us write M_G for the matrix of coefficients of this system of equations. M_G is the *totally magic equation matrix* of G. For convenience, we refer to the row corresponding to the weight equation for vertex X_i or edge Y_j as "row X_i' or "row Y_j" respectively. Similarly we refer to columns by the appropriate vertex, edge or magic constant.

The problem of determining whether G is a TMI graph can thus be stated as the problem of finding positive integer solutions to

$$M_G[x_1, x_2, \ldots, x_v, y_1, \ldots, y_e, h, k]^T = 0 \tag{4}$$

where all the x's and y's are distinct (and then setting $\lambda(X_i) = x_i$, etc.).

We write M_G^- for the matrix derived from M_G by deleting the columns for h and k.

Theorem 10 *If β is a totally magic injection on a graph G, and M_G^- is invertible, then each edge label has the form*

$$\beta(Y_i) = \frac{n_i}{\det M_G^-}(2h - k), \tag{5}$$

where the n_i are integers.

Proof. The equations (4) can be interpreted as the set of equations

$$M_G^-[x_1, x_2, \ldots, x_v, y_1, \ldots, y_e]^T = b,$$

where b is the vector with its first v elements k and its other e elements h.

Given the edge Y_k joining vertices X_i and X_j, consider row Y_k of M_G^-. It contains 1's in columns Y_k, X_i and X_j and 0's elsewhere. Rows X_i and X_j will contain 1's in columns X_i and X_j respectively and in the columns of edges incident with the relevant vertex. Subtracting rows X_i and X_j from row Y_k and then negating yields a row with 0 in each vertex column, 1 in each column representing an edge incident with X_i and X_j (or both), and 0 in the other edge columns. The right hand will be $2h - k$. If we repeat for all edge rows, we finish with the last e equations in the form $\sum y_i = 2h - k$.

If we continue Gaussian elimination until the first $v + e$ columns form an identity matrix (possible because M_G^- is invertible), only edge rows will be added to the edge rows, so the right-hand sides of the final e equations will all be multiples of $2h - k$. Cramer's rule tells us that the multiplier must be an integer multiple of the inverse of $\det M_G^-$. □

In the process of proving Theorem 10, we in fact showed

Corollary 10.1 *If X_iX_j is an edge of a TMI graph, the sum of the weights of all edges incident with X_i or X_j or both is $2h - k$.*

If G is regular, an analogous proof applies to the vertex weights.

Theorem 11 *If β is a totally magic injection on a regular graph G of degree d, and M_G^- is invertible, then each vertex label has the form*

$$\beta(X_i) = \frac{m_i}{\det M_G^-}(dk - h), \tag{6}$$

where the m_i are integers.

6 Survivors on seven vertices

In [3] it was observed that the only graphs with fewer than seven vertices that were not eliminated by the above theorems were those we have already listed in Section 2. There were 42 connected graphs on 7 vertices, other than the star, not eliminated by the theorems; these were called *survivors* in [3], and were eliminated by computer-aided techniques. Those techniques did not rule out the possibility that the graphs might have totally magic injections. We examine these 42 graphs. To identify the graphs, we use the labeling in the database of graphs on seven or fewer vertices given in [1].

$$\begin{aligned} x_i &= \frac{h_i}{\det M_G^-}h + \frac{k_i}{\det M_G^-}k, 1 \le i \le v, \\ y_j &= \frac{n_j}{\det M_G^-}(2h - k), 1 \le j \le k. \end{aligned}$$

For a graph G, we write

$$\begin{aligned} P_G^h &= [h_1, h_2, \ldots, h_v], \\ P_G^k &= [k_1, k_2, \ldots, k_v] \text{ and} \\ N_G &= [n_1, n_2, \ldots, n_e]. \end{aligned}$$

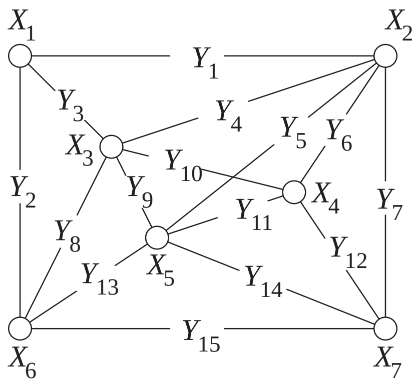

Figure 1: The graph G_{1200}

Figure 1 shows G_{1200}, in the notation of [1]. We found that this is a TMI graph. It has $\det M^-_{G_{1200}} = -789$, and satisfies

$$\begin{aligned} P^h_{G_{1200}} &= [359, -7, 64, 169, 97, 47, 161], \\ P^k_{G_{1200}} &= [-574, -391, -410, -479, -443, -418, -475] \text{ and} \\ N_{G_{1200}} &= [-176, -203, -195, -12, -45, -81, -77, -39, -64, \\ &\qquad -100, -133, -165, -72, -129, -104]. \end{aligned}$$

To ensure that all labels are integers, we need $2h-k \equiv 0 (\bmod 789)$. If $2h-k \leq 0$ the edge-labels will be non-positive, so we try the minimum feasible value, $2h-k = 789$. Then $h = k = 789$ is an obvious solution, and others can be found by adding t to each vertex label, giving $h = 789+t$ and $k = 789+2t$; the edge labels are unchanged. The smallest feasible value is $t = -214$, which gives $x_1 = 1$ (x_1 is the smallest vertex label), but in this case $x_7 = y_{10} = 100$. Similarly, $t = -213$ forces $x_5 = y_{11} = 133$. The next case, $t = -212$, gives a minimal totally magic injection with $h = 577$, $k = 365$ and deficiency 181. It is shown in Figure 2.

It remains to show that there are no further TMI graphs of order 7. In Table 1 we exhibit the vector N_G for the remaining 41 graphs. In each case it is seen that every case N_G contains two equal entries (which implies that two edge labels are equal), a zero (implying a zero edge label) or both positive and negative entries (so that there must be a negative edge label). So we have:

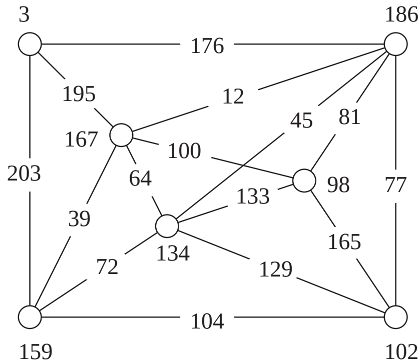

Figure 2: A totally magic injection for G_{1200}

Theorem 12 *There are precisely five TMI graphs with seven vertices, of which two (G_{1200} and $K_{1,6}$) are connected.*

n	N_{G_n}	$\det M^-_{G_n}$
839	$[0, 3, 3, 0, -6, -3, -6, 3, 6, 6, 3]$	3
964	$[21, 21, 21, 0, 0, 0, 21, 21, 0, 21, 0, 0]$	63
971	$[15, 15, 6, 6, 9, 9, 0, 0, 9, 12, 12, 3]$	45
980	$[7, 21, 12, 1, 4, -6, -2, 9, 27, -10, -1, 8]$	39
981	$[15, -6, 2, -1, 10, -3, 2, 1, 4, 12, 9, -5]$	21
983	$[-1, -2, 3, 3, -2, -3, -3, 2, 2, 2, 2, -3]$	1
1000	$[15, 6, 6, 15, 18, 9, 18, 21, 12, 12, 21, 15]$	81
1005	$[0, 9, 0, 9, 0, 9, 9, 0, 9, 9, 0, 0]$	27
1050	$[30, 26, -1, -21 - 6, 5, -2, 7, 24, -8, 18, 9, -22]$	33
1052	$[-22, -34, 24, 42, 14, -36, 2, -26, -2, 30, -20, 36, 12]$	24
1064	$[-28, -22, 30, 18, 8, 14, 24, 18, -24, -12, -2, -20, -8]$	6
1090	$[-6, -14, -8, 12, -10, -4, 4, -12, -6, -10, 6, -4, -12]$	−30
1!093	$[-20, 10, -15, 10, -20, -35, -30, -30, -35, -25, 5, 5, -25]$	−95
1095	$[-24, -12, -24, -6, 18, -6, -18, -30 - 30, -18, 6, 6, 18]$	−54
1096	$[0, 18, 0, 18, 0, 0, 18, 0, 0, 0, 18, 18, 18]$	54
1100	$[-21, -7, -7, -7, -7, -21, 21, 0, 0, 21, -14, -14, 7]$	−21
1101	$[-21, -7, -7, -21, -35, -35, -28, -14, -14, -14, -14, -28, -7]$	−119
1104	$[-9, -15, -9, -12, -12, -9, -15, -12, -12, -9, -15, -15, -21]$	−81
1146	$[-70, 15, 15, -70, -90, -5, -90, -30, -80, -80, 30, -45, -55, -55]$	−255
1149	$[-48, -48, 36, -24, -60, -60, -72, 12, 12, -36, -48, -48, -72, 0]$	−216
1150	$[-48, -48, -96, -12, -12, -96, -48, -60, -60, -96, -12, -12, -96, -48]$	−360
1153	$[-36, -21, -78, 21, -18, 45, -48, 36, -60, 3, -36, -3, -66, -33]$	−135

n	N_{G_n}	$\det M^-_{G_n}$
1154	[−60, −20, −60, 0, −20, −100, −80, −80, −40, −100, −40, −20, −40, −40]	−340
1155	[−84, −63, −63, −!63, −63, −84, 35, 14, −35, −35, 14, −56, −56, −7]	−259
1156	[−63, −14, −14, −63, −70, −21 − 70, −77, −28, −28, −28, −28, −77, −35]	−301
1159	[−33, −24, −24, −33, −39, −18, −18, −18, −18, −39, −9, −30 − 30, −51]	−189
1166	[−76, −32, −60, −28, −32, −60, −28, −48, −12, −20, −16, −68, −100, −64]	−316
1167	[−40, −60, −60, −40, −45, −65, −45, −35, −35, −85, −15, −15, −85, −75]	−345
1168	[−18, −36, −24, −66, −66, −24, −36, −60, −30, −18, −30, −42, −42, −54]	−270
1170	[−49, −49, −49, −49, −49, −49, −49, −49, −49, −49, −49, −49, −49, −49]	−343
1186	[0, −200, 50, −200, 0, −200, −150, −150, −200, 50, −150, 50, −200, −200, 0]	−700
1189	[−200, −152, −162, 30, −24, −128, 78, −154, 14, −90, −208, −144, 24, −144, −80]	−636
1190	[−96, −96, −192, −48, 0, −48, −240, −96, −144, −144, −48, 0, −240, −192, −96]	−816
1194	[−144, −183, −195, −24, −60, −60, −75, −21, −153 − 21 − 189, −57, −72, −189, −72]	−741
1199	[−200, −150, −200, −30, −30, −80, −120, −70, −30, −70, −120, −80, −160, −120, −120]	−780
1205	[−175, −100, −100, −175, −25, −100, ! − 25, −225, −225, −25, −100, −225, −100, −100, −175]	−925
1207	[−48, −48, −48, −24, −24, −48, −48, −24, −24, −24, −24, −48, −48, −48, −48]	−288
1209	[−161, −140, −105, −105, −105, −105, −140, −49, −84, −112, −112, −84, −147, −147, −119]	−861
1222	[−400, −240, −180, −320, −320, −180, −240, −160, −100, −160, −20 − 420, −20, −480, −480, −80]	−1880
1228	[−336, −240, −384, −180, −180, −384 − 240, −228, −84, −84, −228, −288, −288, −288, −288, −144]	−1944
1232	[−192, −216, −216, −168, −288, −312, −312, −288, −168, −264, −264, −264, −264, −144, −168, −168]	−1872

Table 1: The vector N_G and $\det M^-_G$ for 41 survivors G

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