AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **32** (2005), Pages 243–252

A-efficient balanced treatment incomplete block designs

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Abstract

The purpose of this paper is to present a large number of highly A-efficient incomplete block designs for making comparisons among a set of test treatments and a control treatment. These designs are BTIB designs. A simple method of construction of BTIB designs, based on BIB designs, is proposed. The advantage of this method is that one can use the vast literature on BIB designs to obtain a large number of highly A-efficient BTIB designs. In several cases, for a given number of test treatments and given block size, these efficient designs require far fewer numbers of blocks than the corresponding A-optimal designs available in the literature.

1 Introduction

In many industrial, agricultural and biological experiments, one often encounters the following problem: p new or *test* treatments are available and an existing old treatment (called *control*) is to be eventually replaced by one of the test treatments. It is desired to conduct an efficient experiment for comparing each of the test treatments with control and also the test treatments among themselves; the primary interest however is in the control-test treatment comparisons. The issue of obtaining optimal designs for this problem has received a great deal of attention due to their wide applicability. For excellent reviews on the subject up to different stages, see Hedayat, Jacroux and Majumdar (1988) and Majumdar (1996), where more references can be found.

This communication provides highly efficient designs for comparing p test treatments with a control, using b blocks each of size $k \leq p$. Under the usual additive and homoscedastic linear model, the aim is to find block designs that allow the unbiased estimation of the elementary contrasts among the p test treatments and the control with maximum efficiency. Among the various optimality criteria that are available in the literature, the most appealing one in the present context is the A-optimality criterion for which the sum of the variances of the best linear unbiased estimators for the p elementary contrasts among each of the test treatments and the control is a minimum. As such, we use the A-criterion as the basis of our choice for a good design for the problem under consideration. Throughout, we shall denote the class of all connected designs (i.e., designs permitting the estimability of all elementary treatment contrasts among the test treatments and the control) having p test treatments, b blocks and block size k by $\mathcal{D}(p, b, k)$. The control treatment will be denoted by 0 and the test treatments will be labelled $1, 2, \ldots, p$. Clearly, a connected design allows the estimability of all contrasts among test treatments as well.

A useful class of designs for planning test treatments-control experiments is the class of balanced treatment incomplete block (BTIB) designs, introduced by Bechhofer and Tamhane (1981). According to Bechhofer and Tamhane (1981), a design $d \in \mathcal{D}(p, b, k)$ is called a BTIB design if

(a) d is incomplete, i.e., no block contains all the $p+1$ treatments,
(b) $\lambda_{0i} = \lambda_c,\ 1 \leq i \leq p$ and $\lambda_{i_1 i_2} = \lambda,\ 1 \leq i_1 \neq i_2 \leq p$, where $\lambda_{uu'} = \sum_{j=1}^{b} n_{uj} n_{u'j},\ 0 \leq u \neq u' \leq p$ and n_{xy} denotes the number of times the xth treatment appears in the yth block, $0 \leq x \leq p,\ 1 \leq y \leq b$.

The parameters of a BTIB design are denoted by $p, b, k, r, r_c, \lambda, \lambda_c$. Here r is the replication number of each of the test treatments and r_c, that of the control treatment. A perusal of the existing literature shows that an A-optimal design in $\mathcal{D}(p, b, k)$ belongs to a subclass of BTIB designs, called BTIB $(p, b, k; t, s)$ designs and it is for this reason that BTIB$(p, b, k; t, s)$ designs have been studied extensively in the literature. A design $d \in \mathcal{D}(p, b, k)$ is called a BTIB$(p, b, k; t, s)$ design if

(i) d is a BTIB design which is binary in test treatments, and
(ii) there are s blocks in d each of which contains exactly $t+1$ replications of the control, while each of the remaining $b - s$ blocks contains exactly t replications of the control.

The construction of BTIB$(p, b, k; t, s)$ designs has been addressed among others, by Hedayat and Majumdar (1984), Stufken (1987), Cheng, Majumdar, Stufken and Türe (1988) and Parsad, Gupta and Prasad (1995). A BTIB$(p, b, k; t, s)$ design may not exist for all values of the parameters. Also, highly A-efficient BTIB designs not belonging to the class of BTIB$(p, b, k; t, s)$ designs might exist. These considerations motivate one to find highly efficient BTIB designs not necessarily belonging to the class of BTIB$(p, b, k; t, s)$ designs. In Section 2 of this paper, we give a simple method of construction of BTIB designs, using balanced incomplete block (BIB) designs. The advantage of this method is that one can use the extremely rich literature on BIB designs to construct BTIB designs. Using this method, a large number of highly A-efficient BTIB designs are obtained. These designs in most cases require far fewer number of blocks than an available A-optimal design for the same value of p and k. In view of this, the proposed designs are likely to be useful in practice as the A-efficiency of these designs is close to unity (the A-efficiency of an A-optimal design is unity) and at the same time there is considerable saving in terms of experimental units.

From practical considerations, it is useful to have a catalog of efficient designs. In Section 3, we present a comprehensive catalog of highly A-efficient BTIB designs in the practically useful ranges $2 \leq k \leq 10,\ r \leq 10,\ k \leq p \leq b \leq 50$.

2 Construction of BTIB designs

Consider a BIB design d_0 with usual parameters $v^*, b^*, r^*, k^*, \lambda^*$. Replace i $(0 \leq i \leq v^* - 2)$ of the treatments in d_0 by the control treatment and call the resultant design BIB$_i(v^*, b^*, k^*)$. Finally, augment each block of the design BIB$_i(v^*, b^*, k^*)$ by $t \geq 0$ replications of the control, such that $(i, t) \neq (0, 0)$ and call this design d. Then, it is easy to see that d is a BTIB design with parameters $p = v^* - i, b = b^*, k = k^* + t, r = r^*, r_c = ir^* + b^*t, \lambda = \lambda^*, \lambda_c = i\lambda^* + r^*t,\ 0 \leq i \leq v^* - 2,\ t \geq 0$. For convenience, in the catalog of designs that follows later, the design d is denoted by BIB$_i(v^*, b^*, k^*; t)$. Note that a BIB$_0(v^*, b^*, k^*; t)$ design is a BTIB$(v^*, b^*, k^* + t; t, 0)$ design (of the R-type) while a BIB$_1(v^*, b^*, k^*; t)$ is a BTIB$(v^* - 1, b^*, k^* + t; t, s(= b^*k^*/v^*))$design (of the S-type). For a definition of R- and S-type BTIB designs, see Hedayat and Majumdar (1984). It may be remarked here that systematic methods of construction of A-optimal (or, highly A-efficient) S-type BTIB designs are largely not available. The present method of construction gives a fairly large class of S-type BTIB designs.

A few designs in the catalog are obtained through partially balanced incomplete block (PBIB) designs. It is therefore thought necessary to describe this method of construction of BTIB designs as well. Consider two PBIB designs d_1 and d_2, with two associate classes such that both these designs are based on the same association scheme. Suppose the parameters of d_1 and d_2 respectively are $v, b_i,\ r_i,\ k_i,\ \lambda_{1i},\ \lambda_{2i}$, $i = 1, 2$. Assume without loss of generality that $k_2 > k_1$. If d_1 and d_2 are such that $\lambda_{11} + \lambda_{12} = \lambda_{21} + \lambda_{22} = \lambda$, then the design obtained by taking the union of the blocks of d_1 and d_2, and adding the control treatment $k_2 - k_1$ times to the blocks of size k_1, is a BTIB design with $p = v,\ b = b_1 + b_2,\ k = k_2,\ r = r_1 + r_2,\ r_c = b_1(k_2 - k_1),\ \lambda$, $\lambda_c = r_1(k_2 - k_1)$. A result similar to the above was obtained earlier by Parsad, Gupta and Prasad (1995); however they restrict attention only to those PBIB designs for which $k_2 = k_1 + 1$.

3 A catalog of A-efficient BTIB designs

The A-efficiency of a design for making test treatments-control comparisons is computed following the procedure described by Stufken (1988). As before, we denote by $\mathcal{D}(p,b,k)$ the class of all connected designs with p test treatments, one control, b blocks and block size k. Let $(\hat{\tau}_{d0}-\hat{\tau}_{di}), i=1,\ldots,p$, be the best linear unbiased estimator of $(\tau_0-\tau_i)$ under a design $d\in\mathcal{D}(p,b,k)$ where τ_0 and τ_i respectively denote the effect of the control and ith test treatment. A design is called A-optimal if it minimizes $\Sigma_{i=1}^{p} Var(\hat{\tau}_{d0}-\hat{\tau}_{di})$ as d varies over $\mathcal{D}(p,b,k)$. Let $a=(p-1)^2, c=bpk(k-1), q=p(k-1)+k, \Lambda=\{(x,z), x=0,\ldots,[k/2]-1; z=0,1,\ldots,b$ with $z>0$ when $x=0\}$. Here $[\cdot]$ is the greatest integer function. Furthermore, let $g(x,z)=a/\{c-q(bx+z)+(bx^2+2xz+z)\}+1/\{k(bx+z)-(bx^2+2xz+z)\}$. A *lower bound* to the A-efficiency of a BTIB design d with parameters $p,b,k,r,r_c,\lambda,\lambda_c$ is then given by

$$e=\frac{\min\limits_{(x,z)\in\Lambda} g(x,z)}{B_d}$$

where

$$B_d=\frac{(\lambda_c+\lambda)}{\lambda_c(\lambda_c+p\lambda)}.$$

If $\min\limits_{(x,z)\in\Lambda} g(x,z)$ is attained at $x=t,\ z=s$, then the corresponding design has $e=1$ and the design is A-optimal. Using the expression given above, we have computed lower bounds to the A-efficiency of BTIB designs constructed in this paper.

In Table 1, we present a catalog of highly A-efficient BTIB designs ($e\geq 0.950$) in the practically useful ranges $2\leq k\leq 10, r\leq 10, k\leq p\leq b\leq 50$. Among designs with the same values of p and k, there may exist several designs with $e\geq 0.950$. In Table 1, among such designs with same values of p and k we do *not* list designs that satisfy both the following conditions: (i) small value of e and (ii) large number of blocks. That is, if for the same values of p and k, there are two designs, say d_1 and d_2 having b_1 and b_2 blocks and A-efficiencies e_1 and e_2 respectively, such that $b_1\geq b_2$, then d_1 is not included in the catalog if $e_1\leq e_2$.

Furthermore, for some combinations of the parameters p and k, no A-optimal designs have been reported by Hedayat and Majumdar (1984). Nearly A-optimal designs (that is, designs with A-efficiency close to unity) for such situations are also reported in Table 1. For instance, no A-optimal design is reported in Hedayat and Majumdar (1984) for $p=10,\ k=3$. For these values of p and k, we report a design (No. 10 in Table 1) with A-efficiency at least 0.986. (Recall that e is a *lower bound* to the A-efficiency.) It is also noted that two A-optimal designs, obtained through trial and error by Hedayat and Majumdar (1984), can also be obtained by following the method described in this paper. The parameters of these designs are $p=14, b=35, k=7$ and $p=15, b=16, k=7$; these are exhibited as Design S6 and S7 respectively in Hedayat and Majumdar (1984) and can in fact, be obtained as $\text{BIB}_1(15,35,6;1)$ and $\text{BIB}_1(16,16,6;1)$ respectively.

Under the 'Reference' column in Table 1, S, SR, R, T and LS refer to PBIB designs in Clatworthy (1973). In some cases, the trivial disconnected PBIB design with m blocks each of size k have been used. This fact is exhibited as (m, k). Among the 10 designs (Nos. 5, 6, 10, 17, 19, 21, 25, 26, 41, 45) in Table 1 constructed using PBIB designs, design numbers 6 and 26 have not been reported earlier. The rest of the designs can also be found in Gupta, Pandey and Parsad (1998). An A-optimal design with $p = 9$ and $k = 3$, has been obtained earlier by Hedayat and Majumdar (1984) and requires only 24 blocks as compared to 36 blocks of design number 9 in Table 1.

The catalog of designs presented in Table 1 contains 155 designs in the ranges of parameters specified earlier. Out of these, 45 are R-type BTIB designs and 10 are obtained using PBIB designs. The remaining 100 designs are apparently new.

The A-optimal designs given in Hedayat and Majumdar (1984) often require a large number of blocks. For the same values of p and k, we are able to give designs in smaller number of blocks, and with high A-efficiencies. For example, Hedayat and Majumdar (1984) reported an A-optimal design with $p = 6$ and $k = 3$ in 37 blocks whereas we have a design for same (p, k) in 11 blocks, the A-efficiency of this design being at least 0.997. Thus, in this case there is considerable saving in terms of the number of experimental units with no appreciable loss in efficiency. For several values of p and k, we have designs with fewer blocks than the corresponding A-optimal designs reported by Hedayat and Majumdar (1984). These designs are listed in Table 2; in this table, b_0 denotes the number of blocks required for an A-optimal design.

TABLE 1

Catalog of A-efficient BTIB designs with $2 \le k \le 10,\ r \le 10,\ k \le p \le b \le 50$

No.	p	b	k	r	r_c	λ	λ_c	e	Reference
1	2	3	2	2	2	1	1	1	$\text{BIB}_1(3,3,2;0)$
2	3	3	3	2	3	1	2	1	$\text{BIB}_0(3,3,2;1)$
3	4	6	3	3	6	1	3	1	$\text{BIB}_0(4,6,2;1)$
4	5	10	3	4	10	1	4	1	$\text{BIB}_0(5,10,2;1)$
5	6	11	3	4	9	1	3	0.997	SR6, (2,3)
6	6	26	3	9	24	2	8	0.999	R24, (2,3)
7	7	21	3	6	21	1	6	0.985	$\text{BIB}_0(7,21,2;1)$
8	8	28	3	7	28	1	7	0.977	$\text{BIB}_0(8,28,2;1)$
9	9	36	3	8	36	1	8	0.969	$\text{BIB}_0(9,36,2;1)$
10	10	25	3	6	15	1	3	0.986	T2, T9
11	12	35	3	7	21	1	3	0.953	$\text{BIB}_3(15,35,3;0)$
12	4	4	4	3	4	2	3	1	$\text{BIB}_0(4,4,3;1)$
13	5	7	4	4	8	2	4	0.953	$\text{BIB}_2(7,7,4;0)$
14	5	10	4	6	10	3	6	1	$\text{BIB}_0(5,10,3;1)$
15	6	10	4	5	10	2	5	1	$\text{BIB}_0(6,10,3;1)$
16	7	7	4	3	7	1	3	1	$\text{BIB}_0(7,7,3;1)$
17	8	26	4	10	24	3	9	0.993	R58, (2,4)
18	9	12	4	4	12	1	4	1	$\text{BIB}_0(9,12,3;1)$
19	10	25	4	8	20	2	6	0.986	T12, T28

TABLE 1 (Contd.)

No.	p	b	k	r	r_c	λ	λ_c	e	Reference
20	10	30	4	9	30	2	9	0.999	$\text{BIB}_0(10, 30, 3; 1)$
21	12	19	4	5	16	1	4	0.998	SR26, (3,4)
22	13	20	4	5	15	1	3	0.954	$\text{BIB}_3(16, 20, 4; 0)$
23	13	26	4	6	26	1	6	0.995	$\text{BIB}_0(13, 26, 3; 1)$
24	15	35	4	7	35	1	7	0.991	$\text{BIB}_0(15, 35, 3; 1)$
25	16	28	4	6	16	1	3	0.969	LS18, LS29
26	16	36	4	7	32	1	6	0.998	R86, (4,4)
27	20	50	4	8	40	1	5	0.966	$\text{BIB}_5(25, 50, 4; 0)$
28	21	50	4	8	32	1	4	0.968	$\text{BIB}_4(25, 50, 4; 0)$
29	5	5	5	4	5	3	4	0.970	$\text{BIB}_0(5, 5, 4; 1)$
30	5	15	5	10	25	6	16	0.995	$\text{BIB}_1(6, 15, 4; 1)$
31	6	7	5	4	11	2	6	0.992	$\text{BIB}_1(7, 7, 4; 1)$
32	7	7	5	4	7	2	4	0.996	$\text{BIB}_0(7, 7, 4; 1)$
33	8	11	5	5	15	2	6	0.967	$\text{BIB}_3(11, 11, 5; 0)$
34	8	14	5	7	14	3	7	0.999	$\text{BIB}_0(8, 14, 4; 1)$
35	9	15	5	6	21	2	8	0.981	$\text{BIB}_1(10, 15, 4; 1)$
36	9	18	5	8	18	3	8	1	$\text{BIB}_0(9, 18, 4; 1)$
37	10	15	5	6	15	2	6	1	$\text{BIB}_0(10, 15, 4; 1)$
38	12	13	5	4	17	1	5	0.974	$\text{BIB}_1(13, 13, 4; 1)$
39	13	13	5	4	13	1	4	1	$\text{BIB}_0(13, 13, 4; 1)$
40	15	20	5	5	25	1	6	0.973	$\text{BIB}_1(16, 20, 4; 1)$
41	15	33	5	9	30	2	8	0.995	R117, (3,5)
42	16	20	5	5	20	1	5	1	$\text{BIB}_0(16, 20, 4; 1)$
43	17	21	5	5	20	1	4	0.965	$\text{BIB}_4(21, 21, 5; 0)$
44	18	21	5	5	15	1	3	0.951	$\text{BIB}_3(21, 21, 5; 0)$
45	20	29	5	6	25	1	5	0.999	SR46, (4,5)
46	21	30	5	6	24	1	4	0.968	$\text{BIB}_4(25, 30, 5; 0)$
47	24	50	5	8	58	1	9	0.971	$\text{BIB}_1(25, 50, 4; 1)$
48	25	50	5	8	50	1	8	0.994	$\text{BIB}_0(25, 50, 4; 1)$
49	6	12	6	8	24	5	15	0.973	$\text{BIB}_3(9, 12, 6; 0)$
50	6	15	6	10	30	6	20	0.993	$\text{BIB}_0(6, 15, 4; 2)$
51	7	7	6	4	14	2	8	0.985	$\text{BIB}_0(7, 7, 4; 2)$
52	8	11	6	6	18	3	9	0.982	$\text{BIB}_3(11, 11, 6; 0)$
53	8	18	6	10	28	5	15	0.999	$\text{BIB}_1(9, 18, 5; 1)$
54	9	11	6	5	21	2	9	0.964	$\text{BIB}_2(11, 11, 5; 1)$
55	9	18	6	9	27	4	13	0.999	$\text{BIB}_1(10, 18, 5; 1)$
56	10	11	6	5	16	2	7	0.999	$\text{BIB}_1(11, 11, 5; 1)$
57	11	11	6	5	11	2	5	0.991	$\text{BIB}_0(11, 11, 5; 1)$
58	12	16	6	6	24	2	8	0.972	$\text{BIB}_4(16, 16, 6; 0)$
59	13	16	6	6	18	2	6	0.973	$\text{BIB}_3(16, 16, 6; 0)$
60	19	21	6	5	31	1	7	0.964	$\text{BIB}_2(21, 21, 5; 1)$

Table 1 (Contd.)

No.	p	b	k	r	r_c	λ	λ_c	e	Reference
61	20	21	6	5	26	1	6	0.986	$\text{BIB}_1(21,21,5;1)$
62	21	21	6	5	21	1	5	1	$\text{BIB}_0(21,21,5;1)$
63	23	30	6	6	42	1	8	0.964	$\text{BIB}_2(25,30,5;1)$
64	24	30	6	6	36	1	7	0.985	$\text{BIB}_1(25,30,5;1)$
65	25	30	6	6	30	1	6	1	$\text{BIB}_0(25,30,5;1)$
66	26	31	6	6	30	1	5	0.975	$\text{BIB}_5(31,31,6;0)$
67	27	31	6	6	24	1	4	0.967	$\text{BIB}_4(31,31,6;0)$
68	7	12	7	8	28	5	18	0.968	$\text{BIB}_2(9,12,6;1)$
69	8	12	7	8	20	5	13	0.997	$\text{BIB}_1(9,12,6;1)$
70	9	11	7	6	23	3	12	0.977	$\text{BIB}_2(11,11,6;1)$
71	9	15	7	9	24	5	14	0.998	$\text{BIB}_1(10,15,6;1)$
72	10	11	7	6	17	3	9	0.999	$\text{BIB}_1(11,11,6;1)$
73	11	11	7	5	22	2	10	0.986	$\text{BIB}_0(11,11,5;2)$
74	12	15	7	7	21	3	9	0.983	$\text{BIB}_3(15,15,7;0)$
75	13	16	7	6	34	2	12	0.951	$\text{BIB}_3(16,16,6;1)$
76	14	16	7	6	28	2	10	0.984	$\text{BIB}_2(16,16,6;1)$
77	15	16	7	6	22	2	8	1	$\text{BIB}_1(16,16,6;1)$
78	16	16	7	6	16	2	6	0.990	$\text{BIB}_0(16,16,6;1)$
79	17	30	7	10	40	3	12	0.982	$\text{BIB}_4(21,30,7;0)$
80	18	30	7	10	30	3	9	0.967	$\text{BIB}_3(21,30,7;0)$
81	21	36	7	9	63	2	14	0.952	$\text{BIB}_7(28,36,7;0)$
82	22	36	7	9	54	2	12	0.970	$\text{BIB}_6(28,36,7;0)$
83	23	36	7	9	45	2	10	0.979	$\text{BIB}_5(28,36,7;0)$
84	24	36	7	9	36	2	8	0.975	$\text{BIB}_4(28,36,7;0)$
85	28	31	7	6	49	1	9	0.961	$\text{BIB}_3(31,31,6;1)$
86	29	31	7	6	43	1	8	0.978	$\text{BIB}_2(31,31,6;1)$
87	30	31	7	6	37	1	7	0.992	$\text{BIB}_1(31,31,6;1)$
88	31	31	7	6	31	1	6	1	$\text{BIB}_0(31,31,6;1)$
89	8	12	8	8	32	5	21	0.963	$\text{BIB}_1(9,12,6;2)$
90	9	12	8	8	24	5	16	1	$\text{BIB}_0(9,12,6;2)$
91	10	11	8	6	28	3	15	0.961	$\text{BIB}_1(11,11,6;2)$
92	10	15	8	9	30	5	18	0.999	$\text{BIB}_0(10,15,6;2)$
93	11	11	8	6	22	3	12	0.999	$\text{BIB}_0(11,11,6;2)$
94	12	15	8	8	24	4	12	0.985	$\text{BIB}_3(15,15,8;0)$
95	13	15	8	7	29	3	13	0.989	$\text{BIB}_2(15,15,7;1)$
96	14	15	8	7	22	3	10	0.997	$\text{BIB}_1(15,15,7;1)$
97	15	15	8	7	15	3	7	0.961	$\text{BIB}_0(15,15,7;1)$
98	16	16	8	6	32	2	12	0.986	$\text{BIB}_0(16,16,6;2)$
99	16	24	8	9	48	3	18	0.986	$\text{BIB}_0(16,24,6;2)$
100	18	30	8	10	60	3	19	0.972	$\text{BIB}_3(21,30,7;1)$
101	19	30	8	10	50	3	16	0.992	$\text{BIB}_2(21,30,7;1)$
102	20	30	8	10	40	3	13	0.999	$\text{BIB}_1(21,30,7;1)$

TABLE 1 (Contd.)

No.	p	b	k	r	r_c	λ	λ_c	e	Reference
103	21	30	8	10	30	3	10	0.987	$\text{BIB}_0(21,30,7;1)$
104	24	36	8	9	72	2	17	0.955	$\text{BIB}_4(28,36,7;1)$
105	25	36	8	9	63	2	15	0.976	$\text{BIB}_3(28,36,7;1)$
106	26	36	8	9	54	2	13	0.991	$\text{BIB}_2(28,36,7;1)$
107	27	36	8	9	45	2	11	0.999	$\text{BIB}_1(28,36,7;1)$
108	28	36	8	9	36	2	9	0.998	$\text{BIB}_0(28,36,7;1)$
109	9	12	9	8	36	5	24	0.962	$\text{BIB}_0(9,12,6;3)$
110	9	13	9	9	36	6	24	0.970	$\text{BIB}_4(13,13,9;0)$
111	10	13	9	9	27	6	18	0.989	$\text{BIB}_3(13,13,9;0)$
112	11	13	9	9	18	6	12	0.951	$\text{BIB}_2(13,13,9;0)$
113	12	15	9	8	39	4	20	0.965	$\text{BIB}_3(15,15,8;1)$
114	13	15	9	8	31	4	16	0.991	$\text{BIB}_2(15,15,8;1)$
115	14	15	9	8	23	4	12	0.989	$\text{BIB}_1(15,15,8;1)$
116	15	15	9	7	30	3	14	0.999	$\text{BIB}_0(15,15,7;2)$
117	16	19	9	9	27	4	12	0.977	$\text{BIB}_3(19,19,9;0)$
118	19	25	9	9	54	3	18	0.970	$\text{BIB}_6(25,25,9;0)$
119	20	25	9	9	45	3	15	0.986	$\text{BIB}_5(25,25,9;0)$
120	21	25	9	9	36	3	12	0.987	$\text{BIB}_4(25,25,9;0)$
121	21	30	9	10	60	3	20	0.989	$\text{BIB}_0(21,30,7;2)$
122	22	25	9	9	27	3	9	0.963	$\text{BIB}_3(25,25,9;0)$
123	27	36	9	9	81	2	20	0.954	$\text{BIB}_1(28,36,7;2)$
124	28	36	9	9	72	2	18	0.975	$\text{BIB}_0(28,36,7;2)$
125	29	37	9	9	72	2	16	0.966	$\text{BIB}_8(37,37,9;0)$
126	30	37	9	9	63	2	14	0.980	$\text{BIB}_7(37,37,9;0)$
127	31	37	9	9	54	2	12	0.987	$\text{BIB}_6(37,37,9;0)$
128	32	37	9	9	45	2	10	0.987	$\text{BIB}_5(37,37,9;0)$
129	33	37	9	9	36	2	8	0.973	$\text{BIB}_4(37,37,9;0)$
130	10	13	10	9	40	6	27	0.970	$\text{BIB}_3(13,13,9;1)$
131	11	13	10	9	31	6	21	0.996	$\text{BIB}_2(13,13,9;1)$
132	12	13	10	9	22	6	15	0.981	$\text{BIB}_1(13,13,9;1)$
133	12	16	10	10	40	6	24	0.985	$\text{BIB}_4(16,16,10;0)$
134	13	15	10	8	46	4	24	0.952	$\text{BIB}_2(15,15,8;2)$
135	13	16	10	10	30	6	18	0.986	$\text{BIB}_3(16,16,10;0)$
136	14	15	10	8	38	4	20	0.984	$\text{BIB}_1(15,15,8;2)$
137	15	15	10	8	30	4	16	1	$\text{BIB}_0(15,15,8;2)$
138	16	19	10	9	46	4	21	0.979	$\text{BIB}_3(19,19,9;1)$
139	17	19	10	9	37	4	17	0.994	$\text{BIB}_2(19,19,9;1)$
140	18	19	10	9	28	4	13	0.986	$\text{BIB}_1(19,19,9;1)$
141	21	25	10	9	61	3	21	0.968	$\text{BIB}_4(25,25,9;1)$
142	22	25	10	9	52	3	18	0.987	$\text{BIB}_3(25,25,9;1)$
143	23	25	10	9	43	3	15	0.997	$\text{BIB}_2(25,25,9;1)$

TABLE 1 (Contd.)

No.	p	b	k	r	r_c	λ	λ_c	e	Reference
144	24	25	10	9	34	3	12	0.993	$\text{BIB}_1(25, 25, 9; 1)$
145	25	25	10	9	25	3	9	0.965	$\text{BIB}_0(25, 25, 9; 1)$
146	25	31	10	10	60	3	18	0.984	$\text{BIB}_6(31, 31, 10; 0)$
147	26	31	10	10	50	3	15	0.990	$\text{BIB}_5(31, 31, 10; 0)$
148	27	31	10	10	40	3	12	0.982	$\text{BIB}_4(31, 31, 10; 0)$
149	28	31	10	10	30	3	9	0.951	$\text{BIB}_3(31, 31, 10; 0)$
150	32	37	10	9	82	2	19	0.964	$\text{BIB}_5(37, 37, 9; 1)$
151	33	37	10	9	73	2	17	0.979	$\text{BIB}_4(37, 37, 9; 1)$
152	34	37	10	9	64	2	15	0.991	$\text{BIB}_3(37, 37, 9; 1)$
153	35	37	10	9	55	2	13	0.998	$\text{BIB}_2(37, 37, 9; 1)$
154	36	37	10	9	46	2	11	0.999	$\text{BIB}_1(37, 37, 9; 1)$
155	37	37	10	9	37	2	9	0.990	$\text{BIB}_0(37, 37, 9; 1)$

TABLE 2

Comparison of *A*-efficient and *A*-optimal designs
with respect to number of blocks

No.	p	b	k	b_o	e
1	6	11	3	37	0.997
2	6	26	3	37	0.999
3	5	7	4	10	0.953
4	6	7	5	18	0.992
5	7	7	5	35	0.996
6	9	15	5	18	0.981
7	12	13	5	33	0.974
8	9	11	7	48	0.977
9	9	15	7	48	0.998
10	14	16	7	35	0.984
11	8	12	8	28	0.963

Acknowledgement

The authors wish to thank a referee for useful comments on an earlier version.

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(Received 16 Oct 2003; revised 20 Feb 2004)