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An algorithm for determining connectedness of tetravalent metacirculant graphs

Ngo Dac Tan

Institute of Mathematics
18 Hoang Quoc Viet Road
10307 Hanoi, Vietnam
ndtan@math.ac.vn

Tran Minh Tuoc

Pedagogic University of Hanoi II
Xuanhoa, Melinh, Vinhphuc, Vietnam
minhtuoc@hn.vnn.vn

Abstract

In this paper, we will prove necessary and sufficient conditions for tetravalent metacirculant graphs, the first symbol of which is empty, to be connected. The case where the first symbol is nonempty was treated previously by the authors. Based on these results we develop an algorithm for determining connectedness of tetravalent metacirculant graphs.

1 Introduction

In the last decades vertex-transitive graphs have been paid attention by many researchers. The reason for this is the high symmetry of these graphs, which makes them have many pleasant properties and to have use in designing parallel-processing computers and interconnection networks (see, for example, [3]). Metacirculant graphs, introduced in [1] by Alspach and Parsons, are interesting vertex-transitive graphs. They have a rather simple transitive subgroup of automorphisms. These graphs are not necessarily connected. But for many applications, we need to use only connected metacirculant graphs. So a natural question raised here is to develop an algorithm for determining connectedness of a given metacirculant graph. For this purpose, we try to find necessary and sufficient conditions for these graphs to be connected.

The necessary and sufficient conditions for cubic metacirculant graphs and for tetravalent ones with the non-empty first symbol to be connected have been obtained in [4] and [6], respectively. In this paper we continue to consider connectedness of tetravalent metacirculant graphs for the case of the empty first symbol. We use successfully here general techniques, which were developed in [6] and [4], to obtain the necessary and sufficient conditions for tetravalent metacirculant graphs with the empty first symbol to be connected (see Theorem 3.5 in Section 3). Based on this result and the result obtained in [6], we get an algorithm for determining connectedness of tetravalent metacirculant graphs. The results obtained in this paper and in [6] are useful not only for practical but also for theoretical problems. For example, they may be applied in the Hamilton problem for tetravalent metacirculant graphs: in [7] we have used the conditions for tetravalent metacirculant graphs with the nonempty first symbol to be connected to obtain some results on the existence of Hamilton cycles in these graphs.

2 Preliminaries

All graphs considered in this paper are finite undirected graphs without loops and multiple edges. Unless otherwise indicated, our graph-theoretic terminology will follow [2], and our group-theoretic terminology will follow [8]. For a graph G we will denote by $V(G)$, $E(G)$ and $\mathrm{Aut}(G)$ the vertex-set, the edge-set and the automorphism group of G, respectively. If $W \subseteq V(G)$ then we denote by $G[W]$ the subgraph of G induced by W. For a positive integer n, we will denote the ring of integers modulo n by $\mathbb{Z}_n$ and the multiplicative group of units in $\mathbb{Z}_n$ by $\mathbb{Z}_n^*$.

Let n be a positive integer and S be a subset of $\mathbb{Z}_n$ such that $0 \notin S = -S$. Then we define the *circulant graph* $G = C(n, S)$ to be the graph with vertex-set $V(G) = \{v_y \mid y \in \mathbb{Z}_n\}$ and edge-set $E(G) = \{v_y v_h \mid y, h \in \mathbb{Z}_n;\ (h - y) \in S\}$, where subscripts are always reduced modulo n. The subset S is called the *symbol* of $C(n, S)$.

Let m and n be two positive integers, $\alpha \in \mathbb{Z}_n^*, \mu = \lfloor m/2 \rfloor$ and $S_0, S_1, ..., S_\mu$ be subsets of $\mathbb{Z}_n$, satisfying the following conditions:

1) $0 \notin S_0 = -S_0$;

2) $\alpha^m S_r = S_r$ for $0 \le r \le \mu$;

3) If m is even, then $\alpha^\mu S_\mu = -S_\mu$.

Then we define the *metacirculant graph* $G = MC(m, n, \alpha, S_0, ..., S_\mu)$ to be the graph with vertex-set $V(G) = \{v_j^i \mid i \in \mathbb{Z}_m; j \in \mathbb{Z}_n\}$ and edge-set

$$E(G) = \{v_j^i v_h^{i+r} \mid 0 \le r \le \mu; i \in \mathbb{Z}_m; j, h \in \mathbb{Z}_n \ \&\ (h - j) \in \alpha^i S_r\},$$

where superscripts and subscripts are always reduced modulo m and modulo n, respectively. The subset S_i is called $(i + 1)$*-th symbol* of G.

It is easy to see that the permutations ρ and τ on $V(G)$ with $\rho(v_j^i) = v_{j+1}^i$ and $\tau(v_j^i) = v_{\alpha j}^{i+1}$ are automorphisms of G and the subgroup $\langle \rho, \tau \rangle$ generated by ρ and τ is a transitive subgroup of $\mathrm{Aut}(G)$. Thus, metacirculant graphs are vertex-transitive.

Denote the *degree* of a vertex v of a graph G by $\deg(v)$. It is not difficult to see that for any vertex $v \in V(G)$ of a metacirculant graph $G = MC(m, n, \alpha, S_0, ..., S_\mu)$

$$deg(v) = \begin{cases} |S_0| + 2|S_1| + \cdots + 2|S_\mu| & \text{if } m \text{ is odd,} \\ |S_0| + 2|S_1| + \cdots + 2|S_{\mu-1}| + |S_\mu| & \text{if } m \text{ is even.} \end{cases} \quad (1)$$

A graph G is called *cubic* if for any $v \in V(G)$, $\deg(v) = 3$ and it is called *tetravalent* if for any $v \in V(G)$, $\deg(v) = 4$.

Let $W = v_{j_1}^{i_1} v_{j_2}^{i_2} ... v_{j_t}^{i_t}$ be a walk in a metacirculant graph $G = MC(m,n,\alpha, S_0, ..., S_\mu)$. Then the value $(j_t - j_1)$ modulo n is called the *change* (in subscripts) of W and is denoted by $ch(W)$. The walk $W^{-1} = v_{j_t}^{i_t} ... v_{j_2}^{i_2} v_{j_1}^{i_1}$ is called the *inverse walk* of W. Let $U = v_{j_t}^{i_t} v_{j_{t+1}}^{i_{t+1}} ... v_{j_{t+l}}^{i_{t+l}}$ be another walk in G, which starts at the vertex where W terminates. Then the walk $P = v_{j_1}^{i_1} v_{j_2}^{i_2} ... v_{j_t}^{i_t} v_{j_{t+1}}^{i_{t+1}} ... v_{j_{t+l}}^{i_{t+l}}$ is called the *concatenation* of W and U and is denoted by $W * U$. It is easy to see that the concatenation operation of walks is associative, i. e. , $(W_1 * W_2) * W_3 = W_1 * (W_2 * W_3)$. Further we have $ch(W^{-1}) \equiv -ch(W) \pmod n$, $ch(W * U) \equiv ch(W) + ch(U) \pmod n$ and if a walk W has the form $W = W_1 * Q * Q^{-1} * W_2$ then $ch(W) = ch(W_1 * W_2)$.

Let $G = MC(m,n,\alpha,S_0,...,S_\mu)$ be a metacirculant graph and s be an element of S_i. Then an edge of G is called an *s^+-edge* if it has the type $v_y^x v_{y+\alpha^x s}^{x+i}$ and an *s^--edge* if it has the type $v_y^x v_{y-\alpha^{x-i}s}^{x-i}$. An edge is called *$s$-edge* if it is either an s^+-edge or s^--edge and it is called an *S_i^+-edge* (resp. *S_i^--edge*, *S_i-edge*) if it is an s^+-edge (resp. s^--edge, s-edge) for some $s \in S_i$. If all edges of a walk W are s^+-edges (resp. s^--edges, s-edges, S_i^+-edges, S_i^--edges, S_i-edges) then W is called an *s^+-walk* (resp. *s^--walk, s-walk, S_i^+-walk, S_i^--walk, S_i-walk*). A maximal s^+-subwalk (resp. s^--subwalk, s-subwalk, S_i^+-subwalk, S_i^--subwalk, S_i-subwalk) of W is called an *s^+-interval* (resp. *s^--interval, s-interval, S_i^+-interval , S_i^--interval, S_i-interval*) of W. A subwalk W' of a walk W is called an *interval* of W if it is an S_i-interval for some $i \in \{0, 1, ..., \mu\}$. So each walk W in G can be represented in the form $W = W_1 * W_2 * ... * W_k$, where $W_1, W_2, ..., W_k$ are intervals of W.

Let $G = MC(m,n,\alpha,S_0,...,S_\mu)$ be a metacirculant graph. Denote $V^i = \{v_j^i \mid j \in \mathbb{Z}_n\}$. We define graphs $\overline{G}$ and G^i as follows. The graph $\overline{G}$ has the vertex-set $V(\overline{G}) = \overline{V} = \{V^0, V^1, ..., V^{m-1}\}$ and the edge-set $E(\overline{G}) = \overline{E} = \{V^i V^j \mid i \neq j$ and there exists $v_p^i v_q^j \in E(G)$ for some $p, q \in \mathbb{Z}_n\}$. The graph G^i has the vertex-set $V(G^i) = V^i$ and the edge-set $E(G^i) = E^i = \{v_k^i v_l^i \mid k \neq l$ and there exists a walk in G joining v_k^i to $v_l^i\}$, $i \in \{0, 1, ..., m-1\}$.

The following results, proved in [4] and [6], will be useful for considering connectedness of tetravalent metacirculant graphs.

Lemma 2.1 ([6]). *Let $G = MC(m,n,\alpha,S_0,...,S_\mu)$ be a metacirculant graph. Then*
(1) $\overline{G}$ is isomorphic to $C(m, \overline{S})$, where $\overline{S} = \{h \in \mathbb{Z}_m \mid V^0 V^h \in \overline{E}\}$;
(2) G^i is isomorphic to $C(n, S^i)$, where $S^i = \{j \in \mathbb{Z}_n \mid v_0^i v_j^i \in E^i\}$;
(3) All graphs G^i, $i \in \mathbb{Z}_m$, are isomorphic to each other.

By this lemma, we can identify $\overline{G}$ with $C(m, \overline{S})$, G^i with $C(n, S^i)$ and may write $\overline{G} = C(m, \overline{S})$ and $G^i = C(n, S^i)$.

Lemma 2.2 ([6]). *Let $G = MC(m,n,\alpha,S_0,...,S_\mu)$ be a metacirculant graph. Then G is connected if and only if both $\overline{G}$ and G^0 are connected.*

Lemma 2.3 ([4]). *Let $G = C(n, S)$ be a circulant graph with symbol $S = \{\pm s_1, \pm s_2, ..., \pm s_k\}$. Then G is connected if and only if $gcd(s_1, s_2, ..., s_k, n) = 1$.*

Let $G = C(n, S)$ be a circulant graph and R be a subset of S satisfying the following conditions:

(i) $R = -R$;

(ii) For each $s \in S$, we can write $s = \sum_{i=1}^{h} t_i r_i$, where $t_i \in \mathbb{Z}$, $r_i \in R$.

Then we say that S is generated by R and denote this fact by $S = \langle R \rangle$.

Lemma 2.4 ([6]). *Let $G = C(n, S)$ be a circulant graph with $S = \langle R \rangle$. Then G is connected if and only if $C(n, R)$ is connected.*

Lemma 2.5. *Let $G = MC(m, n, \alpha, S_0, ..., S_\mu)$ be a metacirculant graph. Then G is tetravalent if and only if one of the following cases holds:*

1. $|S_0| = 4$ *and* $S_1 = \cdots = S_\mu = \emptyset$;
2. m *and* n *are even,* $|S_0| = 3$, $S_j = \emptyset$ *for* $j \in \{1, 2, \ldots, \mu - 1\}$ *and* $|S_\mu| = 1$;
3. m *is even,* $|S_0| = 2$, $S_j = \emptyset$ *for* $j \in \{1, 2, \ldots, \mu - 1\}$ *and* $|S_\mu| = 2$;
4. $m > 2$, $|S_0| = 2$, $|S_i| = 1$ *for some* $i \in \{1, 2, \ldots, \mu\}$ *if* m *is odd or* $i \in \{1, 2, \ldots, \mu - 1\}$ *if* m *is even and* $S_j = \emptyset$ *for* $i \neq j \in \{1, 2, \ldots, \mu\}$;
5. m *and* n *are even,* $|S_0| = 1$, $S_j = \emptyset$ *for* $j \in \{1, 2, \ldots, \mu - 1\}$ *and* $|S_\mu| = 3$;
6. $m > 2$, m *and* n *are even,* $|S_0| = 1$, $|S_i| = 1$ *for some* $i \in \{1, 2, \ldots, \mu - 1\}$, $S_j = \emptyset$ *for* $i \neq j \in \{1, 2, \ldots, \mu - 1\}$ *and* $|S_\mu| = 1$.
7. m *is even,* $S_0 = \cdots = S_{\mu-1} = \emptyset$ *and* $|S_\mu| = 4$;
8. $m > 2$ *is even,* $|S_i| = 1$ *for some* $i \in \{1, \ldots, \mu - 1\}$, $S_j = \emptyset$ *for all* $j \in \{0, \ldots, \mu - 1\} \setminus \{i\}$ *and* $|S_\mu| = 2$;
9. $m > 2$, $|S_i| = |S_j| = 1$ *for some* $i, j \in \{1, \ldots, \mu\}$ *if* m *is odd or* $i, j \in \{1, \ldots, \mu - 1\}$ *if* m *is even with* $i \neq j$ *and* $S_k = \emptyset$ *for all* $k \in \{0, \ldots, \mu\} \setminus \{i, j\}$;
10. $m > 2$, $|S_i| = 2$ *for some* $i \in \{1, \ldots, \mu\}$ *if* m *is odd or* $i \in \{1, \ldots, \mu - 1\}$ *if* m *is even,* $S_j = \emptyset$ *for all* $j \in \{0, \ldots, \mu\} \setminus \{i\}$.

Proof. The lemma follows immediately from Formula (1). □

In [6] we have obtained the following result, which we need for describing the algorithm for determining connectedness of tetravalent metacirculant graphs.

Theorem 2.6 ([6]). *Let $G = MC(m, n, \alpha, S_0, \ldots, S_\mu)$ be a tetravalent metacirculant graph with $S_0 \neq \emptyset$. Then the graph G is connected if and only if one of the following conditions holds:*

1. $m = 1$, $S_0 = \{\pm s, \pm r\}$ *and* $gcd(s, r, n) = 1$;
2. $m = 2$, n *is even,* $S_0 = \{\pm s, \frac{n}{2}\}$, $S_1 = \{k\}$ *and* $gcd(s, \frac{n}{2}) = 1$;
3. $m = 2$, $S_0 = \{\pm s\}$, $S_1 = \{k, l\}$ *and* $gcd(s, k - l, n) = 1$;

4. *$m > 2$, $S_0 = \{\pm s\}$, $S_i = \{k\}$ for some $i \in \{1, 2, \ldots, \mu\}$ if m is odd or $i \in \{1, 2, \ldots, \mu - 1\}$ if m is even such that $gcd(i, m) = 1$, $S_j = \emptyset$ for any $i \neq j \in \{1, 2, \ldots, \mu\}$ and $gcd(s, r, n) = 1$, where $r = k(1 + \alpha^i + \cdots + \alpha^{(m-1)i})$;*

5. *$m = 2$, n is even, $S_0 = \{\frac{n}{2}\}$, $S_1 = \{h, k, l\}$ and $gcd(h - k, k - l, \frac{n}{2}) = 1$;*

6. *$m > 2$ is even, n is even, $S_0 = \{\frac{n}{2}\}$, $S_i = \{s\}$ where i is odd and $gcd(i, m) = 1$, $S_j = \emptyset$ for any $i \neq j \in \{1, 2, \ldots, \mu - 1\}$, $S_\mu = \{r\}$ and $gcd(p, \frac{n}{2}) = 1$, where p is $\left[r - s(1 + \alpha^i + \alpha^{2i} + \cdots + \alpha^{(\mu-1)i})\right]$ reduced modulo n;*

7. *$m > 2$ is even, but $\mu = \frac{m}{2}$ is odd, n is even, $S_0 = \{\frac{n}{2}\}$, $S_i = \{s\}$ where i is even and $gcd(i, m) = 2$, $S_j = \emptyset$ for any $i \neq j \in \{1, 2, \ldots, \mu - 1\}$, $S_\mu = \{r\}$ and $gcd(q, \frac{n}{2}) = 1$, where $i = 2^t i'$ with i' is odd and q is $\left[r(1 + \alpha^{i'} + \alpha^{2i'} + \cdots + \alpha^{(2^t-1)i'}) - s(1 + \alpha^{i'} + \alpha^{2i'} + \cdots + \alpha^{(\mu-1)i'})\right]$ reduced modulo n.*

3 Results

Necessary and sufficient conditions for tetravalent metacirculant graphs with the non-empty first symbol to be connected have been obtained in [6]. In this section, we give necessary and sufficient conditions for tetravalent metacirculant graphs with the empty first symbol to be connected. Based on these two results we get an algorithm for determining connectedness of tetravalent metacirculant graphs. We assume that all tetravalent metacirculant graphs considered in this section have the first symbol $S_0 = \emptyset$.

In Lemma 3.1 below we get a necessary and sufficient condition for a tetravalent metacirculant graph in Case 7 of Lemma 2.5 to be connected.

Lemma 3.1. *Let $G = MC(m, n, \alpha, S_0, ..., S_\mu)$ be a tetravalent metacirculant graph with m even, $S_0 = \cdots = S_{\mu-1} = \emptyset$ and $S_\mu = \{s_1, s_2, s_3, s_4\}$. Then G is connected if and only if $m = 2$ and $gcd(s_1 - s_2, s_2 - s_3, s_3 - s_4, n) = 1$.*

Proof. It is easy to see that $\overline{G} = C(m, \overline{S})$ with $\overline{S} = \{\pm\mu\}$. We will prove that $G^0 = C(n, S^0)$ with $S^0 =< R >$ where $R = \{\pm(s_1 - s_2), \pm(s_2 - s_3), \pm(s_3 - s_4)\}$.

Let W be a walk in G starting at a vertex v_x^0 and terminating at a vertex v_y^0 of the block V^0. Then $W = Q_1 * Q_2 * \cdots * Q_p$, where Q_i is of the form $v_{x_i}^0 v_{x_i+s}^\mu$ with $s \in S_\mu$ if i is odd and of the form $v_{x_i}^\mu v_{x_i-s}^0$ with $s \in S_\mu$ if i is even. It is clear that the number p of Q_i must be even. We will prove that $ch(W) \in \langle R \rangle$ by induction on p.

If $p = 2$ then $W = Q_1 * Q_2$, where $Q_1 = v_a^0 v_{a+s_i}^\mu$ and $Q_2 = v_{a+s_i}^\mu v_{a+s_i-s_k}^0$ for some $i, k \in \{1, 2, 3, 4\}$. Therefore $ch(W) = ch(Q_1) + ch(Q_2) = s_i - s_k \in \langle R \rangle$.

Assume now that $ch(P) \in \langle R \rangle$ for any walk P which has its endvertices in V^0 and the number of edges Q_i of which is less than or equal to $2t$. Let $W = Q_1 * Q_2 * \cdots * Q_{2(t+1)}$ be a walk of G, which has its endvertices in V^0 and the number of edges Q_i of which is $2(t+1)$. Set $W_1 = Q_1 * Q_2 * \ldots * Q_{2t}$ and $W_2 = Q_{2t+1} * Q_{2t+2}$. Then W_1 and W_2 are subwalks of W, which have their endvertices in V^0 and the numbers of edges Q_i in both W_1 and W_2 are less than or equal to $2t$. By the induction hypothesis, $ch(W_1), ch(W_2) \in \langle R \rangle$. This implies that $ch(W) = ch(W_1) + ch(W_2) \in \langle R \rangle$.

Thus, $ch(W) \in \langle R \rangle$ for any walk W with its endvertices in V^0. This implies that $G^0 = C(n, S^0)$ with $S^0 =< R >$.

Since $\overline{G} = C(m, \overline{S})$ with $\overline{S} = \{\pm\mu\}$, by Lemma 2.3, $\overline{G}$ is connected if and only if $\gcd(\mu, m) = 1$. We have $\mu = m/2$ because m is even. So $\gcd(\mu, m) = \mu$. Therefore $\overline{G}$ is connected if and only if $m = 2$. Since $G^0 = C(n, S^0)$ with $S^0 =< R >$, by Lemmas 2.3 and 2.4, G^0 is connected if and only if $\gcd(s_1 - s_2, s_2 - s_3, s_3 - s_4, n) = 1$. Now by Lemma 2.2, we may conclude that G is connected if and only if $m = 2$ and $\gcd(s_1 - s_2, s_2 - s_3, s_3 - s_4, n) = 1$. □

Necessary and sufficient conditions for a tetravalent metacirculant graph in Case 8 of Lemma 2.5 to be connected are obtained in the following lemma.

Lemma 3.2. *Let $G = MC(m, n, \alpha, S_0, ..., S_\mu)$ be a tetravalent metacirculant graph with $m > 2$ even, $S_0 = \emptyset$, $S_i = \{k\}$ for some $i \in \{1, \dots, \mu - 1\}$, $S_j = \emptyset$ for any $i \neq j \in \{1, \dots, \mu - 1\}$ and $S_\mu = \{s, r\}$. Then*

1. *If G is connected, then either i is odd and gcd$(i, m) = 1$ or i is even, μ is odd and gcd$(i, m) = 2$.*
2. *If i is odd and gcd$(i, m) = 1$, then G is connected if and only if gcd$(p, u, n) = 1$, where $u = s - r$ and $p = k(1 + \alpha^i + \cdots + \alpha^{(\mu-1)i}) - s$.*
3. *If i is even, μ is odd and gcd$(i, m) = 2$, then G is connected if and only if gcd$(\xi, u, n) = 1$, where $u = s - r$ and $\xi = \big[k(1 + \alpha^{i'} + \alpha^{2i'} + \cdots + \alpha^{(\mu-1)i'}) - s(1 + \alpha^{i'} + \alpha^{2i'} + \cdots + \alpha^{(2^t-1)i'})\big]$ with $t \geq 1$ and i' an odd integer such that $i = 2^t i'$.*

Proof. (1) Since G is connected, by Lemma 2.2, the graph $\overline{G}$ is connected. But $\overline{G} = C(m, \overline{S})$ with $\overline{S} = \{\pm i, \mu\}$ by Lemma 2.1. So by Lemma 2.3, we have $\gcd(i, \mu, m) = \gcd(i, \mu) = 1$. Therefore either i is odd and $\gcd(i, m) = 1$ or i is even, μ is odd and $\gcd(i, m) = 2$. Assertion (1) is proved.

(2) Suppose that $G = MC(m, n, \alpha, S_0, ..., S_\mu)$ is a tetravalent metacirculant graph with $m > 2$ even, $S_0 = \emptyset$, $S_i = \{k\}$ for some odd $i \in \{1, \dots, \mu - 1\}$ such that $\gcd(i, m) = 1$, $S_j = \emptyset$ for any $i \neq j \in \{1, \dots, \mu - 1\}$ and $S_\mu = \{s, r\}$. Since i is odd and $\gcd(i, m) = 1$, it is not difficult to see that the smallest positive integer d such that $di \equiv \mu \pmod{m}$ is $d = \mu$. Let $p = k(1 + \alpha^i + \cdots + \alpha^{(\mu-1)i}) - s$ and $u = s - r$. It is clear that $\overline{G} = C(m, \overline{S})$ with $\overline{S} = \{\pm i, \mu\}$. We show that $G^0 = C(n, S^0)$ with $S^0 = \langle \{\pm p, \pm u\} \rangle$.

Let P be a walk in G starting at v_x^0 and terminating at a vertex $v_y^0 \neq v_x^0$ of the block V^0. Let $z(P)$ be the number of S_μ-edges in P. We will prove that $ch(P) \in S^0$ by induction on $z(P)$. Without loss of generality we may assume that P has no subwalks of the type $Q * Q^{-1}$ and the only vertices of P in V^0 are its endvertices.

If $z(P) = 0$ then P is either an S_i^+-walk or an S_i^--walk. In the former case, P has the form $P = v_x^0 v_{x+k}^i v_{x+k(1+\alpha^i)}^{2i} \cdots v_{x+k(1+\alpha^i+\cdots+\alpha^{(2\mu-1)i})}^0$. So $ch(P) = k(1 + \alpha^i + \cdots + \alpha^{(2\mu-1)i}) = k(1 + \alpha^i + \cdots + \alpha^{(\mu-1)i}) + k(\alpha^{\mu i} + \cdots + \alpha^{(2\mu-1)i}) = k(1 + \alpha^i + \cdots + \alpha^{(\mu-1)i}) + \alpha^{\mu i} k(1 + \alpha^i + \cdots + \alpha^{(\mu-1)i})$.

On the other hand, $\mu i \equiv \mu \pmod{m}$ because i is odd and $\mu = m/2$. It follows that $\mu i = \mu + am$ for some $a \in \mathbb{Z}$. So $\alpha^{\mu i} S_\mu = \alpha^\mu \alpha^{am} S_\mu = \alpha^\mu S_\mu = -S_\mu$. Since $S_\mu = \{s, r\}$, we have either $\alpha^{\mu i} s = -s$ or $\alpha^{\mu i} s = -r$ in $\mathbb{Z}_n$. If $\alpha^{\mu i} s = -s \pmod{n}$ then $ch(P) = k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i})+\alpha^{\mu i}k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i}) = k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i})-s-\alpha^{\mu i}s+\alpha^{\mu i}k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i}) = [k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i})-s]+\alpha^{\mu i}[k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i})-s] = [k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i})-s]\big(1+\alpha^{\mu i}\big)$. Therefore $ch(P) = p(1+\alpha^{\mu i}) \in S^0$. If $\alpha^{\mu i} s = -r \pmod{n}$ then $ch(P) = k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i})+\alpha^{\mu i}k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i}) = k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i})-r-\alpha^{\mu i}s+\alpha^{\mu i}k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i}) = [k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i})-s+s-r]+\alpha^{\mu i}[k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i})-s] = [k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i})-s]+(s-r)+\alpha^{\mu i}[k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i})-s] = p+u+\alpha^{\mu i}p = (1+\alpha^{\mu i})p+u$. Therefore $ch(P)$ is in S^0 again. Thus, $ch(P) \in S^0$ if $z(P) = 0$.

Let $z(P) = 1$. Then the following cases may happen for P:

1. $P = P_1 * P_2$, where either
   (a) P_1 is an S_i^+-interval and P_2 is an S_μ-edge or
   (b) P_1 is an S_i^--interval and P_2 is an S_μ-edge or
   (c) P_1 is an S_μ-edge and P_2 is an S_i^--interval or
   (d) P_1 is an S_μ-edge and P_2 is an S_i^+-interval.
2. $P = P_1 * P_2 * P_3$, where either
   (a) P_1 and P_3 are S_i^+-intervals and P_2 is an S_μ-edge or
   (b) P_1 is an S_i^+-interval, P_2 is an S_μ-edge and P_3 is an S_i^--interval or
   (c) P_1 and P_3 are S_i^--intervals and P_2 is an S_μ-edge or
   (d) P_1 is an S_i^--interval, P_2 is an S_μ-edge and P_3 is an S_i^+-interval.

We now consider the above cases in turn.

(1.a) $P = P_1 * P_2$ with P_1 an S_i^+-interval and P_2 an S_μ-edge.

In this case, since s and r play the equal role in S_μ, we may assume that P has the form

$$P = v_x^0 v_{x+k}^i v_{k(1+\alpha^i)}^{2i} \cdots v_{x+k(1+\alpha^i+\cdots+\alpha^{\mu-1})}^{\mu i} v_{x+k(1+\alpha^i+\cdots+\alpha^{\mu-1})+\alpha^{\mu i}s}^0.$$

As before, we can show that either $\alpha^{\mu i} s = -s$ or $\alpha^{\mu i} s = -r$ in $\mathbb{Z}_n$. So $ch(P) = p$ or $ch(P) = p + u$. Thus $ch(P) \in S^0$.

(1.b) P_1 is an S_i^--interval and P_2 is an S_μ-edge.

Without loss of generality we may assume that

$$P = v_x^0 v_{x-\alpha^{-i}k}^{-i} \cdots v_{x-\alpha^{-i}k-\alpha^{-2i}k-\cdots-\alpha^{-\mu i}k}^{-\mu i} v_{x-\alpha^{-i}k-\alpha^{-2i}k-\cdots-\alpha^{-\mu i}k-s}^0.$$

So $ch(P) = -\alpha^{-i}k - \alpha^{-2i}k - \cdots - \alpha^{-\mu i}k - s = -s - \alpha^{-\mu i}k - \alpha^{-\mu i+i}k - \cdots - \alpha^{-\mu i+(\mu-1)i}k$. But we have again $\alpha^{-\mu i}k \equiv \alpha^{-\mu}k \equiv \alpha^\mu k \pmod{n}$ and either $-s = \alpha^\mu s$ or $-s = \alpha^\mu r$ in $\mathbb{Z}_n$. If $-s = \alpha^\mu s$ then $ch(P) = \alpha^\mu s - \alpha^{-\mu i}k(1+\alpha^i+\cdots+\alpha^{(\mu-1)i}) = \alpha^\mu s - \alpha^\mu k(1+$

$\alpha^i + \cdots + \alpha^{(\mu-1)i}) = -\alpha^\mu p \in S^0$. If $-s = \alpha^\mu r$ then $ch(P) = \alpha^\mu r - \alpha^{-\mu i}k(1 + \alpha^i + \cdots + \alpha^{(\mu-1)i}) = \alpha^\mu(r - s + s) - \alpha^\mu k(1 + \alpha^i + \cdots + \alpha^{(\mu-1)i}) = -\alpha^\mu u - \alpha^\mu p \in S^0$.

The proof of the assertion that $ch(P) \in S^0$ for Cases (1.c) and (1.d) is reduced to that of (1.a) and (1.b), respectively, by replacing the walk P with the walk P^{-1}. We omit them here.

Let $v^i_j v^k_h$ be an edge of G. When the subscript h is completely determined by i, j, k, we will write the edge $v^i_j v^k_h$ simply as $v^i_j v^k$.

(2.a) P_1 and P_3 are S^+_i-intervals and P_2 is an S_μ-edge.

We can write $P = v^0_x v^i v^{2i} \dots v^{ti}_a v^{ti+\mu}_b v^{ti+\mu+i} v^{ti+\mu+2i} \dots v^0_y$ for some $t \neq \mu$ and $0 < t < m$.

First suppose that $0 < t < \mu$. Then since $\mu i \equiv \mu \pmod m$, we can rewrite $P = v^0_x v^i v^{2i} \dots v^{ti}_a v^{ti+\mu i}_b v^{ti+\mu i+i} v^{ti+\mu i+2i} \dots v^{ti+\mu i+(\mu-t)i}_y$. Then $P_1 = v^0_x v^i v^{2i} \dots v^{ti}_a$, $P_2 = v^{ti}_a v^{ti+\mu i}_b$ and $P_3 = v^{ti+\mu i}_b v^{ti+\mu i+i} v^{ti+\mu i+2i} \dots v^{ti+\mu i+(\mu-t)i}_y$. Let $Q = v^{ti+\mu i}_b v^{ti+\mu i-i} v^{ti+\mu i-2i} \dots v^{ti}_c$ and $P' = P_1 * P_2 * Q * Q^{-1} * P_3$. Then $ch(P') = ch(P)$. We will show that $ch(P') \in S^0$.

We have $ch(P') = \big[ch(P_1) + ch(P_3) + ch(Q^{-1})\big] + \big[ch(P_2) + ch(Q)\big]$. But the walk $P_2 * Q$ has the type similar to Case (1.c). So it is not difficult to see that $ch(P_2 * Q) = ch(P_2) + ch(Q) \in S^0$. On the other hand $ch(P_1) = k + \alpha^i k + \cdots + \alpha^{(t-1)i}k$; $ch(P_3) = \alpha^{ti+\mu i}k + \cdots + \alpha^{ti+\mu i+(\mu-t-1)i}k$ and $ch(Q^{-1}) = \alpha^{ti}k + \alpha^{(t+1)i}k + \cdots + \alpha^{(t+\mu-1)i}k$. This implies that $ch(P_1) + ch(P_3) + ch(Q^{-1}) = k + \alpha^i k + \cdots + \alpha^{(2\mu-1)i}k$. By arguments similar to that of the case $z(P) = 0$, we have $k + \alpha^i k + \cdots + \alpha^{(2\mu-1)i}k \in S^0$. Thus $ch(P) = ch(P') = \big[ch(P_1) + ch(P_3) + ch(Q^{-1})\big] + \big[ch(P_2) + ch(Q)\big] \in S^0$.

By similar arguments we can prove (2.a) when $\mu < t < m$. The detailed proof is left to the reader.

(2.b) $P = P_1 * P_2 * P_3$, where P_1 is an S^+_i-interval, P_2 is an S_μ-edge and P_3 is an S^-_i-interval.

In this case $P = v^0_x v^i v^{2i} \dots v^{ti}_a v^{ti+\mu}_b v^{ti+\mu-i} v^{ti+\mu-2i} \dots v^0_y$ for some $t \neq \mu$ and $0 < t < m$. First suppose that $0 < t < \mu$. Then since $\mu i \equiv \mu \pmod m$, we can rewrite $P = v^0_x v^i v^{2i} \dots v^{ti}_a v^{ti+\mu i}_b v^{ti+\mu i-i} v^{ti+\mu i-2i} \dots v^0_y$. Then $P_1 = v^0_x v^i v^{2i} \dots v^{ti}_a$, $P_2 = v^{ti}_a v^{ti+\mu i}_b$ and $P_3 = v^{ti+\mu i}_b v^{ti+\mu i-i} v^{ti+\mu i-2i} \dots v^0_y$. Let $Q = v^{ti+\mu i}_b v^{ti+\mu i+i} v^{ti+\mu i+2i} \dots v^0_c$ and $P' = P_1 * P_2 * Q * Q^{-1} * P_3$.Then $ch(P) = ch(P') = ch(P_1 * P_2 * Q) + ch(Q^{-1} * P_3)$. It is clear that $P_1 * P_2 * Q$ has the type similar to walks in Case (2.a) where $0 < t < \mu$ which we have considered above and $Q^{-1} * P_3$ has the type similar to one of the walks in the case $z(P) = 0$. Therefore $ch(P) = ch(P') = ch(P_1 * P_2 * Q) + ch(Q^{-1} * P_3)$ is in S^0.

By similar arguments we can prove (2.b) when $\mu < t < m$. Also, the proof that $ch(P) \in S^0$ for Cases (2.c) and (2.d) is similar to that of (2.a) and (2.b). So we omit them here.

Suppose now that the assertion $ch(W) \in S^0$ has been proved for any walk W with the number of S_μ-edges less than h ($h \geq 2$) and let P be a walk with $z(P) = h$. We show that $ch(P) \in S^0$.

Let e be the first S_μ-edge we encounter going along P from its beginning vertex and let v^a_b be the last vertex of e. Then we can write $P = P_1 * e * P_2$. Since μ is the smallest positive integer d such that $di \equiv \mu \pmod{m}$ and $\gcd(i,m) = 1$, we can construct a walk Q which starts at v_b^a and terminates at a vertex of V^0 and consists of only S_i-edges. Consider the walk $P' = (P_1 * e * Q) * (Q^{-1} * P_2)$. We have $ch(P') = ch(P)$, both $(P_1 * e * Q)$ and $(Q^{-1} * P_2)$ have their endvertices in V^0 and the number of S_μ-edges in these walks less than h. By the induction hypothesis, $ch(P_1 * e * Q)$ and $ch(Q^{-1} * P_2)$ are in S^0. Therefore $ch(P) = ch(P') = ch(P_1 * e * Q) + ch(Q^{-1} * P_2) \in S^0$.

Thus, the assertion $ch(P) \in S^0$ has been proved for any walk P, the only vertices of which in V^0 are its endvertices.

By Lemmas 2.2, 2.3, and 2.4 we see that, if i is odd and $\gcd(i,m) = 1$ then the graph G is connected if and only if $\gcd(p,u,n) = 1$.

(3) Let $G = MC(m,n,\alpha,S_0,...,S_\mu)$ be a tetravalent metacirculant graph, where $m > 2$ is even, but $\mu = m/2$ is odd, $S_0 = \emptyset$, $S_i = \{k\}$ for some even $i \in \{1,\dots,\mu-1\}$ such that $\gcd(i,m) = 2$, $S_j = \emptyset$ for any $i \neq j \in \{1,\dots,\mu-1\}$ and $S_\mu = \{s,r\}$. First we prove the following claims.

Claim 1. *If i is even, μ is odd, gcd(i,m) = 2 and $i = 2^t i'$ with $t \geq 1$ and i' odd, then the graph G is isomorphic to the metacirculant graph $G' = MC(m,n,\alpha',S_0',\dots,S_\mu')$, where $\alpha' = \alpha^{i'}$, $S_0' = \cdots = S_{2^t-1}' = \emptyset$, $S_{2^t}' = \{k\}$, $S_{2^t+1}' = \cdots = S_{\mu-1}' = \emptyset$ and $S_\mu' = \{s,r\}$.*

Proof of Claim 1. Since i is even, μ is odd and $\gcd(i,m) = 2$, the integers $0, i, 2i, \dots, (\mu-1)i$ are all distinct even integers and $\mu, i+\mu, 2i+\mu, \dots, (\mu-1)i+\mu$ are all distinct odd integers in $\mathbb{Z}_n$. Since i' is odd, $\mu i' \equiv \mu \pmod{m}$. Let $\varphi : V(G) \to V(G')$: $v_y^{xi} \mapsto v_y^{x2^t}$ and $v_y^{xi+\mu} \mapsto v_y^{x2^t+\mu}$. Then φ is an isomorphism between G and G'. The detailed verification is not difficult. So we omit it here.

Claim 2. *Let $G = MC(m,n,\alpha,S_0,...,S_\mu)$ be a tetravalent metacirculant graph, where $m > 2$ is even, $\mu = m/2$ is odd, $S_0 = \cdots = S_{2^t-1} = \emptyset$ with $t \geq 1$, $S_{2^t} = \{k\}$, $S_{2^t+1} = \cdots = S_{\mu-1} = \emptyset$ and $S_\mu = \{s,r\}$. If a walk P of G joins two vertices of V^0 then $ch(P) \in \langle \pm p, \pm u \rangle$, where $u = s-r$ and $p = k(1+\alpha+\alpha^2+\cdots+\alpha^{\mu-1}) - s(1+\alpha+\alpha^2+\cdots+\alpha^{2^t-1})$ modulo n.*

Proof of Claim 2. Let G be such a tetravalent metacirculant graph, P be a walk in G joining a vertex v_a^0 to a vertex v_f^0 of V^0. Without loss of generality, we may assume that P has no subwalks of the type $Q * Q^{-1}$ and the only vertices of which in V^0 are its endvertices. Let $z(P)$ be the number of S_μ-edges in P. We will prove this claim by induction on $z(P)$. We note that $z(P)$ must be even because only S_μ-edges can join vertices of blocks with even superscripts to vertices of blocks with odd superscripts.

If $z(P) = 0$ then P can be represented by $P = v_a^0 v^{2^t} v^{2.2^t} \dots v^{(\mu-1)2^t} v_f^0$. So $ch(P) = k(1+\alpha^{2^t}+\alpha^{2.2^t}+\cdots+\alpha^{(\mu-1)2^t})$. Since μ is odd, we have $\gcd(2^t,m) = 2$. It follows that $0, 2^t, 2.2^t, \dots, (\mu-1)2^t$ are all even integers modulo m. Therefore $ch(P) = k(1+\alpha^{2^t}+\alpha^{2.2^t}+\cdots+\alpha^{(\mu-1)2^t}) = k(1+\alpha^2+\alpha^4+\cdots+\alpha^{m-2}) \equiv k(1+\alpha+\alpha^2+\cdots+\alpha^{(\mu-1)})(1-\alpha+\alpha^2-\cdots+\alpha^{(\mu-1)}) \pmod{n}$. By definition of metacirculant graphs, we have $\alpha^\mu S_\mu = -S_\mu$, i.e., $\alpha^\mu\{s,r\} = -\{s,r\}$. This means $\alpha^\mu s \equiv -s \pmod{n}$ or $\alpha^\mu s \equiv -r \pmod{n}$.

If $\alpha^\mu s \equiv -s \pmod n$ then $(\alpha^\mu + 1)s \equiv 0 \pmod n$. So we can write $0 \equiv s(\alpha^\mu + 1)(1+\alpha^2)(1+\alpha^{2^2})\dots(1+\alpha^{2^{t-1}}) \equiv s(1+\alpha+\alpha^2+\dots+\alpha^{(2^t-1)})(1-\alpha+\alpha^2-\dots+\alpha^{(\mu-1)}) \pmod n$.

From the above formulas, $ch(P) = -s(1+\alpha+\alpha^2+\dots+\alpha^{(2^t-1)})(1-\alpha+\alpha^2-\dots+\alpha^{(\mu-1)}) + k(1+\alpha+\alpha^2+\dots+\alpha^{(\mu-1)})(1-\alpha+\alpha^2-\dots+\alpha^{(\mu-1)}) = (1-\alpha+\alpha^2-\dots+\alpha^{(\mu-1)})\big[k(1+\alpha+\alpha^2+\dots+\alpha^{(\mu-1)}) - s(1+\alpha+\alpha^2+\dots+\alpha^{(2^t-1)})\big] = (1-\alpha+\alpha^2-\dots+\alpha^{(\mu-1)})p \pmod n$. So $ch(P) \in \langle \pm p, \pm u\rangle$.

If $\alpha^\mu s \equiv -r \pmod n$ then $\alpha^\mu s + s - s + r = s(\alpha^\mu+1) + (r-s) \equiv 0 \pmod n$. Therefore, we also have $0 \equiv \big[s(\alpha^\mu+1)+(r-s)\big](1+\alpha^2)(1+\alpha^{2^2})\dots(1+\alpha^{2^{t-1}}) \equiv s(\alpha^\mu+1)(1+\alpha^2)(1+\alpha^{2^2})\dots(1+\alpha^{2^{t-1}}) + (r-s)(1+\alpha^2)(1+\alpha^{2^2})\dots(1+\alpha^{2^{t-1}}) \pmod n$. So $ch(P) \equiv k(1+\alpha+\alpha^2+\dots+\alpha^{(\mu-1)})(1-\alpha+\alpha^2-\dots+\alpha^{(\mu-1)}) - s(\alpha^\mu+1)(1+\alpha^2)(1+\alpha^{2^2})\dots(1+\alpha^{2^{t-1}}) - (r-s)(1+\alpha^2)(1+\alpha^{2^2})\dots(1+\alpha^{2^{t-1}}) \equiv (1-\alpha+\alpha^2-\dots+\alpha^{(\mu-1)})\big[k(1+\alpha+\alpha^2+\dots+\alpha^{(\mu-1)}) - s(1+\alpha+\alpha^2+\dots+\alpha^{(2^t-1)})\big] + (s-r)(1+\alpha^2)(1+\alpha^{2^2})\dots(1+\alpha^{2^{t-1}}) \equiv (1-\alpha+\alpha^2-\dots+\alpha^{(\mu-1)})p + \big[(1+\alpha^2)(1+\alpha^{2^2})\dots(1+\alpha^{2^{t-1}})\big]u \pmod n$. So $ch(P)$ is also in $\langle \pm p, \pm u\rangle$. Thus if $z(P)=0$, we have $ch(P) \in \langle \pm p, \pm u\rangle$.

If a walk P contains only two S_μ-edges, i.e. $z(P)=2$, then we construct the subwalks P_1, P_2, P_3 and P_4 of P as follows. P_1 starts at the beginning vertex v_a^0 of P and terminates with the first S_μ-edge $v_b^{x2^t} v_{b'}^{x2^t+\mu}$ contained in P, P_2 starts at $v_{b'}^{x2^t+\mu}$ and terminates at $v_c^{x2^t+\mu}$, which is the last vertex of P with superscript $x2^t+\mu$. The subwalk P_3 starts at $v_c^{x2^t+\mu}$ and terminates with the second S_μ-edge $v_d^{y2^t+\mu} v_{d'}^{y2^t}$ contained in P. Finally, start P_4 at $v_{d'}^{y2^t}$ and terminate it with the last vertex v_f^0 of P. Thus $P = P_1 * P_2 * P_3 * P_4$. Moreover, P_2 is a walk joining two vertices of the same block $V^{x2^t+\mu}$ and having no S_μ-edges. By the same arguments used for the case $z(P)=0$, we have $ch(P_2) \in \langle \pm p, \pm u\rangle$. So $ch(P) \in \langle \pm p, \pm u\rangle$ if and only if $ch(P_1) + ch(P_3) + ch(P_4) \in \langle \pm p, \pm u\rangle$.

By the constructions, P_4 is an S_{2^t}-walk, all edges but the last one of P_1 and P_3 are also S_{2^t}-edges. The orientations of S_{2^t}-portions of P_1 and P_3 and the orientation of P_4 may be positive or negative. But we can verify that in all cases $ch(P_1) + ch(P_3) + ch(P_4)$ reduced modulo n is always in $\langle \pm p, \pm u\rangle$. Here we will demonstrate calculations only for the case when the S_{2^t}-portions of P_1, P_3 and P_4 have positive orientations.

Let

$$\begin{aligned} P_1 &= v_a^0 v^{2^t} v^{2.2^t} \dots v_b^{x2^t} v_{b'}^{x2^t+\mu}, \\ P_3 &= v_c^{x2^t+\mu} v^{(x+1)2^t+\mu} \dots v_d^{y2^t+\mu} v_{d'}^{y2^t}, \\ P_4 &= v_{d'}^{y2^t} v^{(y+1)2^t} \dots v^{(\mu-1)2^t} v_f^0. \end{aligned}$$

Since $S_\mu = \{s, r\}$, S_μ-edges may be either s-edges or r-edges. Therefore, there are four possibilities for $ch(P_1) + ch(P_3) + ch(P_4)$ to consider.

(a) $ch(P_1) + ch(P_3) + ch(P_4) \equiv \big[k(1+\alpha^{2^t}+\dots+\alpha^{(x-1)2^t}) + \alpha^{x2^t}s\big] + \big[\alpha^\mu k(\alpha^{x2^t} + \alpha^{(x+1)2^t} + \dots + \alpha^{(y-1)2^t}) + \alpha^\mu \alpha^{y2^t} s\big] + \big[k(\alpha^{y2^t} + \alpha^{(y+1)2^t} + \dots + \alpha^{(\mu-1)2^t})\big] \equiv \big[k(1 + \alpha^{2^t} + \dots + \alpha^{(\mu-1)2^t})\big] + \big[\alpha^{x2^t}s + \alpha^\mu k(\alpha^{x2^t} + \alpha^{(x+1)2^t} + \dots + \alpha^{(y-1)2^t}) + \alpha^\mu\alpha^{y2^t}s\big] - \alpha^{(y-1)2^t}k - \alpha^{(y-2)2^t}k - \dots - \alpha^{x2^t}k \pmod n$. By the calculations in the case $z(P)=0$, the first term is in $\langle \pm p, \pm u\rangle$. Consider the remainder. If $\alpha^\mu s = -s$ then we can see

that $\left[\alpha^{x2^t}s+\alpha^{\mu}k(\alpha^{x2^t}+\alpha^{(x+1)2^t}+\cdots+\alpha^{(y-1)2^t})+\alpha^{\mu}\alpha^{y2^t}s\right]-\alpha^{(y-1)2^t}k-\alpha^{(y-2)2^t}k-\cdots-\alpha^{x2^t}k\equiv\alpha^{x2^t}(1-\alpha)(1+\alpha^{2^t}+\cdots+\alpha^{(y-x-1)2^t})\left[s(1+\alpha+\alpha^2+\cdots+\alpha^{(2^t-1)})-k(1+\alpha+\alpha^2+\cdots+\alpha^{(\mu-1)})\right]$ (mod n), i.e., the remainder is in $\langle\pm p,\pm u\rangle$. Similarly, it is not difficult to verify that if $\alpha^{\mu}s=-r$ then $ch(P_1)+ch(P_3)+ch(P_4)\in\langle\pm p,\pm u\rangle$.

(b) $ch(P_1)+ch(P_3)+ch(P_4)\equiv\left[k(1+\alpha^{2^t}+\cdots+\alpha^{(x-1)2^t})+\alpha^{x2^t}r\right]+\left[\alpha^{\mu}k(\alpha^{x2^t}+\alpha^{(x+1)2^t}+\cdots+\alpha^{(y-1)2^t})+\alpha^{\mu}\alpha^{y2^t}r\right]+\left[k(\alpha^{y2^t}+\alpha^{(y+1)2^t}+\cdots+\alpha^{(\mu-1)2^t})\right]$. By exchanging the role of s-edges and r-edges, with the calculation similar to *(a)*, we can show that the remainder is equivalent to $\alpha^{x2^t}(1-\alpha)(1+\alpha^{2^t}+\cdots+\alpha^{(y-x-1)2^t})\left[r(1+\alpha+\alpha^2+\cdots+\alpha^{(2^t-1)})-k(1+\alpha+\alpha^2+\cdots+\alpha^{(\mu-1)})\right]$. On the other hand, we have $r(1+\alpha+\alpha^2+\cdots+\alpha^{(2^t-1)})-k(1+\alpha+\alpha^2+\cdots+\alpha^{(\mu-1)})=[(r-s)+s](1+\alpha+\alpha^2+\cdots+\alpha^{(2^t-1)})-k(1+\alpha+\alpha^2+\cdots+\alpha^{(\mu-1)})=(r-s)(1+\alpha+\alpha^2+\cdots+\alpha^{(2^t-1)})+\left[s(1+\alpha+\alpha^2+\cdots+\alpha^{(2^t-1)})-k(1+\alpha+\alpha^2+\cdots+\alpha^{(\mu-1)})\right]$. From this, we see that $ch(P_1)+ch(P_3)+ch(P_4)\in\langle\pm p,\pm u\rangle$.

(c) $ch(P_1)+ch(P_3)+ch(P_4)\equiv\left[k(1+\alpha^{2^t}+\cdots+\alpha^{(x-1)2^t})+\alpha^{x2^t}s\right]+\left[\alpha^{\mu}k(\alpha^{x2^t}+\alpha^{(x+1)2^t}+\cdots+\alpha^{(y-1)2^t})+\alpha^{\mu}\alpha^{y2^t}r\right]+\left[k(\alpha^{y2^t}+\alpha^{(y+1)2^t}+\cdots+\alpha^{(\mu-1)2^t})\right]\equiv\left[k(1+\alpha^{2^t}+\cdots+\alpha^{(\mu-1)2^t})\right]+\left[\alpha^{x2^t}s+\alpha^{\mu}k(\alpha^{x2^t}+\alpha^{(x+1)2^t}+\cdots+\alpha^{(y-1)2^t})+\alpha^{\mu}\alpha^{y2^t}r\right]-\alpha^{(y-1)2^t}k-\alpha^{(y-2)2^t}k-\cdots-\alpha^{x2^t}k$ (mod n).

We have $\left[\alpha^{x2^t}s+\alpha^{\mu}k(\alpha^{x2^t}+\alpha^{(x+1)2^t}+\cdots+\alpha^{(y-1)2^t})+\alpha^{\mu}\alpha^{y2^t}r\right]-\alpha^{(y-1)2^t}k-\alpha^{(y-2)2^t}k-\cdots-\alpha^{x2^t}k\equiv-\alpha^{x2^t}k(1+\alpha^{2^t}+\cdots+\alpha^{(y-x-1)2^t})(1-\alpha^{\mu})+(\alpha^{x2^t}s+\alpha^{\mu}\alpha^{y2^t}r)$ (mod n). By definition of metacirculant graphs, $\alpha^{\mu}r\equiv-s$ (mod n) or $\alpha^{\mu}r\equiv-r$ (mod n).

If $\alpha^{\mu}r\equiv-s$ (mod n) then $\alpha^{x2^t}s+\alpha^{\mu}\alpha^{y2^t}r\equiv\alpha^{x2^t}s-\alpha^{y2^t}s\equiv\alpha^{x2^t}s(1-\alpha^{(y-x)2^t})\equiv\alpha^{x2^t}s(1-\alpha^{2^t})(1+\alpha^{2^t}+\cdots+\alpha^{(y-x-1)2^t})$ (mod n). So $\left[\alpha^{x2^t}s+\alpha^{\mu}k(\alpha^{x2^t}+\alpha^{(x+1)2^t}+\cdots+\alpha^{(y-1)2^t})+\alpha^{\mu}\alpha^{y2^t}r\right]-\alpha^{(y-1)2^t}k-\alpha^{(y-2)2^t}k-\cdots-\alpha^{x2^t}k\equiv\alpha^{x2^t}(1+\alpha^{2^t}+\cdots+\alpha^{(y-x-1)2^t})\left[s(1-\alpha^{2^t})-k(1-\alpha^{\mu})\right]\equiv\alpha^{x2^t}(1+\alpha^{2^t}+\cdots+\alpha^{(y-x-1)2^t})(1-\alpha)\left[s(1+\alpha+\alpha^2+\cdots+\alpha^{(2^t-1)})-k(1+\alpha+\alpha^2+\cdots+\alpha^{(\mu-1)})\right]$. Therefore $ch(P_1)+ch(P_3)+ch(P_4)\in\langle\pm p,\pm u\rangle$.

If $\alpha^{\mu}r\equiv-r$ (mod n) then $\alpha^{x2^t}s+\alpha^{\mu}\alpha^{y2^t}r\equiv\alpha^{x2^t}s-\alpha^{y2^t}r\equiv\alpha^{x2^t}s-\alpha^{y2^t}s+\alpha^{y2^t}s-\alpha^{y2^t}r\equiv(\alpha^{x2^t}s-\alpha^{y2^t}s)+\alpha^{y2^t}(s-r)$ (mod n). Using the calculations in (c), we obtain $ch(P_1)+ch(P_3)+ch(P_4)\in\langle\pm p,\pm u\rangle$.

(d) $ch(P_1)+ch(P_3)+ch(P_4)\equiv\left[k(1+\alpha^{2^t}+\cdots+\alpha^{(x-1)2^t})+\alpha^{x2^t}r\right]+\left[\alpha^{\mu}k(\alpha^{x2^t}+\alpha^{(x+1)2^t}+\cdots+\alpha^{(y-1)2^t})+\alpha^{\mu}\alpha^{y2^t}s\right]+\left[k(\alpha^{y2^t}+\alpha^{(y+1)2^t}+\cdots+\alpha^{(\mu-1)2^t})\right]$.

By calculations similar to those in *(c)* with the exchanging the role of s-edges and r-edges, we can see that $ch(P_1)+ch(P_3)+ch(P_4)$ is in $\langle\pm p,\pm u\rangle$ in this case.

Thus $ch(P)\in\langle\pm p,\pm u\rangle$ if $z(P)=2$.

Suppose now that the claim is true for any walk joining two vertices of V^0 and having less than or equal to $2h$ S_μ-edges. Let P be a walk joining two vertices of V^0 and having $2(h+1)$ S_μ-edges. We represent P as the concatenation P_1*P_2 of two subwalks P_1 and P_2 such that P_1 contains $2h$ S_μ-edges and P_2 contains only two S_μ-edges. Let v_y^x be the terminal vertex of P_1. Then x must be even. Let Q be a walk joining v_y^x to a vertex of V^0 and having no S_μ-edges. Such a walk Q can be always found. Then $ch(P)=ch(P_1*P_2)=ch(P_1*Q*Q^{-1}*P_2)=ch(P_1*Q)+ch(Q^{-1}*P_2)$.

It is clear that $P_1 * Q$ joins two vertices of V^0 and has $2h$ S_μ-edges and $Q^{-1} * P_2$ also joins two vertices of V^0 and has only two S_μ-edges. By the induction hypothesis both $ch(P_1 * Q)$ and $ch(Q^{-1} * P_2)$ are in $\langle \pm p, \pm u\rangle$. Therefore $ch(P) \in \langle \pm p, \pm u\rangle$. Thus the claim is also true for a walk joining two vertices of V^0 and having $2(h+1)$ S_μ-edges. Claim 2 is proved.

By Claim 1, Claim 2 and Lemmas 2.2, 2.3, 2.4 we conclude that if i is even, μ is odd and $\gcd(i,m) = 2$, then G is connected if and only if $\gcd(\xi, u, n) = 1$. Lemma 3.2 has been proved completely. □

The next Lemma 3.3 deals with a necessary and sufficient condition for a tetravalent metacirculant graph in Case 9 of Lemma 2.5 to be connected.

Lemma 3.3. *Let $G = MC(m, n, \alpha, S_0, ..., S_\mu)$ be a tetravalent metacirculant graph with $m > 2$, $S_i = \{s\}$, $S_j = \{r\}$ for some $i \neq j \in \{1, \dots, \mu - 1\}$ if m is even or $i \neq j \in \{1, \dots, \mu\}$ if m is odd and $S_k = \emptyset$ for any $k \in \{1, \dots, \mu\} \setminus \{i, j\}$. Then G is connected if and only if gcd(i, j, m) $= 1$ and gcd(p, q, t, u, n) $= 1$, where $p = s(1 + \alpha^i + \cdots + \alpha^{(t_i-1)i})$, $q = r(1 + \alpha^j + \cdots + \alpha^{(t_j-1)j})$ with t_i, t_j the smallest positive integers satisfying $it_i \equiv 0$ (mod m), $jt_j \equiv 0$ (mod m), respectively, $u = s(1+\alpha^i+\cdots+\alpha^{(d_i-1)i}) - r(1+\alpha^j+\cdots+\alpha^{(d_j-1)j})$ with $d_i = \dfrac{lcm(i,j)}{i}$, $d_j = \dfrac{lcm(i,j)}{j}$ and $t = s(1 - \alpha^j) + r(\alpha^i - 1)$.*

Proof. It is clear that $\overline{G} = C(m, \overline{S})$ with $\overline{S} = \{\pm i, \pm j\}$. We will show that $G^0 = C(n, S^0)$ with $S^0 = \langle R\rangle$, where $R = \{\pm p, \pm q, \pm t, \pm u\}$.

Let P be a walk in G starting at a vertex v_x^0 and terminating at a vertex v_y^0 with $y \neq x$. Without loss of generality, we may assume that the walk P has no subwalks of the type $Q * Q^{-1}$ and the only vertices of P in V^0 are its endvertices. Then in order to show that $S^0 = \langle R\rangle$, we will prove that $ch(P) \in \langle R\rangle$ by induction on the sum z of the number of S_i-intervals and the number of S_j-intervals in P.

If $z = 1$ then P is either an S_i-interval or an S_j-interval. Suppose that P is an S_i-interval. Then P is an S_i^+-walk or S_i^--walk. So P can be represented in the form

$P = v_x^0 v_{x+s}^i v_{x+s+\alpha^i s}^{2i} \dots v_{x+p}^{t_i i}$, or

$P = v_x^0 v_{x-\alpha^{-i}s}^{-i} v_{x-\alpha^{-i}s-\alpha^{-2i}s}^{-2i} \dots v_{x-\alpha^{-i}s-\alpha^{-2i}s-\dots-\alpha^{-t_i i}s}^{-t_i i}$.

Therefore $ch(P) = p$ or $ch(P) = -p$. By similar arguments we can show that $ch(P) = q$ or $ch(P) = -q$ if P is an S_j-interval. Thus $ch(P) \in \langle R\rangle$ if $z = 1$.

If $z = 2$ then P has one S_i-interval and one S_j-interval. Without loss of generality, we may assume that $P = P_1 * P_2$, where P_1 is an S_i^+-interval and P_2 is an S_j-interval. Let v_b^a be the common vertex of P_1 and P_2. Then a must be a common multiple of i and j. Let $d =$lcm(i, j). Then $a = kd$ for a suitable integer k. Rewriting the vertex v_b^a by $v_{x_k}^{kd}$ and v_x^0 by $v_{x_o}^0$, we can represent P_1 in the form $P_1 = v_{x_o}^0 \dots v_{x_1}^d \dots v_{x_2}^{2d} \dots v_{x_k}^{kd} = W_1 * W_2 * \cdots * W_k$, where $W_l = v_{x_{l-1}}^{(l-1)d} \dots v_{x_l}^{ld}$ is an S_i^+-walk for $l = 1, 2, \dots, k$.

Let Q_l be the S_j^--walk joining vertices $v_{x_l}^{ld}$ and $v_{f_{l-1}}^{(l-1)d}$ of $V^{(l-1)d}$ for a suitable $f_{l-1} \in \mathbb{Z}_n$, $l = 1, 2, \dots, k$. These walks exist because d is a multiple of i and j. Then we construct the walk P_1' from P_1 as follows: $P_1' = W_1 * Q_1 * Q_1^{-1} * W_2 * Q_2 * Q_2^{-1} *$

$\cdots * W_k * Q_k * Q_k^{-1}$. So we have

$$ch(P_1) = ch(P_1^{'}) = \sum_{l=1}^{k} ch(W_l * Q_l * Q_l^{-1}) = \sum_{l=1}^{k} ch(W_l * Q_l) + \sum_{l=1}^{k} ch(Q_l^{-1}).$$

We have $ch(W_l * Q_l)$ is equal to

$$\begin{aligned} & \alpha^{(l-1)d}s + \alpha^{(l-1)d+i}s + \cdots + \alpha^{(l-1)d+(d_i-1)i}s - \alpha^{ld-j}r - \alpha^{ld-2j}r - \cdots \\ & \cdots - \alpha^{(l-1)d}r \\ = \; & \alpha^{(l-1)d}\big[s(1+\alpha^i+\cdots+\alpha^{(d_i-1)i}) - r(\alpha^{d-j}+\alpha^{d-2j}+\cdots+1)\big] \\ = \; & \alpha^{(l-1)d}\big[s(1+\alpha^i+\cdots+\alpha^{(d_i-1)i}) - r(1+\alpha^{j}+\cdots+\alpha^{(d_j-1)j})\big] \\ = \; & \alpha^{(l-1)d}u. \end{aligned}$$

Therefore $\sum_{l=1}^{k} ch(W_l * Q_l) = \sum_{l=1}^{k} \alpha^{(l-1)d}u$. On the other hand,

$$\begin{aligned} \sum_{l=1}^{k} ch(Q_l^{-1}) &= \sum_{l=1}^{k} \left(\alpha^{(l-1)d}r + \alpha^{(l-1)d+j}r + \cdots + \alpha^{(l-1)d+(d_j-1)j}r\right) \\ &= \alpha^0 r + \alpha^j r + \cdots + \alpha^{(d_j-1)j}r + \cdots + \alpha^{(k-1)d+(d_j-1)j}r \\ &= \alpha^0 r + \alpha^j r + \cdots + \alpha^{(kd_j-1)j}r \\ &= r(1+\alpha^j+\cdots+\alpha^{(kd_j-1)j}). \end{aligned}$$

So $ch(P_1) = \sum_{l=1}^{k} \alpha^{(l-1)d}u + r(1+\alpha^j+\cdots+\alpha^{(kd_j-1)j})$.

Consider the walk P_2. Since $|S_j| = 1$ and P has no subwalks of the type $Q * Q^{-1}$, we can see that P_2 is either S_j^+-interval or S_j^--interval. Therefore,

$$ch(P_2) = \begin{cases} \alpha^{kd}r + \alpha^{kd+j}r + \cdots + \alpha^{(t_j-1)j}r, & \text{if } P_2 \text{ is an } S_j^+\text{-interval}, \\ -\alpha^{(kd_j-1)j}r - \alpha^{(kd_j-2)j}r - \cdots - \alpha^0 r, & \text{if } P_2 \text{ is an } S_j^-\text{-interval}. \end{cases}$$

Then

$$ch(P) = ch(P_1) + ch(P_2) = \begin{cases} \sum_{l=1}^{k} \alpha^{(l-1)d}u + q, & \text{if } P_2 \text{ is an } S_j^+\text{-interval}, \\ \sum_{l=1}^{k} \alpha^{(l-1)d}u, & \text{if } P_2 \text{ is an } S_j^-\text{-interval}. \end{cases}$$

Therefore $ch(P) \in \langle R \rangle$.

For the remaining possibilities of this case we can use similar proofs to get the assertion that $ch(P) \in \langle R \rangle$. Thus $ch(P) \in \langle R \rangle$ for any walk P with the sum z of the number of S_i-intervals and the number of S_j-intervals equal to 2.

Now let P be a walk belonging to one of the four following types:

1. P_1 consists of k S_i^+-edges, P_2 consists of l S_j^+-edges, P_3 consists of k S_i^--edges and P_4 consists of l S_j^--edges.

2. P_1 consists of k S_i^+-edges, P_2 consists of l S_j^--edges, P_3 consists of k S_i^--edges and P_4 consists of l S_j^+-edges.

3. P_1 consists of k S_i^--edges, P_2 consists of l S_j^--edges, P_3 consists of k S_i^+-edges and P_4 consists of l S_j^+-edges.

4. P_1 consists of k S_i^--edges, P_2 consists of l S_j^+-edges, P_3 consists of k S_i^+-edges and P_4 consists of l S_j^--edges.

We show that $ch(P) \in < R >$. We do calculations in detail only for types (1) and (2). The remaining types (3) and (4) can be considered similarly and we omit their proof here.

(1) It is easy to see that

$$\begin{aligned} ch(P_1) &= s(1+\alpha^i+\alpha^{2i}+\cdots+\alpha^{(k-1)i}),\\ ch(P_2) &= r(\alpha^{ki}+\alpha^{ki+j}+\cdots+\alpha^{ki+(l-1)j}),\\ ch(P_3) &= s(-\alpha^{(k-1)i+lj}-\alpha^{(k-2)i+lj}-\cdots-\alpha^{lj}),\\ ch(P_4) &= r(-\alpha^{(l-1)j}-\alpha^{(l-2)j}-\cdots-1). \end{aligned}$$

Then $ch(P) = s(1-\alpha^{lj})(1+\alpha^i+\alpha^{2i}+\cdots+\alpha^{(k-1)i}) + r(\alpha^{ki}-1)(1+\alpha^j+\alpha^{2j}+\cdots+\alpha^{(l-1)j}) = s(1-\alpha^j)(1+\alpha^j+\alpha^{2j}+\cdots+\alpha^{(l-1)j})(1+\alpha^i+\alpha^{2i}+\cdots+\alpha^{(k-1)i}) + r(\alpha^i-1)(1+\alpha^j+\alpha^{2j}+\cdots+\alpha^{(l-1)j})(1+\alpha^i+\alpha^{2i}+\cdots+\alpha^{(k-1)i}) = \big[s(1-\alpha^j)+r(\alpha^i-1)\big](1+\alpha^j+\alpha^{2j}+\cdots+\alpha^{(l-1)j})(1+\alpha^i+\alpha^{2i}+\cdots+\alpha^{(k-1)i}) = t(1+\alpha^j+\alpha^{2j}+\cdots+\alpha^{(l-1)j})(1+\alpha^i+\alpha^{2i}+\cdots+\alpha^{(k-1)i})$. So $ch(P) \in \langle R\rangle$.

(2) We have

$$\begin{aligned} ch(P_1) &= s(1+\alpha^i+\alpha^{2i}+\cdots+\alpha^{(k-1)i}),\\ ch(P_2) &= r(-\alpha^{ki-j}-\alpha^{ki-2j}-\cdots-\alpha^{ki-lj}),\\ ch(P_3) &= s(-\alpha^{(k-1)i-lj}-\alpha^{(k-2)i-lj}-\cdots-\alpha^{-lj}),\\ ch(P_4) &= r(\alpha^{-lj}+\alpha^{(-l-1)j}+\cdots+\alpha^{-j}). \end{aligned}$$

Then $ch(P) = s(1-\alpha^{-lj})(1+\alpha^i+\alpha^{2i}+\cdots+\alpha^{(k-1)i}) + r(1-\alpha^{ki})(\alpha^{-j}+\alpha^{-2j}+\cdots+\alpha^{-lj}) = s(1-\alpha^{-j})(1+\alpha^{-j}+\cdots+\alpha^{-(l-1)j})(1+\alpha^i+\alpha^{2i}+\cdots+\alpha^{(k-1)i}) + \alpha^{-j}r(1-\alpha^i)(1+\alpha^i+\alpha^{2i}+\cdots+\alpha^{(k-1)i})(1+\alpha^{-j}+\cdots+\alpha^{-(l-1)j}) = \big[s(1-\alpha^{-j})+\alpha^{-j}r(1-\alpha^i)\big](1+\alpha^i+\alpha^{2i}+\cdots+\alpha^{(k-1)i})(1+\alpha^{-j}+\cdots+\alpha^{-(l-1)j})$. But $\big[s(1-\alpha^{-j})+\alpha^{-j}r(1-\alpha^i)\big] = (-\alpha^{-j})\big[\frac{s(1-\alpha^{-j})}{-\alpha^{-j}} + r(\alpha^i-1)\big] = (-\alpha^{-j})\big[s(1-\alpha^j)+r(\alpha^i-1)\big] = (-\alpha^{-j})t$. So $ch(P) \in \langle R\rangle$.

Assume now that $h \geq 3$ and $ch(X) \in < R >$ for any walk X having the beginning and terminating vertices in V^0 and the number of intervals less than h. Let P be a walk having the beginning and terminating vertices in V^0 and the number z of intervals equal to h. Let $P = P_1 * P_2 * \cdots * P_{h-1} * P_h$. Without loss of generality we may assume that P_1 is an S_i-interval and P_2 is an S_j-interval. Let v_b^a be the common vertex of P_2 and P_3. We can choose an S_i-interval Q_1 and an S_j-interval Q_2 such that $P_1 * P_2 * Q_1 * Q_2$ is a walk belonging to one of the special types just considered above. Then the endvertices of $Q_1 * Q_2$ are v_b^a and v_x^0. Now we insert the subwalk $Q_1 * Q_2 * Q_2^{-1} * Q_1^{-1}$ into P at the vertex v_b^a. Let $P' = P_1 * P_2 * Q_1 * Q_2$

and $P'' = Q_2^{-1} * Q_1^{-1} * P_3 * P_4 * \cdots * P_h$. Then as we have already shown above, $ch(P') \in < R >$. Further, the endvertices of P'' are in V^0 and the number of intervals of P'' is less than h. So by the induction hypothesis, $ch(P'') \in \langle R \rangle$. Since $ch(P) = ch(P') + ch(P''$, it follows that $ch(P) \in < R >$.

Thus, for any walk P starting at $v_x^0 \in V^0$ and terminating at $v_y^0 \in V^0$ with $x \neq y$, we always have $ch(P) \in \langle R \rangle$. So we conclude that $G^0 = C(n, S^0)$ where $S^0 =< R >$ with $R = \{ \pm p, \pm q, \pm t, \pm u\}$.

By Lemmas 2.2, 2.3 and 2.4, we can assert that the graph G is connected if and only if $\gcd(i, j, m) = 1$ and $\gcd(p, q, t, u, n) = 1$, where $p = s(1 + \alpha^i + \cdots + \alpha^{(t_i-1)i})$, $q = r(1 + \alpha^j + \cdots + \alpha^{(t_j-1)j})$ with t_i, t_j are the smallest positive integers satisfying $it_i \equiv 0 \pmod m$, $jt_j \equiv 0 \pmod m$, respectively, $u = s(1+\alpha^i+\cdots+\alpha^{(d_i-1)i}) - r(1+\alpha^j+\cdots+\alpha^{(d_j-1)j})$ with $d_i = \dfrac{lcm(i,j)}{i}$, $d_j = \dfrac{lcm(i,j)}{j}$ and $t = s(1-\alpha^j)+r(\alpha^i-1)$. □

Now we consider tetravalent metacirculant graphs in Case 10 of Lemma 2.5. The following lemma provides a necessary and sufficient condition for these graphs to be connected.

Lemma 3.4. *Let $G = MC(m, n, \alpha, S_0, ..., S_\mu)$ be a tetravalent metacirculant graph with $m > 2$, $S_i = \{s, r\}$ for some $i \in \{1, \dots, \mu - 1\}$ if m is even or $i \in \{1, \dots, \mu\}$ if m is odd and $S_j = \emptyset$ for all $j \in \{0, 1, \dots, \mu\} \setminus \{i\}$. Then G is connected if and only if $gcd(i, m) = 1$ and $gcd(g, u, n) = 1$, where $u = s - r$ and $g = s(1+\alpha^i+\ldots+\alpha^{(m-1)i})$.*

Proof. It is clear that $\overline{G} = C(m, \overline{S})$ with $\overline{S} = \{\pm i\}$. Let d be the smallest positive integer such that $di \equiv 0 \pmod m$, $p = s(1 + \alpha^i + \cdots + \alpha^{(d-1)i})$ and $u = s - r$. We will prove that $G^0 = C(n, S^0)$, where $S^0 = \langle R \rangle$ with $R = \{\pm p, \pm u\}$.

Let P be a walk starting at a vertex v_x^0 and terminating at a vertex v_y^0 of the block V^0 where $x \neq y$. We show that $ch(P) \in \langle R \rangle$. Without loss of generality, we may assume that P does not contain subwalks of the type $Q * Q^{-1}$, and the only vertices of P in V^0 are its endvertices. We also set $q = r(1 + \alpha^i + \cdots + \alpha^{(d-1)i})$. We will prove that $ch(P) \in \langle R \rangle$ by induction on z, where z is the sum of the number of s-intervals and the number of r-intervals in P.

If $z = 1$, then P must be either an s-interval or an r-interval. It is clear that $ch(P) \in \{\pm p\}$ or $ch(P) \in \{\pm q\}$. In the latter case, we can write $q = [s-(s-r)](1+\alpha^i + \cdots + \alpha^{(d-1)i}) = p - u(1 + \alpha^i + \cdots + \alpha^{(d-1)i})$. So $ch(P) \in \langle R \rangle$ in both cases.

If $z = 2$, then P must be a walk belonging to one of the following types:

1) $P = A_1 * A_2$, where A_1 is an s^+-interval and A_2 is an r^--interval ,

2) $P = B_1 * B_2$, where B_1 is an s^+-interval and B_2 is an r^+-interval ,

3) $P = C_1 * C_2$, where C_1 is an s^--interval and C_2 is an r^+-interval ,

4) $P = D_1 * D_2$, where D_1 is an s^--interval and D_2 is an r^--interval .

and four other types, which are similar to the above ones and the first and the second intervals of which are r-intervals and s-intervals, respectively.

We consider these cases in turn.

1) $P = A_1 * A_2$, where A_1 is an s^+-interval and A_2 is an r^--interval .

Suppose that A_1 contains k s^+-edges. Since $s, r \in S_i$, A_2 must contain k r^--edges. Therefore A_1 and A_2 can be represented as follows:

$$\begin{aligned} A_1 &= v_x^0 v_{x+s}^i v_{x+s+\alpha^i s}^{2i} \dots v_{x+s+\alpha^i s+\dots+\alpha^{(k-1)i}s}^{ki}, \\ A_2 &= v_{x+s+\dots+\alpha^{(k-1)i}s}^{ki} v_{x+s+\dots+\alpha^{(k-1)i}s-\alpha^{(k-1)i}r}^{(k-1)i} \dots \\ &\quad \dots v_{x+s+\dots+\alpha^{(k-1)i}s-\alpha^{(k-1)i}r-\dots-r}^{0}. \end{aligned}$$

Then

$$\begin{aligned} ch(P) &= ch(A_1) + ch(A_2) \\ &= s(1+\alpha^i+\dots+\alpha^{(k-1)i}) - r(1+\alpha^i+\dots+\alpha^{(k-1)i}) \\ &= (s-r)(1+\alpha^i+\dots+\alpha^{(k-1)i}) = u(1+\alpha^i+\dots+\alpha^{(k-1)i}). \end{aligned}$$

So $ch(P) \in \langle R \rangle$.

2) $P = B_1 * B_2$, where B_1 is an s^+-interval and B_2 is an r^+-interval.

Denote the common vertex of B_1 and B_2 by v_b^a. We construct the walk P' from P by inserting the subwalk $Q * Q^{-1}$ into P at the vertex v_b^a, where Q is an r^--interval joining v_b^a to a vertex v_f^0 of V^0. It is clear that such a walk Q exists. Then $P' = B_1 * Q * Q^{-1} * B_2$. So $ch(P) = ch(P') = ch(B_1 * Q) + ch(Q^{-1} * B_2)$. Since $B_1 * Q$ is a walk connecting v_x^0 to v_f^0 and containing one s^+-interval and one r^--interval and $Q^{-1} * B_2$ is an r^+-interval from v_f^0 to v_y^0, by the induction basis and Case (1), both $ch(B_1 * Q)$ and $ch(Q^{-1} * B_2)$ are in $\langle R \rangle$. Therefore $ch(P) \in \langle R \rangle$.

3) $P = C_1 * C_2$, where C_1 is an s^--interval and C_2 is an r^+-interval.

Let k be the number of s^--edges in C_1, then C_2 must contain k r^+-edges because s and r are in the same symbol S_i. Therefore C_1 and C_2 can be represented as follows:

$$\begin{aligned} C_1 &= v_x^0 v_{x-\alpha^{-i}s}^{-i} v_{x-\alpha^{-i}s-\alpha^{-2i}s}^{-2i} \dots v_{x-\alpha^{-i}s-\alpha^{-2i}s-\dots-\alpha^{-ki}s}^{-ki}, \\ C_2 &= v_{x-\alpha^{-i}s-\alpha^{-2i}s-\dots-\alpha^{-ki}s}^{-ki} v_{x-\alpha^{-i}s-\alpha^{-2i}s-\dots-\alpha^{-ki}s+\alpha^{-ki}r}^{-ki+i} \dots v_y^0, \end{aligned}$$

where $y = x - \alpha^{-i}s - \alpha^{-2i}s - \dots - \alpha^{-ki}s + \alpha^{-ki}r + \alpha^{-(k-1)i}r + \dots + \alpha^{-i}r$. Then

$$\begin{aligned} ch(P) &= ch(C_1) + ch(C_2) \\ &= -\alpha^{-i}s - \alpha^{-2i}s - \dots - \alpha^{-ki}s + \alpha^{-ki}r + \alpha^{-(k-1)i}r + \dots + \alpha^{-i}r \\ &= \alpha^{-i}(r-s) + \alpha^{-2i}(r-s) + \dots + \alpha^{-ki}(r-s) \\ &= (r-s)\big(\alpha^{-i} + \alpha^{-2i} + \dots + \alpha^{-ki}\big) \\ &= u(-\alpha^{-i} - \alpha^{-2i} - \dots - \alpha^{-ki}). \end{aligned}$$

Thus $ch(P) \in \langle R \rangle$.

4) $P = D_1 * D_2$, where D_1 is an s^--interval and D_2 is an r^--interval.

Denote the common vertex of D_1 and D_2 by v_b^a. We construct the walk P' from P by inserting the subwalk $Q * Q^{-1}$ into P at the vertex v_b^a, where Q is an r^+-interval joining v_b^a to a vertex v_f^0 of V^0. It is clear that such a walk Q exists. Then

$P' = D_1 * Q * Q^{-1} * D_2$. So $ch(P) = ch(P') = ch(D_1 * Q) + ch(Q^{-1} * D_2)$. It is clear that $D_1 * Q$ is a walk from v_x^0 to v_f^0 containing one s^--interval and one r^+-interval and $Q^{-1} * D_2$ is an r^--interval from v_f^0 to v_y^0. By the induction basis and by Case (3), both $ch(D_1 * Q)$ and $ch(Q^{-1} * D_2)$ are in $\langle R\rangle$. Therefore $ch(P) \in \langle R\rangle$.

For the four remaining cases, where P is a walk containing two interval with the first one an r-interval and the second one an s-interval , we consider the inverse walk P^{-1} of P. Then $ch(P^{-1}) \in \langle R\rangle$ by the considered above cases. So $ch(P) = -ch(P^{-1}) \in \langle R\rangle$. Thus, for any walk P in G, which has two intervals and the only vertices of which in V^0 are its endvertices, we have proved that $ch(P) \in \langle R\rangle$.

Assume now that the assertion is true for any walk in G, the only vertices of which in V^0 are its endvertices and the sum of the number of s-intervals and the number of r-intervals in which is less than or equal to z. Let P be a walk, the only vertices of which in V^0 are its endvertices and the sum of the number of s-intervals and the number of r-intervals in which is $z + 1$. Then we represent the walk P in the form $P = S_1 * S_2$, where S_1 is a walk containing the first z intervals of P and S_2 is the last interval of P. Without loss of generality, we may assume that S_2 is an r-interval. Denote the common vertex of S_1 and S_2 by v_b^a. We construct the walk P' from P by inserting the subwalk $Q * Q^{-1}$ into P at the common vertex v_b^a, where Q in an s-interval from v_b^a to v_f^0 of V^0. It is clear that such a walk Q exists. Then $P' = S_1 * Q * Q^{-1} * S_2$. Therefore $ch(P) = ch(P') = ch(S_1 * Q) + ch(Q^{-1} * S_2)$.

We can see that $S_1 * Q$ is a walk, the only vertices of which in V^0 are its endvertices and the sum of the number of s-intervals and the number of r-intervals is z. By the induction hypothesis, $ch(S_1 * Q) \in \langle R\rangle$. Further, $Q^{-1} * S_2$ is a walk, which starts and terminates at vertices of V^0 and has only two intervals. So $ch(Q^{-1} * S_2)$ is also in $\langle R\rangle$. It follows that $ch(P) \in \langle R\rangle$.

Thus, for any walk P starting and terminating at vertices of the same block V^0, we have proved that $ch(P) \in \langle R\rangle$. Therefore, $G^0 = C(n, S^0)$, where $S^0 = \langle R\rangle$ with $R = \{\pm p, \pm u\}$.

By Lemma 2.2, G is connected if and only if both $\overline{G}$ and G^0 are connected. The graph $\overline{G} = C(m, \overline{S})$ has $\overline{S} = \{\pm i\}$. So by Lemma 2.3, $\overline{G}$ is connected if and only if $\gcd(i, m) = 1$. Therefore, the smallest positive integer d for which $di \equiv 0 \pmod m$ is equal to m. It follows that $p = s(1+\alpha^i+\cdots+\alpha^{(m-1)i}) = g$. The graph $G^0 = C(n, S^0)$ has $S^0 = \langle \pm p, \pm u\rangle = \langle \pm g, \pm u\rangle$. By Lemmas 2.3 and 2.4 G^0 is connected if and only if $gcd(g, u, n) = 1$.

So the graph G is connected if and only if $\gcd(i, m) = 1$ and $\gcd(g, u, n) = 1$, where $g = s(1 + \alpha^i + \cdots + \alpha^{(m-1)i})$ and $u = s - r$. The Lemma 3.4 is proved completely. □

From Lemmas 3.1, 3.2, 3.3 and 3.4 we immediately obtain the following theorem in which Condition 1 is for graphs in Case 7, Conditions 2 and 3 are for graphs in Case 8, Condition 4 is for graphs in Case 9 and Condition 5 is for graphs in Case 10 of Lemma 2.5.

Theorem 3.5. *Let $G = MC(m, n, \alpha, S_0, ..., S_\mu)$ be a tetravalent metacirculant graph with $S_0 = \emptyset$. Then G is connected if and only if one of the following conditions holds:*

1. *$m = 2$, $S_1 = \{s_1, s_2, s_3, s_4\}$ and gcd$(s_1 - s_2, s_2 - s_3, s_3 - s_4, n) = 1$;*

2. *$m > 2$ is even, $S_1 = \cdots = S_{i-1} = \emptyset$, $S_i = \{k\}$ with i odd and gcd$(i, m) = 1$, $S_{i+1} = \cdots = S_{\mu-1} = \emptyset$, $S_\mu = \{s, r\}$ and gcd$(p, u, n) = 1$, where $u = s - r$ and $p = k(1 + \alpha^i + \cdots + \alpha^{(\mu-1)i}) - s$.*

3. *$m > 2$ is even, $\mu = m/2$ is odd, $S_1 = \cdots = S_{i-1} = \emptyset$, $S_i = \{k\}$ with i even and gcd$(i, m) = 2$, $S_{i+1} = \cdots = S_{\mu-1} = \emptyset$, $S_\mu = \{s, r\}$ and gcd$(\xi, u, n) = 1$, where $u = s - r$ and $\xi = \left[s(1+\alpha^{i'}+\alpha^{2i'}+\cdots+\alpha^{(2^t-1)i'}) - k(1+\alpha^{i'}+\alpha^{2i'}+\cdots+\alpha^{(\mu-1)i'})\right]$ with $i = 2^t i'$, $t \geq 1$ and i' odd.*

4. *$m > 2$, $S_i = \{s\}$, $S_j = \{r\}$ for some $i, j \in \{1, \ldots, \mu - 1\}$, $i \neq j$ if m is even or $i, j \in \{1, \ldots, \mu\}$, $i \neq j$ if m is odd such that gcd$(i, j, m) = 1$, $S_k = \emptyset$ for any $k \in \{1, \ldots, \mu\} \backslash \{i, j\}$ and gcd$(p, q, t, u, n) = 1$, where $p = s(1+\alpha^i+\cdots+\alpha^{(t_i-1)i})$, $q = r(1+\alpha^j+\cdots+\alpha^{(t_j-1)j})$, $u = s(1+\alpha^i+\cdots+\alpha^{(d_i-1)i}) - r(1+\alpha^j+\cdots+\alpha^{(d_j-1)j})$, $t = s(1-\alpha^j)+r(\alpha^i-1)$ with t_i, t_j the smallest positive integers satisfying $it_i \equiv 0$ (mod m), $jt_j \equiv 0$ (mod m), respectively, and $d_i = \dfrac{lcm(i,j)}{i}$, $d_j = \dfrac{lcm(i,j)}{j}$.*

5. *$m > 2$, $S_i = \{s, r\}$ for some $i \in \{1, \ldots, \mu - 1\}$ if m is even or $i \in \{1, \ldots, \mu\}$ if m is odd such that gcd$(i, m) = 1$, $S_j = \emptyset$ for all $j \in \{0, 1, \ldots, \mu\} \setminus \{i\}$ and gcd$(g, u, n) = 1$, where $u = s - r$ and $g = s(1 + \alpha^i + \ldots + \alpha^{(m-1)i})$.*

Based on the results obtained, we get the following algorithm for determining whether a given metacirculant graph is a connected tetravalent metacirculant graph.

Algorithm 3.6. *Let $G = MC(m, n, \alpha, S_0, ..., S_\mu)$ be a metacirculant graph where $m, n, \alpha, S_0, ..., S_\mu$ are input data to the algorithm.*

1. *Check if G is tetravalent by using Lemma 2.5. If 'Yes' then go to Step 2 else answer: G is not tetravalent'.*

2. *If $S_0 \neq \emptyset$ then we check the connectedness of G by Theorem 2.6 else we check the connectedness of G by Theorem 3.5.*

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