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Notes on the structure of support in BIB designs

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Abstract

The support of a BIB design is the set of all distinct blocks in the design. The notation $BIB(v, b_0t, r_0t, k, \lambda_0 t \mid b^*)$ is used to denote a BIB design with precisely b^* distinct blocks. We present three theorems about the structure of the support of a BIB design. Two of them are about the number and range of occurrences of points and pairs in the support. In third theorem, given a $BIB(v, b_0t, r_0t, k, \lambda_0 t \mid b^*)$ with $b > b_0$, it is shown that $b^* \geq \lceil \frac{\lceil (2b_0/\lambda_0) \rceil + 7}{2} \rceil$ and when k is not a multiple of 4, then $b^* \geq \lceil \frac{\lceil (2b_0/\lambda_0) \rceil}{2} \rceil + 4$. This result when ${b^*}_{\min} = b_0$ and $\lambda_0 = 1$ leads to the nonexistence of a BIB design with the support sizes equal to: ${b^*}_{\min} + 1$, ${b^*}_{\min} + 2$ and ${b^*}_{\min} + 3$. Also, when $v = 9$ and $k = 3$, it is shown that there is no design with support size 19, and this is a missing case in the study of triple systems (according to C.J. Colbourn and A. Rosa, Triple Systems, Oxford University Press, 1999.)

1 Introduction

Let V be a set of v elements (called points). A balanced incomplete block design $D = \mathrm{BIB}(v, b, r, k, \lambda \mid b^*)$, with positive integer variables v, b, r, k, λ and b^*, is a collection of b subsets of V (called blocks), which satisfy the following conditions:

1. Each block contains k points.
2. Each point occurs in r blocks ($r < b$) and each pair of points occurs in λ blocks ($\lambda < b$).
3. There are exactly b^* distinct (as subsets) blocks ($b^* \leq b$).

By considering the necessary relations between the above cited variables of a design D, one can reformulate the parameters as $\mathrm{BIB}(v, b_0t, r_0t, k, \lambda_0 t \mid b^*)$, where b_0, r_0, λ_0 are fixed positive integers and t is a positive integer variable. The support of the design D, denoted by D^* is the set of all distinct blocks of D, where $b^* = |D^*|$ is called the support size of D. The least possible value of b^* for a $\mathrm{BIB}(v, b_0t, r_0t, k, \lambda_0 t \mid b^*)$

is denoted by ${b^*}_{\min}$.

For a family of BIB designs with given parameters v and k, one of the pertinent mathematical problems with interesting statistical applications is to determine the set of all possible b^* [7,8,11]. In this paper, based on some structural theorems about the support of a BIB design in general, we focus on the case of triple systems ($k=3$) with $v=9$, and show that there is no BIB$(9,b,r,3,\lambda \mid b^*)$ with $b^*=13,14,15,16,17,19$ (among them the size $b^*=19$ is a missing case [1, 2, 3, 4, 5, 6, 10, 11]).

Let D be a BIB$(v,b,r,k,\lambda \mid b^*)$ design. If $b^*<b$, then D is called a BIB design with repeated blocks. Let $D^*=\{B_1,B_2,\cdots,B_{b^*}\}$ be the support of D. We denote by f_B the multiplicity or frequency of a block $B\in D^*$ in D. Let f be the greatest common divisor of frequencies of blocks, so $f=\gcd(f_{B_1},\cdots,f_{B_{b^*}})$, and let D be a BIB design with a given support D^*. Then by applying the *Trade-off Method* [9], b can be reduced by dividing it by the greatest common divisor of λ and f (note that $b=f_{B_1}+\cdots+f_{B_{b^*}}$). Therefore we usually assume that $\gcd(f,\lambda)=1$. Let $i\in \mathbb{N}$; we define the following subsets of D^*:

$$E_i \stackrel{def}{=} \{B\in D^* \mid f_B=i\}; \qquad \Delta_i \stackrel{def}{=} \{B\in D^* \mid f_B=ni, n\in \mathbb{N}\}=\bigcup_{j=in} E_j;$$

$$H_i \stackrel{def}{=} \{B\in D^* \mid f_B\geq i\}=\bigcup_{j\geq i} E_j.$$

Also, the notations ${E'}_i, {\Delta'}_i$ and ${H'}_i$ are used for their set complements in D^*, respectively.

Let $E\subseteq D^*$, where D is a BIB design with the element set V. Let α (or $\alpha\beta$) be a point (or pair) in V. By ${r^*}_E(\alpha)$ (or ${\lambda^*}_E(\alpha\beta)$), we denote the number of blocks in E containing α (or $\alpha\beta$). By $r_E(\alpha)$ (or $\lambda_E(\alpha\beta)$), we denote the sum of frequencies of blocks in E containing α (or $\alpha\beta$). When we write r_E (or λ_E), and ${r^*}_E$ (or ${\lambda^*}_E$), without any specification, then a property about all points or pairs is considered.

2 On the structure of support

The first theorem is about the number of occurrences of points and pairs in the blocks of some special subsets of a design.

Theorem 1. *Let D be a BIB$(v,b_0t,r_0t,k,\lambda_0t \mid b^*)$ design. Then we have:*

(1) ${\lambda^*}_{{\Delta'}_t}\neq 1$ *and* $t \mid \lambda_{{\Delta'}_t}$.

(2) $r^*_{{\Delta'}_t}\neq 1,2,3$ *and* $t \mid r_{{\Delta'}_t}$.

(3) $|{\Delta'}_t|=0$ or $|{\Delta'}_t|\geq 7$ *and if* $4\nmid k$, *then* $|{\Delta'}_t|=0$ or $|{\Delta'}_t|\geq 8$.

(4) ${r^*}_{E_t}\neq (r_0-1)$.

(5) *If* $\lambda_0 \mid r_0$, *then* $r^*_{E_\lambda}\neq ((r_0/\lambda_0)-1)$, *and if for a point* ${r^*}_{E_\lambda}=(r_0/\lambda_0)$, *then for all points* ${r^*}_{E_\lambda}\neq 0$.

(6) *If* $\lambda_0=2$ *or* 3 *and* $H_{2t}=\emptyset$, *then for all points* ${r^*}_{H_t}\neq (r_0-1)$.

Proof. **(1)** Since $\lambda=\lambda_0t=\lambda_{\Delta_t}+\lambda_{{\Delta'}_t}$, clearly $t\mid \lambda_{{\Delta'}_t}$ and hence $\lambda^*_{{\Delta'}_t}=0$ or

$\lambda^*_{\Delta'_t} \geq 2$.

(2) Note that $r = r_0 t = r_{\Delta_t} + r_{\Delta'_t}$. So $t \mid r_{\Delta'_t}$ and hence $r^*_{\Delta'_t} = 0$ or $r^*_{\Delta'_t} \geq 2$. The case $r^*_{\Delta'_t} = 2$ by Part (1) leads to the equality of two blocks in D^*, which is impossible. So let α be a point with $r^*_{\Delta'_t} = 3$ occurrences in blocks B_1, B_2, B_3 of Δ'_t. Since $\Delta'_t \subseteq D^*$, by Part (1) every two of these three blocks have a common point, which do not appear in the third one. Let β be a point such that $\beta \in B_1 \cap B_2$ but $\beta \notin B_3$. Hence $\alpha\beta \in B_1 \cap B_2$ and this pair does not appear elsewhere in Δ'_t. By Part (1), $t|(f_{B_1} + f_{B_2})$. Also, $t|(f_{B_1} + f_{B_2} + f_{B_3})$; consequently $t|f_{B_3}$, which is impossible, since $B_3 \in \Delta'_t$.

(3) Let $\Delta'_t \neq \emptyset$ and consider a point with maximum $r^*_{\Delta'_t}$, where by Part (2) is greater than or equal to 4. We study three cases, $r^*_{\Delta'_t} = 4, 5$ and $r^*_{\Delta'_t} \geq 6$, and show that in each case the claim holds. Let $r^*_{\Delta'_t} = 4$, so Δ'_t has at least 4 blocks, which contain at least one common point. By Parts (1), (2) and our hypotheses about $r^*_{\Delta'_t}$, no point, except the common assumed point, appears more than 2 times in these 4 blocks, and when it appears, it appears exactly 2 times. By Part (2) these points in these 4 blocks have two other occurrences in Δ'_t. Hence Δ'_t has at least 6 blocks. Considering the distinctness of the blocks 5 and 6 yields $\Delta'_t \geq 7$. If $|\Delta'_t| = 7$, then the number of different points in Δ'_t is $7k/4$ (by assumption each point appears 4 times in Δ'_t). So, if $4 \not| k$, then necessarily $|\Delta'_t| \geq 8$.
For the other two cases, $r^*_{\Delta'_t} = 5$ and $r^*_{\Delta'_t} \geq 6$, the same kind of argument implies the claim.

(4) Let $r^*_{E_t}(\alpha) = (r_0 - 1)$ for a point $\alpha \in V$. Then $r_{E'_t}(\alpha) = t$, which implies the equality of at least two blocks in D^* and this is a contradiction.

(5) Let $\alpha \in V$ be a point with $r^*_{E_\lambda}(\alpha) = ((r_0/\lambda_0) - 1)$. Since each pair appears in at most one block of E_λ, the number of joint pairs with α not appearing in E_λ is $(v-1) - ((r_0/\lambda_0) - 1)(k-1) = k-1$. Consider α, which has at least two occurrences in E'_λ (note that $r - r_{E_\lambda} = \lambda$), which leads to the common occurrences of all $k-1$ points with α in these blocks; this leads to the equality of at least two blocks in E'_λ and this is impossible in D^*. Now, let $r^*_{E_\lambda}(\alpha) = (r_0/\lambda_0)$, so $r_{E_\lambda}(\alpha) = r$ and α does not appear in E'_λ. This implies that all v points appear in E_λ or $r^*_{E_\lambda} \neq 0$ for all points.

(6) Let $\lambda_0 = 2$ and $\alpha \in V$ be a point with $r^*_{H_t}(\alpha) = r_0 - 1$. Since $\lambda = 2t$, no pair appears more than two times in H_t. Let m and n be the number of joint pairs with α appearing (respectively) in one and two of $(r_0 - 1)$ blocks in H_t which contain α. Computing the number of joint pairs with α in these blocks, we have $m + 2n = (r_0 - 1)(k - 1)$. Also all of the $v - 1$ joint pairs with α should appear in H_t, and hence clearly $m + n = v - 1$. Solving these equations simultaneously yields $m = k - 1,\ n = v - k$. The m pairs should appear in H'_t but they can make exactly one block with α. In other words, α can not appear more than one time in H_t'. Hence we cannot have $r_{H_t'}(\alpha) = r - r_{H_t}(\alpha) = r - (r_0 - 1)t = t$. The same argument holds for the case $\lambda_0 = 3$. □

The following theorem presents some nonexistence support size values.

Theorem 2. *Let D be a BIB$(v, b_0t, r_0t, k, \lambda_0 t, |\, b^*)$ design. Then we have:*

- **(i)** *If $b > b_0$, then $b^* \geq \lceil \frac{\lceil (2b_0/\lambda_0)\rceil + 7}{2} \rceil$; also when $4 \not| k$, then $b^* \geq \lceil \frac{\lceil (2b_0/\lambda_0)\rceil}{2} \rceil + 4$.*
- **(ii)** *If $H_t = \emptyset$, then $b^* \geq \lceil b_0/\lambda_0 \rceil + b_0$.*
- **(iii)** *If ${b^*}_{\min} = b_0$ and $\lambda_0 = 1$, then there do not exist BIB designs with the support sizes equal to ${b^*}_{\min} + 1$, ${b^*}_{\min} + 2$, ${b^*}_{\min} + 3$.*

Proof. **(i)** If $b > b_0$, then clearly $t > 1$; hence ${\Delta'}_t \neq \emptyset$. For otherwise we have $D^* = \Delta_t$ and this contradicts our general assumption (in the introduction) that $\gcd(f, \lambda) = 1$. In this situation at least $(\binom{v}{2} - \binom{k}{2}|\Delta_t|)$ pairs do not appear in Δ_t. By Part (1) of Theorem 1 these pairs have at least two occurrences in ${\Delta'}_t$; hence $|{\Delta'}_t| \geq \lceil 2\frac{\binom{v}{2} - \binom{k}{2}|\Delta_t|}{\binom{k}{2}} \rceil$ or $|{\Delta'}_t| \geq \lceil 2(b_0/\lambda_0)\rceil - 2|\Delta_t|$, so $b^* = |{\Delta'}_t| + |\Delta_t| \geq \lceil (2b_0/\lambda_0)\rceil - |\Delta_t|$. By Part (3) of Theorem 1, $|{\Delta'}_t| \geq 7$ (or $|{\Delta'}_t| \geq 8$, when $4 \not| k$), therefore $b^* \geq 7 + |\Delta_t|$(or $b^* \geq 8 + |\Delta_t|$ when $4 \not| k$). Summing up the inequalities for b^*, we get $b^* \geq \lceil \frac{\lceil (2b_0/\lambda_0)\rceil + 7}{2} \rceil$ (or $b^* \geq \lceil \frac{\lceil (2b_0/\lambda_0)\rceil + 8}{2} \rceil = \lceil \frac{\lceil (2b_0/\lambda_0)\rceil}{2} \rceil + 4$, when $4 \not| k$).

(ii) If $H_t = \emptyset$, then $D^* = {H'}_t$ and each pair has at least $\lambda_0 + 1$ occurrences in D^*. Hence D^* should contain at least $\binom{v}{2}$ $(\lambda_0 + 1)$ pairs. Consequently

$$b^* \geq \left\lceil \binom{v}{2}(\lambda_0 + 1)/\binom{k}{2} \right\rceil = \lceil b_0 + (b_0/\lambda_0)\rceil = b_0 + \lceil b_0/\lambda_0 \rceil .$$

(iii) Setting $b_0 = {b^*}_{\min}$ and $\lambda_0 = 1$ in Case (i), the claim is clear. □

In the following we study the range and dependence of the number of occurrences of points and pairs in the support of a design.

Proposition 3. **(i)** *Let r^* and λ^* be (respectively) the number of occurrences of a point and a pair in the support of a design $D = \mathrm{BIB}(v, b, r, k, \lambda | b^*)$; then*

$$\lceil r/h \rceil \leq r^* \leq \min\left\{ r, \left(\begin{array}{c} v-1 \\ k-1 \end{array} \right) \right\} \quad ; \quad \lceil \lambda/h \rceil \leq \lambda^* \leq \min\left\{ \lambda, \left(\begin{array}{c} v-2 \\ k-2 \end{array} \right) \right\},$$

where $h = \min\{\lambda,\ b/v\}$.

(ii) *Let A_i and B_i be, respectively, the set of points and pairs with i occurrences in D^*. Also let $a_i = |A_i|$ and $b_i = |B_i|$. Then*

$$\left\{ \begin{array}{ll} \sum_{i=r^*} i a_i & = kb^* \\ \sum_{i=r^*} a_i & = v \end{array} \right. ; \left\{ \begin{array}{ll} \sum_{i=\lambda^*} i b_i & = \binom{k}{2} b^* \\ \sum_{i=\lambda^*} b_i & = \binom{v}{2} \end{array} \right. ,$$

where r^* and λ^* vary over the range given in (i).

Proof. (i) The least number of occurrences of a point in the support of a design is obtained if it occurs in blocks with the most probable frequency, where by Mann's inequality [12] is at most equal to h, so clearly $\lceil r/h \rceil \leq r^*$. On the other hand, the least possible frequency of blocks is 1. This gives at most r occurrences of a point in D^*. Also the maximum number of times that a point occurs in D^* is just the maximum number of blocks that could be made by a specified point which is clearly equal to $\binom{v-1}{k-1}$. Therefore $r^* \leq \min\{r, \binom{v-1}{k-1}\}$. The same argument holds for λ^*.

(ii) The first equation in these two systems is obtained by computing the total number of "occurrences" of points or pairs that appear in the support. The second equation is obtained by computing the total number of points and pairs that appear in the support. □

Remark. If we apply the above systems for special subsets of D^*, then the values of the right-hand sides of the equalities and the ranges of r^* and λ^* change. This is what we go through in the next section.

3 More about BIB$(9, 12t, 4t, 3, t|b^*)$

When $v = 9$ and $k = 3$, then ${b^*}_{\min} = b_0 = 12$ and therefore by Part (i) of Theorem 2, for a design D with $b > b_0$, we have $b^* \geq 16$. In this section we consider this design and show that there is a BIB$(9, 12t, 4t, 3, t)$ with the support size $b^* = 18$ and there is no BIB$(9, 12t, 4t, 3, t)$ with the support sizes $b^* = 16, 17, 19$.
First of all, note that by Part (ii) of Theorem 2, if $E_\lambda = H_t = \emptyset$ then $b^* \geq 2b_0 = 24$. Consequently for the cases $b^* = 16, 17, 18, 19$, necessarily $E_\lambda \neq \emptyset$. Also, by Part (3) of Theorem 1, $|\Delta_t{}'| = |E_\lambda'| \geq 8$, hence $b^* = |E_\lambda| + |E_\lambda'| \geq |E_\lambda| + 8$. Setting $b^* = 16, 17, 18, 19$, we have $|E_\lambda| \leq 11$. By Part (5) of Theorem 1, no point occurs 3 times in E_λ. Now, let A, B and C be (respectively) the sets of all points with $1, 2$ and 4 occurrences in E_λ. Also, let $a = |A|, b = |B|$ and $c = |C|$. Then the first system of equations in Proposition 3 for E_λ is as follows:

$$(*) \qquad \begin{cases} a + 2b + 4c = 3|E_\lambda| = u \\ a + b + c = s, \end{cases}$$

where u varies over the set $\{3, 6, \cdots, 33\}$ (for $|E_\lambda| = 1, 2, \cdots, 11$) and s varies over the set $\{3, 4, \cdots, 9\}$. The following lemma reduces the solutions of this system to more suitable ones.

Lemma. *Consider the above system of equations. Then,*

(1) $s \leq 5 \Longrightarrow b^* \geq 21$.
(2) $s = 6 \Longrightarrow b^* \geq |E_\lambda| + 18$.
(3) $s = 7 \Longrightarrow b^* \geq |E_\lambda| + l + 14,\ l = \lceil 2a/3 \rceil$.
(4) $s = 8 \Longrightarrow b^* \geq |E_\lambda| + l + 8,\ l = \lceil l_1/3 \rceil,\ l_1 = 4a + 2b$.
(5) $s \leq 8 \Longrightarrow\ c = 0$.
(6) $6 \leq s \leq 9 \Longrightarrow b^* \geq |E_\lambda| + h,\ h = \lceil h_1/3 \rceil,\ h_1 = 6a + 4b + 8(9 - s)$.

(7) $c \geq 5 \Longrightarrow a = b = 0.$
(8) $3 \leq c \leq 4 \implies a = 0$.
(9) $c \geq 3 \implies |E_\lambda| \geq 9.$
(10) $c = 2 \implies |E_\lambda| \geq 7.$
(11) $c = 1 \implies |E_\lambda| \geq 4.$

Proof. **(1)** If $s \leq 5$, then all occurrences of at least 4 points, say $\{1, 2, 3, 4\}$, have all their occurrences in E'_λ. Each element of this set must occur in at least 8 blocks of E'_λ to fulfill the least necessary occurrences of its joint pairs. Considering the minimum possible size for D^*, this makes E'_λ to be as follows (from now on we show the blocks in a vertical triple array.)

$$E'_\lambda = \left\{ \begin{array}{cccccccccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 \\ 2 & 2 & 3 & 3 & 4 & 4 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \end{array} \right\}.$$

We see that $|E'_\lambda| \geq 20$, which implies $b^* \geq 21$ and this is beyond the scope of our investigation.

(2) Let $s = 6$; iterating the above argument leads to $|E'_\lambda| \geq 18$ or $b^* \geq |E_\lambda| + 18$. In this case only $|E_\lambda| = 1$ may give a suitable situation.

(3) Let $s = 7$. An argument similar to that described in Part (1) shows that $|E'_\lambda| \geq 14$. Putting the condition on D^* to be of minimum size, every element of the set A must occur at least 6 times in E'_λ in such a way that each element appears up to four times in the 14 given blocks of E'_λ. The remaining two occurrences of each element of A need at least $l = \lceil 2a/3 \rceil$ blocks different from the above 14 given blocks. In other words, $|E'_\lambda| \geq 14 + l$ and $b^* \geq |E_\lambda| + 14 + l$.

(4) Let $s = 8$; then clearly $|E'_\lambda| \geq 8$. All elements of A and B need (respectively) at least $4a$ and $2b$ spaces outside the minimum 8 blocks of E'_λ to fulfill the least number of occurrences for their pair joints. Therefore, we need at least $l = \lceil l_1/3 \rceil$ further blocks in E'_λ, where $l_1 = 4a + 2b$. In other words, $b^* \geq |E_\lambda| + 8 + l$.

(5) Let $c \neq 0$; then by Part (5) of Theorem 1, necessarily $s = 9$.

(6) In this case $(9 - s)$ elements have no occurrence in E_λ, so they have at least $8(9 - s)$ occurrences in E'_λ. Also, all elements of A and B need (respectively) at least $6a$ and $4b$ spaces in the blocks of E'_λ. Therefore, $b^* \geq |E_\lambda| + h$, where $h = \lceil h_1/3 \rceil$, $h_1 = 8(9 - s) + 6a + 4b$.

(7) Let $c \neq 0$; then by Part (5) of Theorem 1, $s = 9$. Now, let $c \geq 5$; then each of v points have at least 3 occurrences in E_λ to fulfill all joint pairs of elements of the set C. In other words, $a = 0$, $b = 0$.

(8) As above.

(9) Let $C = \{1, 2, 3, \cdots\}$; then the least size of E_λ is as follows:

$$E_\lambda = \left\{ \begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 3 & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{array} \right\}.$$

(10, 11) As above. □

Based on the above lemma, appropriate solutions of the system of equations (∗) are arranged vertically, in six cases:

Case	1	2	3	4	5	6
s	8	9	9	9	9	9
a	1	0	1	2	3	5
b	7	9	6	6	6	3
c	0	0	2	1	0	1
$\|E_\lambda\|$	5	6	7	6	5	5

In what follows each of these cases is studied separately to find the possible situation for the support sizes they may present.

Case 1. Let "9" be the point that has no occurrence in E_λ and so it occurs at least 8 times in E'_λ. Let $A = \{8\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$. Let (876) be one of the blocks in E_λ; then no pair of this block occurs in E'_λ. As we observed before the elements of A and B, respectively, occur at least 6 and 4 times in E'_λ. Therefore, the minimum size of E'_λ is:

$$E'_\lambda = \left\{ \begin{array}{cccccccccccccccc} 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 8 & 8 & 8 & 8 & 7 & 7 & 6 & 6 \\ 8 & 8 & 7 & 7 & 6 & 6 & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \end{array} \right\}.$$

Thus $|E'_\lambda| \geq 16$ and $b^* = |E_\lambda| + |E'_\lambda| \geq 5 + 16 = 21$, which is beyond the scope of our study.

Case 2. Let (123) be a block in E_λ. Then the least size of E_λ is:

$$E_\lambda = \left\{ \begin{array}{cccccc} 1 & 1 & 2 & 3 & . & . \\ 2 & . & . & . & . & . \\ 3 & . & . & . & . & . \end{array} \right\}.$$

If in the second, third and fourth block there exists a repeated point such as "4", then the associated blocks of four points "1, 2, 3, 4, " in E'_λ are:

$$E'_\lambda = \left\{ \begin{array}{cccccccccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \end{array} \right\}.$$

Thus $|E'_\lambda| \geq 14$ and $b^* = |E_\lambda| + |E'_\lambda| \geq 6 + 14 = 20$, which is out of our range of study. Therefore, the second, third and fourth blocks of E_λ have different points and the structure of D^* with minimum size is:

$$E_\lambda = \left\{ \begin{array}{cccccc} 1 & 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 6 & 7 \\ 3 & 5 & 7 & 9 & . & . \end{array} \right\}; \quad E'_\lambda = \left\{ \begin{array}{ccccccc} 1 & 1 & 1 & 1 & & & \\ 6 & 6 & 7 & 7 & . & . & . \\ 8 & 9 & 8 & 9 & & & \end{array} \right\};$$

forcibly filling the empty positions in E_λ leads to the contradiction of the occurrence of a common pair in E_λ and E'_λ.

Case 3. Let $A = \{9\}, B = \{8, 7, \cdots, 3\}$ and $C = \{2, 1\}$. Since all of the joint pairs of two points "1, 2" appear in E_λ and the point "9" has just one occurrence in E_λ, the structure of D^* with minimum size is:

$$E_\lambda = \left\{ \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 3 & 4 & 5 & 3 & . & . \\ 9 & 6 & 7 & 8 & 4 & . & . \end{array} \right\}; \quad E'_\lambda = \left\{ \begin{array}{cccccccc} 3 & 3 & 3 & 3 & & & & \\ 9 & 9 & . & . & . & . & . & . \\ . & . & . & . & & & & \end{array} \right\}.$$

The third and fourth block of E'_λ should be filled with the three points "5, 7, 8". But there do not exist two different pairs from the $\binom{3}{2} = 3$ possible pairs of these three points, which do not appear in E_λ. So we have a contradiction.

Case 4. Let $C = \{1\}, B = \{2, 3, \cdots, 7\}$ and $A = \{8, 9\}$. In this case the following two situations may happen: either both elements of A occur in different blocks of E_λ or both occur in one block of E_λ. The following structures I and II show these situations:

$$\mathbf{I:}\ E_\lambda = \left\{ \begin{array}{cccccc} 1 & 1 & 1 & 1 & . & . \\ 8 & 9 & . & . & . & . \\ . & . & . & . & . & . \end{array} \right\}; \quad \mathbf{II:}\ E_\lambda = \left\{ \begin{array}{cccccc} 1 & 1 & 1 & 1 & . & . \\ 8 & . & . & . & . & . \\ 9 & . & . & . & . & . \end{array} \right\}.$$

We show that structure (I) leads to a contradiction and structure (II) leads to a design with the support size $b^* = 18$. Our study here shows that this is the unique possible case for the support size 18.

I. Forcibly filling the blocks of E_λ leads to the following situation for D^*:

$$E_\lambda = \left\{ \begin{array}{cccccc} 1 & 1 & 1 & 1 & 4 & 6 \\ 8 & 9 & 4 & 5 & 5 & 7 \\ 2 & 3 & 6 & 7 & 2 & 3 \end{array} \right\}; \quad E'_\lambda = \left\{ \begin{array}{ccccccccc} 2 & 2 & 2 & 2 & 2 & 2 & & & \\ 6 & 6 & 7 & 7 & 3 & 3 & . & . & . \\ . & . & . & . & . & . & & & \end{array} \right\}.$$

Only the point "9" can occur in the third empty positions in the above given blocks of E'_λ, and this leads to the contradiction of equality of blocks in E'_λ.

II. The rest of the empty positions in blocks of E_λ should be filled with the elements of B, which needs 9 pairs of the $\binom{6}{2} = 15$ possible pairs of these elements. The remaining 6 pairs occur at least two times in E'_λ and this situation make the following structure for D^*:

$$E_\lambda = \left\{ \begin{array}{cccccc} 1 & 1 & 1 & 1 & 2 & 3 \\ 8 & 2 & 4 & 6 & 4 & 5 \\ 9 & 3 & 5 & 7 & 6 & 7 \end{array} \right\}; \quad E'_\lambda = \left\{ \begin{array}{cccccccccccc} 8 & 8 & 8 & 8 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 \\ 2 & 2 & 3 & 3 & 4 & 5 & 2 & 2 & 3 & 3 & 4 & 5 \\ 5 & 7 & 4 & 6 & 7 & 6 & 5 & 7 & 4 & 6 & 7 & 6 \end{array} \right\}.$$

Let $\lambda = t = 2$; then $D^* = E_\lambda \cup E'_\lambda$ is a design with $b^* = 18$. Furthermore, it is not difficult to deduce a contradiction by assuming a larger size for E'_λ. In other words, this case leads exactly to a design with $b^* = 18$.

Case 5. Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7, 8, 9\}$. If a pair from the elements of A occurs in a block of E_λ, then the blocks associated with the elements of that block make at least 16 blocks in E'_λ. So, $b^* \geq 16 + 5 = 21$, which is beyond the scope of our studies. Therefore the elements of A occur in different blocks of E_λ, say in the first, second and the third block. If in these three blocks there exists a point with two occurrences, such as the point 4 in three blocks (145), (246), (378), then the blocks associated with the points "1, 2, 4, 5" make at least 16 blocks in E'_λ and so $b^* \geq 16 + 5 = 21$, which is out of our range of study. Consequently, these three blocks have no point with two occurrences and the structure of E_λ is:

$$E_\lambda = \left\{ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 4 & 6 & 8 & 6 & 7 \\ 5 & 7 & 9 & 8 & 9 \end{array} \right\}$$

The least number of blocks associated with the points of the fourth (or fifth) block make at least 12 blocks in E'_λ. At least two occurrences of any element of A occur in blocks different from these 12 blocks. This causes the existence of at least three other blocks in E'_λ and so $b^* \geq 5 + 15 = 20$, which is beyond the range of our studies.

Case 6. Let $C = \{1\}, A = \{2, 3, 4, 5, 6\}$ and $B = \{7, 8, 9\}$. Clearly we have E_λ as follows

$$E_\lambda = \left\{ \begin{array}{ccccc} 1 & 1 & 1 & 1 & 7 \\ 7 & 8 & 9 & 2 & 8 \\ . & . & . & 3 & 9 \end{array} \right\}.$$

The least number of blocks associated with the points "2, 3" causes the existence of at least 12 blocks, six blocks associated to each one, in E'_λ . On the other hand, at least two occurrences of points "4, 5, 6" occur in blocks different from these 12 blocks. This leads to the existence of at least three more blocks in E'_λ. Thus $b^* \geq 15+5 = 20$, which is beyond the range of our study.

Now, we are at the end of our method and the claim holds. □

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