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Note on a closure concept and matching extension

Tsuyoshi Nishimura

Department of Mathematics
Shibaura Institute of Technology
Minuma-Ku, Saitama 330-8570
Japan

Abstract

We prove the following theorems:
(i) Let G be a graph and let x be a locally $2n$-connected vertex. Let $\{u, v\}$ be a pair of vertices in $V(G) - \{x\}$ such that $uv \notin E(G)$, $x \in N_G(u) \cap N_G(v)$, and $N_G(x) \subset N_G(u) \cup N_G(v) \cup \{u, v\}$. Then if $G + uv$ is n-extendable, then G is n-extendable or G is a member of the exceptional family $\mathcal{F}$ of graphs described.
(ii) Let G be a $(2n + 1)$-connected graph. Let $\{u, v, x\}$ be a three-vertex subset of $V(G)$ such that $uv \notin E(G), x \in N_G(u) \cap N_G(v)$, and $N_G(x) \subset N_G(u) \cup N_G(v) \cup \{u, v\}$. If $G+uv$ is n-extendable, then G is n-extendable or G is a member of the exceptional family $\mathcal{F}$ of graphs described.

We consider only finite simple graphs and follow Chartrand and Lesniak [2] for general terminology and notation. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $A \subset V(G)$, $G[A]$ and $G - A$ are the subgraphs of G induced by A and $V(G) - A$, respectively. Further, if F is a subgraph of G, we will write simply $G[F]$ and $G - F$ instead of $G[V(F)]$ and $G - V(F)$, respectively. For $A, B \subset V(G)$, if $A \cap B = \emptyset$, then $E_G(A, B)$ denotes the set of edges such that one endvertex is in A and the other is in B. The set of endvertices of an edge e is denoted by $V(e)$ and for a matching M, let $V(M) = \bigcup_{e \in M} V(e)$. For a vertex $v \in V(G)$, $N_G(v)$ denotes the neighborhood of v in G and let $\deg_G(v) = |N_G(v)|$ denote the degree of v. Further, let $N_G[v]$ denote the closed neighborhood of v, that is, $N_G[v] = N_G(v) \cup \{v\}$. If $G[N_G(v)]$ is k-connected, then v is called *locally k-connected*.

Let $k \geq 0$ and $p > 0$ be integers with $k \leq p - 1$ and G a graph with $2p$ vertices having a 1-factor. Then G is said to be *k-extendable* if every matching of size k in G can be extended to a 1-factor (a perfect matching). A graph G of order p is *k-factor-critical*, where k is an integer of the same parity as p with $0 \leq k \leq p$, if $G - X$ has a 1-factor for any set X of k vertices of G. In particular, G is 0-factor-critical or 0-extendable if and only if G has a 1-factor.

In [4], we proved the following theorem about n-factor-criticality.

Theorem A. *Let G be a graph and let x be a locally n-connected vertex. Let $\{u,v\}$ be a pair of vertices in $V(G)-\{x\}$ such that $uv \notin E(G), x \in N_G(u) \cap N_G(v)$, and $N_G(x) \subset N_G[u] \cup N_G[v]$. Then $G+uv$ is n-factor-critical if and only if G is n-factor-critical.* □

Further, we conjectured the following statement holds: Let G be a graph and let x be a locally $2n$-connected vertex. Let $\{u,v\}$ be a pair of vertices in $V(G)-\{x\}$ such that $uv \notin E(G), x \in N_G(u) \cap N_G(v)$, and $N_G(x) \subset N_G[u] \cup N_G[v]$. If $G+uv$ is n-extendable, then G is n-extendable. But in [4], we showed this conjecture does not hold in contrast to many parallel results for extendability and factor-criticality. That is, there exists a graph G satisfying $x \in N_G(u) \cap N_G(v)$ which is locally $2n$-connected, $uv \notin E(G)$, $N_G(x) \subset N_G(u) \cap N_G(v)$ such that G is not n-extendable, but $G+uv$ is n-extendable.

The purpose of this paper is to show that this conjecture is true, unless the graph is a member of a special exceptional family. Before we present our theorem, we define a family of graphs.

Let G be a graph satisfying the following properties:

(i) there exists a subgraph B of G of order at least $2n+1$ and there exists a vertex $x \in B$ such that B has an n-matching M (a matching of size n) but $B-\{x\}$ does not have an n-matching,

(ii) $G-B$ has $|B|-2n+2$ odd components $C_1, ..., C_{|B|-2n+2}$,

(iii) $\exists i$ and $j \in \{1,2,...,|B|-2n+2\}$ with $i \neq j$ such that there exist two vertices $u \in C_i \cap N_G(x)$ and $v \in C_j \cap N_G(x)$ such that $N_G(x) \subset N_G[u] \cup N_G[v]$.

Let $\mathcal{F}$ be a family of graphs G satisfying the above properties.

Theorem 1. *Let G be a graph and let x be a locally $2n$-connected vertex. Let $\{u,v\}$ be a pair of vertices in $V(G)-\{x\}$ such that $uv \notin E(G), x \in N_G(u) \cap N_G(v)$, and $N_G(x) \subset N_G[u] \cup N_G[v]$. If $G+uv$ is n-extendable, then G is n-extendable or $G \in \mathcal{F}$.*

Theorem 2. *Let G be a $(2n+1)$-connected graph. Let $\{u,v,x\}$ be a three-vertex subset of $V(G)$ such that $uv \notin E(G), x \in N_G(u) \cap N_G(v)$, and $N_G(x) \subset N_G[u] \cup N_G[v]$. If $G+uv$ is n-extendable, then G is n-extendable or $G \in \mathcal{F}$.*

We use the following lemma that is a variation of Tutte's theorem. Let $o(H)$ denote the number of odd components of a graph H.

Lemma B (Chen[3]). Let G be a graph. Then G is n-extendable if and only if $o(G-S) \leq |S|-2n$ for all vertex subsets S containing an n-matching. □

Proof of Theorem 1. Let G, x, u, and v be as in the statement of the theorem. Suppose $G+uv$ is n-extendable but G is not n-extendable. Then there exists a matching M of size n such that $(G+uv)-V(M)$ has a perfect matching and $G-V(M)$ does not have a perfect matching. Therefore, by Lemma B, there exists a vertex subset B with $V(M) \subset B$ such that $o(G-B) > |B|-2n \geq o[(G+uv)-B]$.

Notice that $|V(G)| \equiv 0 \pmod 2$ since $G+uv$ is n-extendable. Since $o(G-B)+|B| \equiv 0 \pmod 2$ and $o[(G+uv)-B] \geq o(G-B)-2$, we have $o(G-B)-2 = o[(G+uv)-B] = |B|-2n$. We may assume $u \in C_1$ and $v \in C_2$, where C_1 and C_2 are odd components of $G-B$. Now let $C_3,\dots,C_{|B|-2n+2}$ be the other odd components of $G-B$. These components are also odd components of $(G+uv)-B$. Now since $E_G(C_1,C_2)=\emptyset$ and $x \in N_G(u) \cap N_G(v)$, we may assume $x \in B$.

Case 1. $|B| = 2n$.

In this case, the two vertices u and $v \in N_G(x)$ are separated by $N_G(x) \cap B$ in $G[N_G(x)]$. Since $|N_G(x) \cap B| < 2n$, this contradicts the assumption that x is locally $2n$-connected.

Case 2. $|B| > 2n$.

If there is a vertex subset $S \subset B-\{x\}$ with $|S| = 2n$ and $V(M) \subset S$, since $G+uv$ is n-extendable, $(G+uv)-S$ has a perfect matching so that every vertex of $B-S$ is matched with a vertex of distinct components $C_3,\dots,C_{|B|-2n+2}$. In particular, we may assume x is matched with a vertex w of C_3. However, since $N_G(x) \subset N_G[u] \cup N_G[v]$, w is adjacent to u or v in G. This is impossible since $E_G(C_1 \cup C_2, C_3) = \emptyset$, a contradiction.

Thus we may assume $x \in V(M)$ and $E(B-V(M)) = \emptyset$. Now let $xx' \in M$. Notice that for any matching M' of size n in B, we have $o(G-B)-2 = o[(G+uv)-B] = |B|-2n$. Now since x is a locally $2n$-connected vertex and $u \in N_G(x) \cap C_1$ and $v \in N_G(x) \cap C_2$, we have $|N_G(x) \cap B| \geq 2n$. Therefore there exists a vertex $y \in N_B(x) - V(M-\{xx'\}) - \{x'\}$. If $x'y \in E(B)$, then we put $M' = (M-\{xx'\}) \cup \{x'y\}$. If $x'y \notin E(B)$ and there exists a vertex $y' \in B-V(M)-\{y\}$ such that yy' is an edge in B, then we put $M' = (M-\{xx'\}) \cup \{yy'\}$. By applying the argument as in the first part of this case for M' instead of M, we have a contradiction. Thus we may assume that $N_B(y) \subset V(M)-\{x'\}$ for every vertex $y \in N_B(x)-V(M)$. Similarly, $N_B(x') \subset V(M)$. Further since $E(B-V(M)) = \emptyset$, $N_B(z) \subset V(M)-\{x\}$ for every vertex $z \in B-V(M)$. Therefore, for any n-matching M in B, $B-[V(M-\{xx'\}) \cup \{x\}]$ induces an empty graph. Hence B has an n-matching M but $B-\{x\}$ does not have an n-matching, which implies G is in $\mathcal{F}$. This completes the proof. □

A Sketch of Proof of Theorem 2. Let G, x, u, and v be as in the statement of Theorem 2. Suppose M is an n-matching such that $(G+uv)-V(M)$ has a perfect matching but $G-V(M)$ does not have a perfect matching. By the same arguments as in the proof of Theorem 1, we have $o(G-B)-2 = o[(G+uv)-B] = |B|-2n$ for some vertex subset B with $V(M) \subset B$. Further, we use the same notation and definitions as in the proof of Theorem 1. Therefore we may assume that $x \in B$, $u \in C_1$ and $v \in C_2$, where $C_1,C_2,\dots,C_{|B|-2n+2}$ are the odd components of $G-B$.

Since G is $(2n+1)$-connected, we may assume $|B| \geq 2n+1$. By the same argument as in the proof of Case 2 of Theorem 1, if there is a vertex subset $S \subset B - \{x\}$ with $|S| = 2n$ and $V(M) \subset S$, then we have a contradiction. Therefore $x \in V(M)$ and $E(B - V(M)) = \emptyset$ for every n-matching M in B. Now let $xx' \in M$. Since $|B| \geq 2n + 1$, there exists a vertex $y \in B - V(M)$. If $x'y \in E(B)$, then when we set $M' = (M - \{xx'\}) \cup \{x'y\}$, we have an n-matching $M' \subset B$ such that $V(M') \subset B - \{x\}$, a contradiction. Thus $N_B(x') \subset V(M) - \{x'\}$. Similarly $N_B(w) \subset V(M) - \{x'\}$ for every vertex $w \in N_B(x) - V(M)$. Further, since $E(B - V(M)) = \emptyset$, $B - V(M - \{xx'\}) - \{x\}$ induces an empty graph. Hence B has an n-matching but $B - \{x\}$ has no n-matching. Then G is in $\mathcal{F}$. □

If $G \in \mathcal{F}$, then G is not n-extendable, because, for $B \supset V(M)$, $o(G - B) = o(G - V(M) - (B - V(M))) = |B| - 2n + 2 > |B| - 2n = |B - V(M)|$. On the other hand, clearly $\mathcal{F}$ contains a graph G such that $G + uv$ is n-extendable. Actually, in [4], we construct such a graph. For the convenience of the reader, we show such a graph here again.

Let w, x, y, z be four vertices. We set $X = (n-1)K_2 \cup \{y, z\}$ and $Y = K_p \cup K_q \cup \{w\}$, where p, q are odd integers greater than n. And let $G = (\{x\} \oplus (X \oplus Y)) - \{xw\}$, where $\oplus$ denotes the *join*. Further, let u (resp. v) be a vertex of K_p (resp. K_q). Then G satisfies the properties that $x \in N_G(u) \cap N_G(v)$ which is locally $2n$-connected, $uv \notin E(G)$, and $N_G(x) \subset N_G[u] \cup N_G[v]$. And if we set $B = X \cup \{x\}$, then B contains an n-matching but $B - \{x\}$ does not contain an n-matching. So clearly G is in $\mathcal{F}$. But one can easily check that $G + uv$ is n-extendable.

The converse statement of Theorem 1 does not hold. Let a, b, l, and m be positive odd integers with $a+b = 2n$ and $l \geq m \geq 2n+1$. Set $G = (K_a \cup K_b) \oplus (K_l \cup K_m)$. Now let x, u,and v be vertices in K_m, K_a , and K_b, respectively. We can easily see that G is n-extendable and that $\{u, v, x\}$ satisfies the hypotheses of the theorem, that is, x is a locally $2n$-connected vertex, $uv \notin E(G), x \in N_G(u) \cap N_G(v)$, $N_G(x) \subset N_G[u] \cup N_G[v]$. However, $G + uv$ is not n-extendable. To see this, choose an n-matching M in $K_a \cup K_b \cup \{uv\}$, then $(G+uv) - V(M) = K_l \cup K_m$ is consisted of two odd components. Therefore, $G + uv$ is not n-extendable.

The connectivity assumption of Theorem 2 cannot be weakened. Let $G = K_{2n} \oplus 2K_{2l+1}$ and $x \in V(K_{2n})$. Let u, v be two vertices in distinct components of $G - K_{2n}$. Clearly $\{u, v, x\}$ satisfies the hypothesis. And one can easily check G is $2n$-connected and $G + uv$ is n-extendable, but G is not n-extendable and $G \notin \mathcal{F}$.

If a graph H is obtained from a graph G by iteratively joining all pairs $\{u, v\}$ satisfying the condition $uv \notin E(G)$, but there exists a locally n-connected vertex x (resp. a vertex x) such that $x \in N_G(u) \cap N_G(v)$ and $N_G(x) \subset N_G[u] \cup N_G[v]$ until there no longer remains any such pair, then H is called an *n-closure* (resp. a *closure*) of G and is denoted by $\mathrm{cl}_n(G)$ (resp. $\mathrm{cl}(G)$). Note that the n-closure of a given graph G is not determined uniquely (see [4]). From our Theorems, we have the following corollaries.

Corollary 3. Let $G \notin \mathcal{F}$ be a graph. If $\mathrm{cl}_{2n}(G)$ is n-extendable, then G is n-extendable. □

Corollary 4. Let $G \notin \mathcal{F}$ be a $(2n+1)$-connected graph. If $\mathrm{cl}(G)$ is n-extendable, then G is n-extendable. □

Note.

The following theorem also holds by a proof similar to that of Theorem A in [4].

Theorem 5. Let G be a $(n+1)$-connected graph. Let $\{u, v, x\}$ be a three-vertex subset of $V(G)$ such that $uv \notin E(G), x \in N_G(u) \cap N_G(v)$, and $N_G(x) \subset N_G[u] \cup N_G[v]$. Then $G + uv$ is n-factor-critical if and only if G is n-factor-critical. □

Here $K_n \oplus 2K_{2l+1}$ shows that the connectivity assumption cannot be weakened.

Corollary 6. Let G be a $(n+1)$-connected graph. G is n-factor-critical if and only if $\mathrm{cl}(G)$ is n-factor-critical. □

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