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Metamorphosis of 2-fold 4-cycle systems into maximum packings of 2-fold 6-cycle systems

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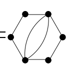
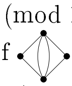
Abstract

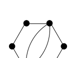
Let $c^* =$  . If we remove the double edge the result is a 6-cycle. Let (X, C) be a 2-fold 4-cycle system without repeated 4-cycles and (X, C^*) a pairing of the 4-cycles into copies of c^*. If C_1^* is the collection of 6-cycles obtained by removing the double edges from each copy of c^* and C_2^* is a reassembly of these double edges into 6-cycles, then $(X, C_1^* \cup C_2^*)$ is a 2-fold 6-cycle system. We show that the spectrum for 2-fold 4-cycle systems of order n having a *metamorphosis* into a 2-fold 6-cycle system as described above is precisely the set of all $n \equiv 0$, 1, 9, or 16 (mod 24). This can be extended to maximum packings of $2K_n$ with 6-cycles.

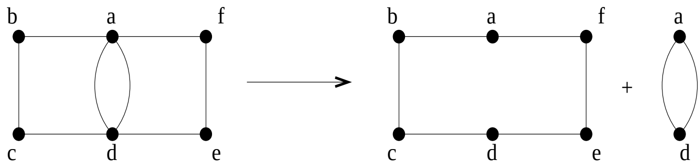
1 Introduction

A λ-fold m-cycle system of order n is a pair (S, C), where C is a collection of edge disjoint m-cycles which partitions the edge set of λK_n (n vertices, each pair joined by λ edges). We will denote the m-cycle consisting of the edges $\{x_1, x_2\}$, $\{x_2, x_3\}$, $\{x_3, x_4\}$, ...,$\{x_{m-1}, x_m\}$, $\{x_m, x_1\}$ by any cyclic shift of $(x_1, x_2, ..., x_m)$ or $(x_1, x_m, x_{m-1}, ..., x_3, x_2)$.

In [1], M. Gionfriddo and C.C. Lindner solved the problem of constructing 2-fold 3-cycle systems having a *metamorphosis* into 2-fold 4-cycle systems.

To be specific, they constructed for every $n \equiv 0$,1,4 or 9 (mod 12) ≥ 9 a 2-fold 3-cycle system so that the triples could be paired into copies of  with the property that the double edges could be removed and organized into 4-cycles. The object of this paper is the generalization of this result to 2-fold 4-cycle systems. Some specifics are in order.

In what follows we will call the graph  a dexagon. Notice that a dexagon consists of two 4-cycles with a common edge and six distinct vertices. (If we remove the double edge from each dexagon the result is a 6-cycle.)



Let (X, C) be a 2-fold 4-cycle system containing an even number of 4-cycles and let D be a pairing of the 4-cycles into dexagons. Denote by E the collection of double edges belonging to the dexagons in D. If we remove the double edges from each dexagon the result is a collection of 6-cycles D^*. If the collection of double edges E can be arranged into a collection of 6-cycles E^*, then $(X, D^* \cup E^*)$ is a 2-fold 6-cycle system and is said to be a *metamorphosis* of the 2-fold 4-cycle system (X, C) into a 2-fold 6-cycle system. The following is an example of a 2-fold 4-cycle system of order 9 having a metamorphosis into a 2-fold 6-cycle system of order 9.

Example 1.1 Let $< a, b, c, d, e, f >$ denote the graph that consists of the union of the 4-cycles (a, b, c, d) and (a, d, e, f). Then the following is a 2-fold 4-cycle system: $< 1, 4, 2, 6, 3, 5 >$, $< 1, 0, 4, 2, 5, 3 >$, $< 2, 0, 8, 3, 7, 5 >$, $< 3, 8, 1, 4, 6, 0 >$, $< 4, 6, 8, 5, 0, 7 >$, $< 5, 7, 0, 6, 8, 1 >$, $< 1, 0, 2, 7, 6, 3 >$, $< 7, 6, 2, 8, 0, 3 >$, $< 8, 5, 0, 4, 7, 2 >$

The double edges can be arranged into the 6-cycles: $(1, 2, 3, 4, 5, 6)$, $(1, 2, 3, 4, 8, 7)$ and $(1, 6, 5, 4, 8, 7)$ □

We will construct for every $n \equiv$ 0,1,9,16 (mod 24) ≥ 9 a 2-fold 4-cycle system having a metamorphosis into a 2-fold 6-cycle system. This is the exact analog of the metamorphosis problem for 2-fold 3-cycle systems. We will prove even more!

A *packing* of a $2K_n$ with 6-cycles is a triple (X, C, L), where C is a collection of edge disjoint 6-cycles of the edge set of $2K_n$ with vertex set X, and L is the set of unused edges called the *leave*. It follows that $E(2K_n) = E(C) \cup L$. If $|C|$ is as large as possible, (X, C, L) is said to be a maximum packing of $2K_n$ with 6-cycles. When $|L| = \emptyset$ we have the definition of a 2-fold 6-cycle system and it is well known that the spectrum for 2-fold 6-cycle systems is the set of all $n \equiv 0$ or $1 (\mathrm{mod}\ 3) \geq 6$ [2]. When $n \equiv 2$ (mod 3) the leave of a maximum packing is a double edge. The object of this paper is the construction of a 2-fold 4-cycle system having a metamorphosis into a maximum packing of $2K_n$ with 6-cycles for every possible $n \equiv 0$ or 1 (mod 4)(The spectrum for 2-fold 4-cycle systems [3])

The 2-fold 4-cycle system (X, C) is said to have a metamorphosis into the maximum packing (X, C^*, L^*) of $2K_n$ with 6-cycles provided it is possible to pair up the 4-cycles in C into dexagons with at most one 4-cycle left over, remove the double edges from the dexagons, and rearrange the deleted edges plus (possibly) the left over 4-cycle into a collection of 6-cycles or a collection of 6-cycles and a double edge.

The resulting maximum packing is said to be a metamorphosis of the 2-fold 4-cycle system we started with. The algorithm of constructing the maximum packing from the 2-fold 4-cycle system is also called a metamorphosis.

2 Metamorphosis of Bipartite graphs

Theorem 2.1 *If* $3|lm$, $m, n \geq 3$, *then* $2K_{2l,2m}$ *has a* 4*-cycle decomposition having a metamorphosis into a* 6*-cycle decomposition of* $2K_{2l,2m}$.

Proof: Let $G = 2K_{2l,2m}$ have parts $\{(x,0)|0 \leq x \leq 2l-1\}$ and $\{(x,1)|0 \leq x \leq 2m-1\}$. Also let $\alpha = (0,1,...,l-1)$ be a derangement on the set $\{0,1,...,l-1\}$ and $\beta = (0,1,...,m-1)$ a derangement on the set $\{0,1,...,m-1\}$

Define a collection of dexagons D as follows: For each $0 \leq a \leq l-1$ and $0 \leq b \leq m-1$, $<(a,0),(b+m,1),(a+l,0),(b,1),(a\alpha+l,0),(b\beta+m,1)> \in D$

We begin by proving that each edge occurs twice in a dexagon of D. Let $((x,0),(y,1))$ be an edge in G.

Case 1 : Let $x \leq l-1$ and $y \leq m-1$. Then $<(x,0),(y+m,1),(x+l,0),(y,1),(x\alpha+l,0),(y\beta+m,1)>$ contains the double edge $((x,0),(y,1))$.

Case 2 : Let $x \geq l$ and $y \geq m$. Then $<(x-l,0),(y,1),(x,0),(y-m,1),((x-l)\alpha+l,0),((y-m)\beta+m,1)>$ and $<((x-l)\alpha^{-1},0),((y-m)\beta^{-1}+m,1),((x-l)\alpha^{-1}+l,0),((y-m)\beta^{-1},1),(x,0),(y,1)>$ contain the edge $((x,0),(y,1))$.

Case 3 : Let $x \leq l-1$ and $y \geq m$. Then $<(x,0),(y,1),(x+l,0),(y-m,1),(x\alpha+l,0),((y-m)\beta+m,1)>$ and $<(x,0),((y-m)\beta^{-1}+m,1),(x+l,0),((y-m)\beta^{-1},1),(x\alpha+l,0),(y,1)>$ contain the edge $((x,0),(y,1))$.

Case 4 : Let $x \geq l$ and $y \leq m$. Then $<(x-l,0),(y+m,1),(x,0),(y,1),((x-l)\alpha+l,0),(y\beta+m,1)>$ and $<(x-l)\alpha^{-1},0),(y+m,1),((x-l)\alpha^{-1}+l,0),(y,1),(x,0),(y\beta+m,1)>$ contain the edge $((x,0),(y,1))$.

It follows that each edge occurs twice. The deleted double edges form a copy of $2K_{l,m}$ (with parts $\{(x,0)|0 \leq x \leq m-1\}$ and $\{(y,1)|0 \leq y \leq l-1\}$). By [4] these edges can be decomposed into 6-cycles. □

3 Important Examples

Recall that $< a,b,c,d,e,f >$ denotes the union of the two 4-cycles (a,b,c,d) and (a,d,e,f). So when we remove the double edge from this union we will have the 6-cycle (a,b,c,d,e,f) and the double edge (ad).

Example 3.1 (The metamorphosis of a 2-fold 4-cycle system of order 8).
$<1,4,6,2,5,3>$, $<2,7,6,3,0,4>$, $<3,5,7,4,2,0>$, $<4,6,0,5,1,7>$, $<5,7,3,6,0,1>$, $<6,7,3,1,0,2>$, $<7,1,4,0,5,2>$.

The double edges can be arranged into the 6-cycles $(1,2,3,4,5,6)$ and $(1,2,3,4,5,6)$ and the double edge (07). □

Example 3.2 (The metamorphosis of a 2-fold 4-cycle system of order 12).
$<1,4,6,2,5,3>$, $<2,7,6,3,8,4>$, $<3,5,7,4,2,8>$, $<4,6,8,5,1,7>$, $<5,7,3,6,8,1>$, $<6,7,3,1,8,2>$, $<7,1,4,8,5,2>$, $<7,9,6,11,2,0>$, $<8,11,9,10,7,0>$, $<10,2,9,0,1,11>$, $<4,10,6,9,8,0>$, $<5,9,1,0,11,10>$, $<3,9,7,10,6,0>$, $<3,9,8,11,6,0>$, $<4,10,9,11,2,0>$, $<5,9,0,11,1,10>$, and the 4-cycle $(1,10,2,9)$. The 6-cycles obtained by rearranging the edges of the left over

4-cycle and the double edges are: $(1,2,3,4,5,6)$, $(1,2,3,4,5,6)$, $(1,10,3,11,4,9)$, $(4,9,2,10,3,11)$, $(10,0,5,11,7,8)$,
$(5,0,10,8,7,11)$. □

Example 3.3 (The metamorphosis of a 2-fold 4-cycle system of order 13)
$<1,4,6,2,5,3>$, $<2,7,6,3,8,4>$, $<3,5,7,4,2,8>$, $<4,6,8,5,1,7>$,
$<5,7,3,6,8,1>$, $<6,7,3,1,8,2>$, $<7,1,4,8,5,2>$, $<10,9,7,12,5,0>$,
$<3,9,8,11,2,10>$, $<2,11,6,9,0,12>$, $<3,9,11,0,5,12>$,
$<2,10,9,0,11,12>$, $<1,9,5,10,6,12>$, $<1,9,12,0,8,11>$,
$<7,10,8,0,6,12>$, $<8,9,4,12,3,10>$, $<7,9,12,11,5,10>$,
$<4,0,10,11,1,12>$, $<4,0,6,10,11,9>$, and the 4-cycle $(5,9,6,11)$. The 6-cycles obtained by rearranging the edges of the left over 4-cycle and the removed double edges are : $(1,2,3,4,5,6)$, $(1,2,3,4,5,6)$, $(1,10,4,11,7,0)$, $(2,9,6,11,3,0)$, $(10,12,8,7,11,4)$, $(8,12,10,1,0,7)$, $(2,9,5,11,3,0)$. □

Example 3.4 (The metamorphosis of a 2-fold 4-cycle system of order 16)
Let $S=\{0,1,2,...,7\}\times\{1,2\}$ and let α be the permutation $\alpha=(0123)$. For each $i\in\{1,2\}$ define the following collection of dexagons:

$<(7,i),(1,i),(6,i),(2,i),(5,i),(4,i)>$, $<(2,i),(0,i),(6,i),(4,i),(3,i),(1,i)>$,
$<(4,i),(5,i),(0,i),(1,i),(2,i),(3,i)>$, $<(1,i),(6,i),(3,i),(5,i),(7,i),(0,i)>$,
$<(5,i),(0,i),(4,i),(6,i),(3,i),(7,i)>$, $<(6,i),(0,i),(4,i),(7,i),(3,i),(2,i)>$,
$<(0,i),(7,i),(1,i),(3,i),(5,i),(2,i)>$,

and for each $0\leq i,j\leq 3$ the dexagons: $<(i,1),(j+4,2),(i+4,1),(j,2),(i\alpha+4,2),(j\alpha+4,2)>$.

The eight 6-cycles obtained from the double edges are: for each $1\leq k\leq 2$ and $i\in Z_4$ (sums are calculated (mod 4)) : $((4,k),(2,k),(7,k),(6,k),(5,k),(1,k))$, $((0+i,1),(0+i,2),(1+i,1),(3+i,2),(3+i,1),(1+i,2))$, $((0,1),(1,2),(2,1),(3,2),(0,2),(3,1))$, and $((0,1),(3,2),(0,2),(1,1),(2,2),(3,1))$. □

Example 3.5 (The metamorphosis of a 2-fold 4-cycle system of order 17).
$<1,4,6,2,5,3>$, $<2,7,6,3,8,4>$, $<3,5,7,4,2,8>$, $<4,6,8,5,1,7>$,
$<5,7,3,6,8,1>$, $<6,7,3,1,8,2>$, $<7,1,4,8,5,2>$ $<10,9,12,13,11,4>$,
$<10,9,13,11,5,12>$, $<3,12,9,11,4,13>$, $<1,12,6,9,5,11>$,
$<1,12,4,10,5,13>$, $<2,10,5,9,4,13>$, $<3,13,6,10,7,9>$,
$<7,9,4,12,5,13>$, $<8,9,6,13,7,11>$, $<8,9,3,12,2,10>$,
$<1,14,4,15,7,16>$, $<2,15,5,0,6,14>$, $<2,15,4,16,6,14>$,
$<3,0,6,15,7,14>$, $<8,15,5,14,7,0>$, $<8,15,6,16,5,0>$,
$<10,7,11,16,9,14>$, $<11,8,10,0,13,14>$, $<11,16,13,15,0,14>$,
$<12,15,0,16,9,14>$, $<12,14,16,0,9,15>$, $<14,0,9,15,16,13>$,
$<10,14,16,15,13,0>$, $<11,9,13,12,10,6>$, $<2,12,6,11,1,13>$,
$<1,16,7,0,4,14>$, $<3,0,4,16,5,14>$.

The 6-cycles obtained from the double edges are:

$(1,2,3,4,5,6)$, $(1,2,3,4,5,6)$, $(3,10,1,9,2,11)$, $(3,10,1,9,2,11)$,
$(7,8,13,10,11,12)$, $(7,8,13,10,11,12)$, $(1,0,2,16,3,15)$, $(1,0,2,16,3,15)$,
$(10,15,11,0,12,16)$, $(10,15,14,8,12,16)$, $(11,15,14,8,12,0)$.

The leave is the double edge (8,16). □

Example 3.6 (The metamorphosis of a 2-fold 4-cycle system of order 24).
Let $S = \{0,1,2,...,7\} \times \{1,2,3\}$ and let α be the permutation $\alpha = (0123)$. Then for each $1 \leq i \leq 3$ define
$< (7,i),(1,i),(6,i),(2,i),(5,i),(4,i) >$, $< (2,i),(0,i),(6,i),(4,i),(3,i),(1,i) >$,
$< (4,i),(5,i),(0,i),(1,i),(2,i),(3,i) >$, $< (1,i),(6,i),(3,i),(5,i),(7,i),(0,i) >$,
$< (5,i),(0,i),(4,i),(6,i),(3,i),(7,i) >$, $< (6,i),(0,i),(4,i),(7,i),(3,i),(2,i) >$,
$< (0,i),(7,i),(1,i),(3,i),(5,i),(2,i) >$.
Also for each $0 \leq i,j \leq 3$ and $1 \leq k,m \leq 3$ define the dexagons $< (i,1),(j+4,2),(i+4,1),(j,2),(i\alpha+4,2),(j\alpha+4,2) >$. The 6-cycles obtained from the double edges are: For each $1 \leq k \leq 3$ and $i \in Z_4$ (sums are calculated (mod 4)): $((4,k),(2,k),(7,k),(6,k),(5,k),(1,k))$, $((0+i,k),(0+i,m),(1+i,k),(3+i,m),(3+i,k),(1+i,m))$, $((0,k),(1,m),(2,k),(3,m),(0,m),(3,k))$, $((0,1),(3,3),(0,2),(1,1),(2,3),(3,2))$ and $((0,3),(1,1),(2,2),(3,1),(2,3),(1,2))$. □

Example 3.7 (The metamorphosis of 2-fold 4-cycle system of order 25). Let $S = \{0,1,2,....,24\}$(where all sums are calculated (mod 25)). For each $i \in Z_{25}$: $< 0+i, 18+i, 13+i, 1+i, 12+i, 21+i >$, $< 0+i, 21+i, 12+i, 2+i, 9+i, 17+i >$ and $< 0+i, 20+i, 14+i, 3+i, 18+i, 12+i >$.

The 6-cycles obtained from rearranging the double edges are: $(0+i, 1+i, 3+i, 6+i, 5+i, 2+i)$, $i \in Z_{25}$. □

Example 3.8 (The metamorphosis of 2-fold 4-cycle system of order 20). Let $S = A \cup B$, where $A = \{(a,0)|0 \leq a \leq 7\}$ and $B = \{b,1)|0 \leq b \leq 11\}$. Put a copy of the design in Example 3.1 on A and a copy of the design in Example 3.2 on B. The edges between the vertices of A and B form a $2K_{8,12}$. By Theorem 2.1 these edges have a 4-cycle decomposition which has a metamorphosis into a 6-cycle decomposition. When we apply the necessary metamorphosis, the leave is a double edge from the decomposition on A. □

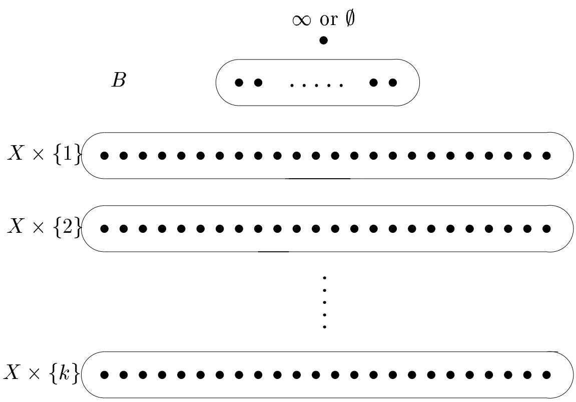
Example 3.9 (The metamorphosis of 2-fold 4-cycle system of order 21). Let $S = A \cup B \cup \{\infty\}$, where $A = \{(a,0)|0 \leq a \leq 7\}$ and $B = \{(b,1)|0 \leq b \leq 11\}$. Put a copy of the design in Example 1.1 on $A \cup \{\infty\}$ and a copy of the design in Example 3.3 on $B \cup \{\infty\}$. The edges between the vertices of A and B form a $2K_{8,12}$. By Theorem 2.1 these edges have a 4-cycle decomposition which has a metamorphosis into a 6-cycle decomposition. Now apply the necessary metamorphosis. □

Example 3.10 (The metamorphosis of 2-fold 4-cycle system of order 28). Let $S = A \cup B$, where $A = \{(a,0)|0 \leq a \leq 11\}$ and $B = \{b,1)|0 \leq b \leq 15\}$. Put a copy of the design in Example 3.2 on A and a copy of the design in Example 3.4 on B. The edges between the vertices of A and B form a $2K_{12,16}$. By Theorem 2.1 these edges have a 4-cycle decomposition which has a metamorphosis into a 6-cycle decomposition. Now apply the necessary metamorphosis. □

Example 3.11 (The metamorphosis of 2-fold 4-cycle system of order 29). Let $S = A \cup B \cup \{\infty\}$, where $A = \{(a,0)|0 \leq a \leq 11\}$ and $B = \{b,1)|0 \leq b \leq 15\}$. Put a copy of the design in Example 3.2 on $A \cup \{\infty\}$ and a copy of the design in Example

3.5 on $B \cup \{\infty\}$. The edges between the vertices of A and B form a $2K_{12,16}$. By Theorem 2.1 these edges have a 4-cycle decomposition which has a metamorphosis into a 6-cycle decomposition. When we apply the necessary metamorphosis, we will have a double edge leave from the decomposition on $B \cup \{\infty\}$. □

4 Main Construction



$24K + |B| + i$ Construction, where $i \in \{0, 1\}$

The $\mathbf{24K + |B|}$ construction, $\mathbf{|B| \in \{0, 8, 12, 16, 20, 28\}}$.

Let $S = \{X \times Y\} \cup B$, where $X = \{1, 2, ..., 24\}$ and $Y = \{1, 2, ..., k\}$. Define the following 4-cycles.

(1) If $|B| \neq 0$ put a design on B from Section 3 of the required size.

(2) For each $i \in Y$ define a copy of the design in Example 3.6 on $X \times \{i\}$.

(3) For each $i, j \in Y$ define a copy of a 4-cycle decomposition of $2K_{24,24}$ with parts $X \times \{i\}$ and $X \times \{j\}$ having a metamorphosis into a 6-cycle decomposition of $2K_{24,24}$. This exists by Theorem 2.1.

(4) For each $i \in Y$ define a copy of a 4-cycle decomposition of $2K_{|B|,24}$ with parts B and $Y \times \{i\}$ having a metamorphosis into a 6-cycle decomposition of $2K_{|B|,24}$. This exists by Theorem 2.1.

To see that this is a 2-fold 4-cycle system of order $24K + |B|$ is straight forward. An easy calculation shows that it has the right number of 4-cycles. The type (1) 4-cycles cover each edge in B. Also each edge in $X \times \{i\}$ is used in a type (2) 4-cycle. For each $i, j \in Y$ each edge between the vertices of $X \times \{i\}$ and the vertices of $X \times \{j\}$ is covered in a type (3) 4-cycle. Finally every edge between the vertices of B and the vertices of $X \times \{i\}$ is used in a type (4) 4-cycle. Now for each type of 4-cycle use the required metamorphosis given in the related examples and Theorem 2.1. The result is a metamorphosis of a 2-fold 4-cycle system of order $24K + |B|$ into a maximum packing with 6-cycles.

Observe that the 4-cycle decomposition defined above will have an odd number of 4-cycles if and only if the decomposition on B has an odd number of 4-cycles. Also the metamorphosis will have a double edge leave if and only if the metamorphosis of B has a double edge leave.

The $\mathbf{24K + |B| + 1}$ Construction, $|\mathbf{B}| \in \{\mathbf{0, 8, 12, 16, 20, 28}\}$.

Let $S = \{X \times Y\} \cup B \cup \{\infty\}$, where $X = \{1, 2, ..., 24\}$ and $Y = \{1, 2, ..., k\}$.

(1) If $|B| \neq 0$ put a design on $B \cup \{\infty\}$ of the required size.

(2) For each $i \in Y$ define a copy of the design in Example 3.7 on $(X \times \{i\}) \cup \{\infty\}$.

(3) For each $i, j \in Y$ define a copy of a 4-cycle decomposition of $2K_{24,24}$ with parts $X \times \{i\}$ and $X \times \{j\}$ having a metamorphosis into a 6-cycle decomposition of $2K_{24,24}$. This exists by Theorem 2.1.

(4) For each $i \in Y$ define a copy of a 4-cycle decomposition of $2K_{|B|,24}$ with parts B and $Y \times \{i\}$ having a metamorphosis into a 6-cycle decomposition of $2K_{|B|,24}$. This exists by Theorem 2.1.

To see that this is a 2-fold 4-cycle system of order $24K+|B|+1$ is straightforward. An easy calculation will show that it has the right number of 4-cycles. Now the type (1) 4-cycles cover each edge in B as well as the edges between ∞ and vertices of B. For all $i \in Y$, each edge in $X \times \{i\}$ and the edges between the vertices of $X \times \{i\}$ and ∞ is used in a type (2) 4-cycle. For each $i, j \in Y$ each edge between the vertices of $X \times \{i\}$ and the vertices of $X \times \{j\}$ is covered in a type (3) 4-cycle. Finally every edge between the vertices of B and the vertices of $X \times \{i\}$ is used in a type (4) 4-cycle. Now for each type of 4-cycle use the required metamorphosis given in the related Examples and Theorem 2.1. The result is a metamorphosis of a 2-fold 4-cycle system of order $24K + |B| + 1$ into a maximum packing with 6-cycles.

Observe that the 4-cycle decomposition defined above will have an odd number of 4-cycles if and only if the decomposition on $B \cup \{\infty\}$ has an odd number of 4-cycles. Also the metamorphosis will have a double edge leave if and only if the metamorphosis of $B \cup \{\infty\}$ has a double edge leave.

5 General Problem

We will now use the $24K + |B|$ and $24K + |B| + 1$ Constructions to solve the general problem. Since the spectrum for 2-fold 4-cycle systems is $n \equiv 0{,}1 \pmod 4$ we will look at $n \equiv 0{,}1{,}4{,}5{,}8{,}9{,}12{,}13{,}16{,}17{,}20{,}21 \pmod{24}$ where $n \geq 8$. We have 4 cases.

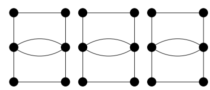
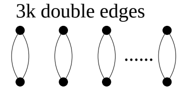
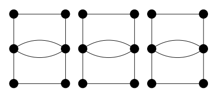
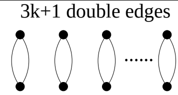
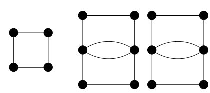
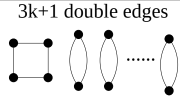
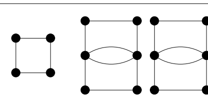
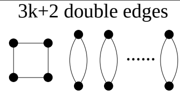
Case 1: $n \equiv 0{,}1{,}9{,}16 \pmod{24}$. Since there are an even number of 4-cycles we can pair them up. When we rearrange the removed double edges we get a 6-cycle system. (For example, Example 1.1 (n=9) and Example 3.4 (n=16) have an even number of 4-cycles and no leave in the 6-cycle decomposition.)

Case 2: $n \equiv 8{,}17 \pmod{24}$. Since there are an even number of 4-cycles we can pair them up. When we rearrange the removed double edges we get a maximum packing of $2K_n$ with 6-cycles with leave a double edge. (For example, Example 3.1 (n=8) and Example 3.5 (n=17) have an even number of 4-cycles and a double edge leave in the metamorphosis into 6-cycles.)

Case 3: $n \equiv 4{,}12{,}13{,}21 \pmod{24}$. Since there are an odd number of 4-cycles when we pair them up we have a 4-cycle left over. When we rearrange the removed double edges and the left over 4-cycle we get a 6-cycle system. (See Examples 3.2 (n=12), 3.3 (n=13), 3.9 (n=21), and 3.10 (n=28).)

Case 4: $n \equiv 5{,}20 \pmod{24}$. Since there are an odd number of 4-cycles when we pair them up we have a 4-cycle left over. When we rearrange the removed double edges and the edges of the left over 4-cycle we get a maximum packing with 6-cycles with leave a double edge. (See Examples 3.8 (n=20) and 3.11 (n=29).)

The solution to the general problem can be summarized with the following table.

n (mod 24)	Decomposition	Take out	Leave
n≡ 0,1,9,16		3k double edges 	Nothing
n≡ 8, 17		3k+1 double edges 	
n≡ 4,12,13,21		3k+1 double edges 	Nothing
n≡ 5,20		3k+2 double edges 	

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