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Lambda-fold complete graph decompositions into perfect four-triple configurations

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Abstract

The four triples or triangles $(a_8, a_1, a_2), (a_2, a_3, a_4), (a_4, a_5, a_6), (a_6, a_7, a_8)$ form a 4-cycle-triple block, and the four triangles (a_6, a_1, a_2), (a_2, a_3, a_4), (a_4, a_5, a_6), (a_6, a_7, a_8) form a kite-triple block. The "interior" of the 4-cycle-triple block consists of the 4-cycle (a_2, a_4, a_6, a_8), and the "kite interior" of the kite-triple block consists of the kite (a_2, a_4, a_6)–a_8 (consisting of a triangle with pendant edge). A 3λ-fold decomposition of K_n into 4-cycle-triple blocks (or into kite-triple blocks) such that the "interiors" of each block form a λ-fold 4-cycle system (or kite system) is said to be a *perfect* 3λ-fold decomposition of K_n. We find such perfect 4-cycle-triple and kite-triple systems for all λ and n.

1 Introduction

We begin with some notation and definitions. An undirected edge with endpoints x and y will be denoted by xy or yx for brevity. An *m-cycle* with edge set $\{a_ia_{i+1} \mid 1 \leq i \leq m-1\} \cup \{a_1a_m\}$ will be denoted by $(a_1, a_2, \dots, a_m)$ or $(a_1, a_m, a_{m-1}, \dots, a_2)$, or any cyclic shift of these.

A *kite* is the simple graph on four vertices with four edges consisting of a 3-cycle (triangle, triple) with a pendant edge. The kite with edge-set $\{a_1a_2, a_2a_3, a_3a_1, a_3a_4\}$ will be denoted by (a_1, a_2, a_3)–a_4 or (a_2, a_1, a_3)–a_4.

A *λ-fold m-cycle system* of order n is a pair (V, C), where V is the vertex set of K_n and C is a collection of edge-disjoint m-cycles which partitions the edge set of λK_n. (Here λK_n means the complete undirected graph on n vertices with λ edges joining each pair of its vertices.) When $\lambda = 1$ we usually omit "1-fold", and when $m = 3$ we use the terms 3-*cycle*, *triangle* and *triple* interchangeably. We also frequently write the 3-cycle (a_1, a_2, a_3) as $a_1a_2a_3$ for short.

A *λ-fold kite system* of order n is a pair (V, K), where V is the vertex set of K_n and K is a collection of edge-disjoint kites which partitions the edge set of λK_n.

The distance 2 graph obtained from the cycle $(a_1, a_2, \dots, a_m)$ has edge set $\{a_ia_{i+2} \mid 1 \leq i \leq m-2\} \cup \{a_{m-1}a_1,\ a_ma_2\}$.

If a λ-fold m-cycle system (V, C) has the extra property that the collection of distance 2 graphs constructed from each of the m-cycles in C is again some cycle system, then (V, C) is said to be 2-*perfect*. See [2] for a survey of cycle systems including perfect cycle systems.

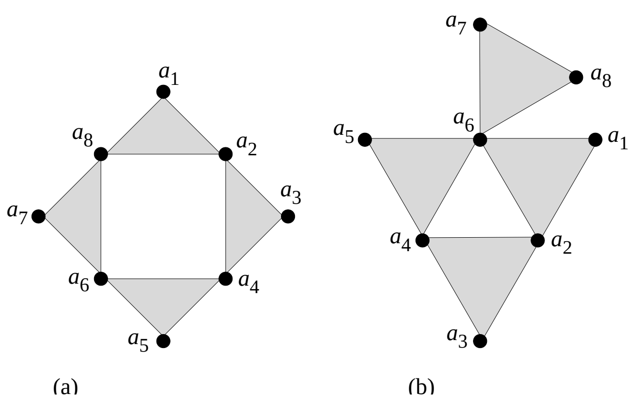

Figure 1. (a) 4-cycle-triple block and (b) kite-triple block

We shall denote the graph in Figure 1(a), consisting of the (shaded) triples $a_8a_1a_2$, $a_2a_3a_4$, $a_4a_5a_6$, $a_6a_7a_8$, by $[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8]_F$ or any cyclic 2-shift of this. Here the subscript F is for 4-cycle, and we shall refer to this configuration as a 4-*cycle-triple block*.

Likewise, the graph in Figure 1(b), consisting of the triples $a_6a_1a_2$, $a_2a_3a_4$, $a_4a_5a_6$, $a_6a_7a_8$, will be denoted by $[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8]_K$ or $[a_5, a_4, a_3, a_2, a_1, a_6, a_7, a_8]_K$ (or either of these with a_7, a_8 interchanged). We shall refer to this configuration as a *kite-triple block*.

A 3-*fold* 4-*cycle-triple system* of order n is a pair (V, F) where V is the vertex set of K_n and F is an edge-disjoint collection of 4-cycle-triple blocks which partitions the edge set of $3K_n$.

Although no perfect 4-cycle system exists (see for instance [2]), we can nevertheless make the following definition.

A *perfect* 3-fold 4-cycle-triple system of order n is a 3-fold 4-cycle-triple system of order n with the additional property that the collection of "inner" 4-cycles (a_2, a_4, a_6, a_8) in each 4-cycle-triple block $[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8]_F$ forms a (1-fold) 4-cycle system of order n.

Likewise, a 3-*fold kite-triple system* of order n is a pair (V, K) where V is the vertex set of K_n and K is an edge-disjoint collection of kite-triple blocks which partitions the edge set of $3K_n$. Furthermore, a *perfect* 3-fold kite-triple system of order n is one with the additional property that the collection of "inner" kites (a_2, a_4, a_6)–a_8 in each kite-triple block $[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8]_K$ form a (1-fold) kite system of order n. (Here the choice of a_7, a_8 does matter, to depict the kite's pendant edge.)

For brevity, we shall henceforth refer to 4-cycle-triple systems and kite-triple systems as FT systems and KT systems respectively. Furthermore, from now on, unless otherwise stated, all such systems we consider will also be *perfect*; we shall not bother to say so every time.

EXAMPLE 1.1
Let $V(K_9) = \mathbb{Z}_9$.

(a) A perfect 3-fold 4-cycle-triple system of order 9.

$[1, 0, 5, 8, 6, 4, 3, 7]_F$ (mod 9).

The triples 017, 734, 468, 850 (mod 9) cover $3K_9$, while the 4-cycle $(0, 7, 4, 8)$ (mod 9) covers K_9 precisely. See Figure 2(a).

(b) A perfect 3-fold kite-triple system of order 9.

$[3, 6, 4, 8, 7, 0, 1, 5]_K$ (mod 9).

The triples 015, 078, 846, 630 (mod 9) cover $3K_9$, while the kites $(8, 6, 0)$–5 (mod 9) cover K_9. See Figure 2(b).

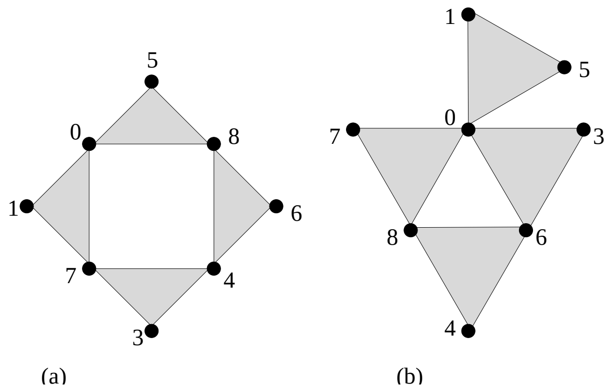

Figure 2. (a) Perfect 3-fold FT system of order 9.
(b) Perfect 3-fold KT system of order 9.

EXAMPLE 1.2
Let $V(K_9) = \mathbb{Z}_9$.

(a) A non-perfect 3-fold 4-cycle-triple system of order 9.

$[0, 8, 7, 4, 6, 2, 3, 5]_F$ (mod 9).

(b) A non-perfect 3-fold kite-triple system of order 9.

$[7, 2, 8, 1, 4, 0, 5, 6]_K$ (mod 9).

Note that in this example the inner 4-cycles and the inner kites do not form a (1-fold) 4-cycle or kite system of order 9; these systems are not perfect. (See Figure 3.)

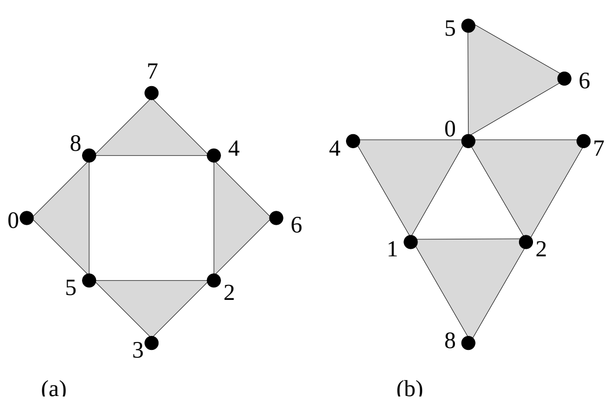

Figure 3. (a) Non-perfect 3-fold FT system of order 9.
(b) Non-perfect 3-fold KT system oforder 9.

The definition of perfect 3-fold 4-cycle-triple and kite-triple systems generalises in the obvious way to perfect 3λ-fold 4-cycle-triple and kite-triple systems, in which the "inner" 4-cycles and kites form a λ-fold 4-cycle system or a λ-fold kite system.

This paper is motivated by Küçükçifçi and Lindner [1]; that paper considers the related problem of perfect *three*-triple configurations, where *three* triangles, with each pair of triangles having a common vertex, were taken. The paper [1] defines a *hexagon triple* to be the graph on six vertices with three 3-cycles $(a, b, c), (c, d, e), (e, f, a)$, and the 3-cycle (a, c, e) is called an "inner" 3-cycle. A perfect hexagon triple system of order n is thus a pair (X, T) where X is the vertex set of K_n and T is an edge-disjoint collection of *hexagon triples* which partitions the edge set of $3K_n$ such that the inner 3-cycles from each hexagon triple form a Steiner triple system. The paper [1] solves the existence problem for λ-fold perfect hexagon triple systems, and also the related maximum packing problem (apart from two unresolved cases of order 17). Here in this paper we consider the natural generalisation from *three* triangles or triples to *four* triples.

The rest of this paper completes the determination of perfect 3λ-fold 4-cycle-triple systems and kite-triple systems for all admissible orders. We consider in turn the cases $\lambda = 1$, 2 and 4, and conclude with a summary.

2 3-fold perfect FT and KT systems

A 3-fold triple system having number of triples a multiple of 4 must have order $n \equiv 1 \pmod 8$. Since a (1-fold) 4-cycle system of order n has $n \equiv 1 \pmod 8$ and a (1-fold) kite system of order n has $n \equiv 0$ or $1 \pmod 8$, a necessary condition for either a perfect FT system or a perfect KT system is that the order n is $1 \pmod 8$.

EXAMPLE 2.1 *3-fold perfect FT and KT systems of order 17.*

Take the vertex set $\mathbb{Z}_{17}$, and starter blocks:

$$\text{FT system:}\quad [16,0,11,8,9,3,10,2]_F,\ [8,0,5,7,14,1,16,4]_F \quad (\text{mod}17).$$

$$\text{KT system:}\quad [10,9,12,7,11,0,2,4]_K,\ [7,6,9,1,5,0,11,3]_K \quad (\text{mod}17).$$

EXAMPLE 2.2(a) *A perfect FT system on the graph* $3K_{4,4,4}$.

Let $V(K_{4,4,4}) = \{1_a, 2_a, 3_a, 4_a\} \cup \{1_b, 2_b, 3_b, 4_b\} \cup \{1_c, 2_c, 3_c, 4_c\}$. Take 4-cycle-triple blocks

$$[1_c, 1_b, 4_c, 2_a, 2_c, 2_b, 3_c, 1_a]_F,\quad [4_c, 3_b, 1_c, 2_a, 3_c, 4_b, 2_c, 1_a]_F,$$
$$[2_c, 1_b, 3_c, 4_a, 1_c, 2_b, 4_c, 3_a]_F,\quad [3_c, 3_b, 2_c, 4_a, 4_c, 4_b, 1_c, 3_a]_F,$$

and eight more blocks obtained from these four via the permutation $(a\,b\,c)$ applied to the subscripts. The 4-cycle system of $K_{4,4,4}$ from the "interior" of these blocks is:

$$(1_b, 2_a, 2_b, 1_a),\ (3_b, 2_a, 4_b, 1_a),\ (1_b, 4_a, 2_b, 3_a),\ (3_b, 4_a, 4_b, 3_a)$$

(and eight more 4-cycles, using the permutation $(a\,b\,c)$ applied to the subscripts).

EXAMPLE 2.2(b) *A perfect KT system on the graph* $3K_{4,4,4}$.

Let $V(K_{4,4,4}) = \{1_a, 2_a, 3_a, 4_a\} \cup \{1_b, 2_b, 3_b, 4_b\} \cup \{1_c, 2_c, 3_c, 4_c\}$. Take kite-triple blocks

$$\begin{array}{ll}
[2_a, 3_c, 3_b, 1_a, 4_c, 2_b, 3_a, 2_c]_K, & [3_a, 1_b, 1_c, 2_a, 2_b, 4_c, 1_a, 4_b]_K, \\
[1_b, 1_c, 4_a, 4_b, 2_c, 3_a, 4_c, 3_b]_K, & [3_c, 3_b, 1_a, 2_c, 4_b, 4_a, 1_b, 4_c]_K, \\
[3_c, 4_b, 2_a, 2_c, 1_b, 1_a, 2_b, 1_c]_K, & [2_c, 2_a, 4_b, 1_c, 3_a, 3_b, 4_a, 3_c]_K, \\
[3_b, 4_c, 4_a, 2_b, 1_c, 3_a, 4_b, 3_c]_K, & [2_b, 3_c, 1_a, 1_b, 4_c, 4_a, 1_c, 4_b]_K, \\
[1_c, 3_b, 2_a, 4_c, 4_b, 1_a, 2_c, 1_b]_K, & [1_b, 3_c, 3_a, 4_b, 4_c, 2_a, 3_b, 2_c]_K, \\
[4_a, 2_c, 2_b, 3_a, 3_c, 1_b, 2_a, 1_c]_K, & [2_c, 4_a, 3_b, 1_c, 1_a, 2_b, 3_c, 2_a]_K.
\end{array}$$

Construction A

Let $n = 8s + 1$, $s \geq 1$, and take

$$V = V(K_n) = \{\infty\} \cup \{(i, j) \mid 1 \leq i \leq 2s,\ 1 \leq j \leq 4\}.$$

We take 4-cycle-triple blocks, F, or kite-triple blocks, K, as follows.

(i) If $s \equiv 0$ or $1 \pmod 3$, on each set $\{\infty\} \cup \{(2i-1, j), (2i, j) \mid 1 \leq j \leq 4\}$, for $1 \leq i \leq s$, take a perfect FT system (KT system) of order 9, and place the blocks in F (in K); see Examples 1.1(a), (b).

(i′) If $s \equiv 2 \pmod 3$, on the set $\{\infty\} \cup \{(i, j) \mid 1 \leq i \leq 4,\ 1 \leq j \leq 4\}$, take a perfect FT system (KT system) of order 17, and place the blocks in F (in K); see Example 2.1. Then repeat (i) for $3 \leq i \leq s$.

(ii) If $s \equiv 0$ or $1 \pmod 3$, take a 3-GDD of type 2^s on $\{1, 2, \ldots, 2s\}$ with groups $\{2i-1, 2i\}$ for $1 \leq i \leq s$. Then for each block xyz in the GDD, take the blocks from an FT system (KT system) on $3K_{4,4,4}$ with vertex set $\{(x, j) \mid 1 \leq j \leq 4\} \cup \{(y, j) \mid 1 \leq j \leq 4\} \cup \{(z, j) \mid 1 \leq j \leq 4\}$, and place these blocks in F (in K); see Examples 2.2(a), (b).

(ii′) If $s \equiv 2 \pmod 3$, take a 3-GDD of type $4^1 2^{s-2}$ on $\{1, 2, \ldots, 2s\}$ with groups $\{1, 2, 3, 4\}$, $\{2i-1, 2i\}$, for $3 \leq i \leq s$. Then repeat (ii) for each block xyz in the GDD.

It is straightforward to verify that the resulting systems (V, F), (V, K) are perfect 3-fold FT, KT systems of order n.

3 6-fold and 12-fold perfect FT and KT systems

A 6-fold triple system of order n exists for all orders $n \neq 2$, but we require the number of triples, $n(n-1)$, to be a multiple of 4, so $n \equiv 0$ or $1 \pmod 4$. Now both a 2-fold 4-cycle system and a 2-fold kite system also have order congruent to 0 or 1 (mod 4).

So we consider orders $n \equiv 0$ or 1 (mod 4) for the 6-fold case, and for convenience we take $n \equiv 0$, 1, 4 or 5 (mod 8), $n \geq 8$. Of course the case $n \equiv 1$ (mod 8) is dealt with by taking two copies of a 3-fold system (see Section 2).

A 12-fold triple system of order n exists for all orders $n \neq 2$, and the number of triples, $2n(n-1)$, is always a multiple of 4, so there is no restriction on n apart from the requirement that $n \geq 8$ in order to have sufficiently many vertices to form blocks.

We give a construction for all cases. Solutions for all subsequent small cases required for the construction are given in the Appendix.

Construction B

Let $n = 8s + \epsilon$, $s \geq 1$, $0 \leq \epsilon \leq 7$. Take $V = V(K_n) = H \cup \{(i,j) \mid 1 \leq i \leq 2s,\ 1 \leq j \leq 4\}$ where $H = \emptyset$ if $\epsilon = 0$, and otherwise $H = \{\infty_i \mid 1 \leq i \leq \epsilon\}$. Then we take 4-cycle-triple blocks, F, or kite-triple blocks, K, as follows, where λ is 6 or 12.

(i) If $s \equiv 0$ or 1 (mod 3), on the set $H \cup \{(1,j),(2,j) \mid 1 \leq j \leq 4\}$ take a perfect λ-fold FT system (KT system) of order $8+\epsilon$, and place the blocks in F (in K).

(i′) If $s \equiv 2$ (mod 3), on the set $H \cup \{(i,j) \mid 1 \leq i \leq 4,\ 1 \leq j \leq 4\}$, take a perfect λ-fold FT system (KT system) of order $16+\epsilon$, and place the blocks in F (in K).

(ii) If $s \equiv 0$ or 1 (mod 3), on each set $H \cup \{(2i-1,j),(2i,j) \mid 1 \leq j \leq 4\}$, for $2 \leq i \leq s$, take a perfect λ-fold FT system (KT system) of order $8+\epsilon$ with a hole of size ϵ on H, and place the blocks in F (in K).

(ii′) If $s \equiv 2$ (mod 3), on each set $H \cup \{(2i-1,j),(2i,j) \mid 1 \leq j \leq 4\}$, for $3 \leq i \leq s$, take a perfect λ-fold FT system (KT system) of order $8+\epsilon$ with a hole of size ϵ on H, and place the blocks in F (in K).

(iii) If $s \equiv 0$ or 1 (mod 3), take a 3-GDD of type 2^s on $\{1,2,\ldots,2s\}$ with groups $\{2i-1,2i\}$ for $1 \leq i \leq s$. Then for each block xyz in the GDD, take the blocks from a perfect λ-fold FT system (KT system) on $3K_{4,4,4}$ with vertex set $\{(x,j) \mid 1 \leq j \leq 4\} \cup \{(y,j) \mid 1 \leq j \leq 4\} \cup \{(z,j) \mid 1 \leq j \leq 4\}$, and place these blocks in F (in K).

(iii′) If $s \equiv 2$ (mod 3), take a 3-GDD of type $4^1 2^{s-2}$ on $\{1,2,\ldots,2s\}$ with groups $\{1,2,3,4\}$, $\{2i-1,2i\}$, for $3 \leq i \leq s$. Then repeat (iii) for each block xyz in the GDD.

The resulting systems (V,F), (V,K) are perfect 3-fold FT, KT systems of order n. Of course Construction B includes Construction A as a special case.

4 Summary

Applying the above construction, with the aid of the designs given in Sections 1, 2 and the Appendix, we now have the following result.

THEOREM 1 *A perfect 3λ-fold 4-cycle-triple system of order n, and a perfect 3λ-fold kite-triple sytem of order n exists if and only if λ and n satisfy:*

λ *(mod 4)*	*order* n
1, 3	1 *(mod 8)*, $n \geq 9$
2	0, 1 *(mod 4)*, $n \geq 8$
0	*any* $n \geq 8$

References

[1] S. Küçükçifçi and C.C. Lindner, *Perfect hexagon triple systems*, Discrete Math. **279** (2004), 325–335.

[2] C.C. Lindner and C.A. Rodger, *"Decomposition into cycles II: Cycle systems"* in *Contemporary design theory: a collection of surveys*, J.H. Dinitz and D.R. Stinson (Editors), Wiley, New York, 1992, pp 325–369.

Appendix

See `http://www.maths.uq.edu.au/~ejb/X006appendix.pdf`
or email `ejb@maths.uq.edu.au` for a copy of the appendix.

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