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A result on $2k$-cycle-free bipartite graphs

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Abstract

Let G be a bipartite graph with vertex parts of orders N and M, and X edges. I prove that if G has no cycles of length $2l$, for all $l \in [2, 2k]$, and $N \geq M$, then $X < M^{\frac{1}{2}} N^{\frac{k+1}{2k}} + O(N)$.

1 Introduction

In [11], I proved that a bipartite graph G without cycles of length $2l$, where $l \in [2, 2k+1]$, has no more than

$$(NM)^{\frac{2k}{2k+1}} + C(N+M)$$

edges. Here N and M are the orders of the two vertex parts of G, and C is a constant which does not depend on N or M.

The result of this paper is the corresponding theorem for the missing cases.

Theorem 1 *Let G be a graph with vertex parts of orders N and M and suppose that $N \geq M$. If G does not contain any cycles of length $2l$, where $l \in [2, 2k]$, then the number of edges in G is no more than*

$$M^{\frac{1}{2}} N^{\frac{k+1}{2k}} + C(N+M),$$

for some constant C depending only on k.

Firstly, we note that the lower order $O(N+M)$ term is required, by considering the case $M = 1$ (and that single vertex joined to all the N other vertices, giving N edges).

There is a large number of existing results for the case $k = 1$, which is simply graphs without quadrilaterals. For that case, Theorem 1 agrees with the best results for the highest order term, and the lower order term is also of the correct order. In fact, for $k = 1$, the constant, 1, for the highest order term $M^{\frac{1}{2}} N$ is easily seen to be sharp.

For $k > 1$, the upper bound results in the literature for this problem are mostly concerned with general graphs rather than bipartite graphs. Thus a main novelty of this and the earlier work [11] is a 'bipartite' version of these bounds for high k. This is similar to (and extends) work of de Caen and Székely [3] who studied bipartite graphs for the girth eight case.

To compare our results with those of the literature let us use the notation $ex(n, S)$ to denote the maximum number of edges in a graph with n vertices and no cycles of length $l \in S$ where $n \in \mathbb{Z}$ and $S \subset \mathbb{Z}$. Theorem 1 can then be interpreted as a bound for $ex(n, \{4, 6, \ldots, 4k\} \cup C^*)$ where $C^* = \{3, 5, 7, \ldots\}$.

The first major result in this direction is that of Bondy and Simonovits [2] who answer questions posed by Erdős by proving that

$$ex(n, \{2k\}) \leq 90kn^{1+1/k}.$$

The best result for $ex(n, \{2k\})$ are due to Verstraëte [15]. He proves that $ex(n, \{2k\}) \leq 8(k-1)n^{1+1/k}$.

It seems very unlikely that the constant $8(k-1)$ is sharp, however. In fact, Erdős and Simonovits conjecture in [7] that

$$ex(n, \{2k\}) = \frac{n^{1+\frac{1}{k}}}{2} + o(n^{1+\frac{1}{k}}).$$

They also prove that

$$ex(n, \{3, 4, \ldots, 4k+1\}) \leq \left(\frac{n}{2}\right)^{1+\frac{1}{2k}} + 2^{2k}\left(\frac{n}{2}\right)^{1-\frac{1}{2k}}. \tag{1}$$

We may compare this result to ours for bipartite graphs by letting $n = N + M$ and setting $M = cN$ for some $0 \leq c \leq 1$. We see that for each k the highest order term of Theorem 1 is superior exactly when $\sqrt{c} < \left(\frac{1+c}{2}\right)^{1+1/2k}$. As k approaches infinity we find that Theorem 1 is superior for nearly all values of c, assuming that N is sufficiently large. When the asymptotics of the leading terms are identical, the lower order term of (1) is superior.

There are many other related results for this problem, many of which study graphs with specified girth and minimum degree. For example, Dutton and Brigham [5] prove that (in our situation)

$$ex(n, \{3, 4, \ldots, 2k+1\}) \leq n\left(1/2 + \sqrt{\frac{n-d-1}{S} + 1/4}\right)/2,$$

where d is the minimum degree, $S = k-1$ if $d \leq 2$ and $S = \frac{(d-1)^{k-1}-1}{d-2}$ if $d \geq 3$. Since we have no requirement on the degree, this bound is of order $O(n^{3/2})$ which is much weaker than Theorem 1 when n is large compared to the minimum degree. Dutton and Brigham also prove that

$$ex(n, \{3, 4, \ldots, 2k+1\} \lesssim (1/2)^{2/k} n^{1+1/k}$$

which is weaker than (1). Dutton and Brigham's results were further refined by Dong and Koh ([4]) for small girth. However, for all the cases that can be compared with

Theorem 1, their results are asymptotically the same as (1). More recently, Hoory [9] has also given bounds relating the number of vertices and the degree for bipartite graphs with high girth.

Unfortunately, at the current moment, there are no lower bound constructions which demonstrate that the order of the leading term $O(n^{1+\frac{1}{2k}})$ or $O(M^{\frac{1}{2}}N^{\frac{k+1}{2k}})$ is correct, much less the constant preceeding it. This is rather different to the situation of the cases covered in [11] and the constructions of Wenger [16] and Benson [1]. Those constructions at least suggest that the upper bound of Theorem 1 could well be of the correct order, and the constant of 1 preceeding this term is the most plausible constant. If, however, the highest order term is incorrect, then my result offers little improvement over the important result of Bondy and Simonovits [2].

The best known lower bounds for the problem of Theorem 1, for most k, are due to Lazebnik, Ustimenko and Woldar [12, 13] in 1995. They find bipartite, q-regular graphs with order $n \leq 2q^{k-t+1}$, where k is an odd integer and $t = \lfloor \frac{k+2}{4} \rfloor$. They then prove that these graphs have girth $g \geq k+5$ (and thus no cycles of length $k+3$ or less), and $\Omega(n^{1+\frac{1}{k-t+1}})$ edges.

For example, for $k = 5$, the girth $g = 10$, their constructions will have around $O(n^{1+\frac{1}{5}})$ edges and no cycles of length less than or equal to 8. This compares with our upper bound value of $O(n^{1+\frac{1}{4}})$.

2 Adjacency matrices, cycles and rook moves

Let G be a bipartite graph with vertex sets V and W, and edge set E. Let the vertices in V and W be $v_1, \ldots, v_K$ and $w_1, \ldots, w_L$ respectively. Then a concise matrix representation of G is the $K \times L$ matrix M with entries defined by:

$$M_{ij} = \begin{cases} 1 & \text{if } (v_i, w_j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Unlike the normal type of adjacency matrices, this one has no redundancy – there is a one to one correspondence between bipartite graphs with vertex parts V and W, and $K \times L$ matrices with entries in $\{0, 1\}$.

A cycle of length $2k$ in a bipartite graph G with corresponding matrix M is represented by a $2k$ sided 'polygon' in M such that:

1. The vertices of the polygon are entries with a 1 in M.
2. Two 'adjacent' vertices of the polygon belong to the same row or column of M. That is, the sides of the polygon are vertical or horizontal.
3. No three vertices belong to the same row or column.

I will thus mean an object of this type when I refer to a *cycle* in a matrix.

I will be using the following definition:

Definition 1 (Rook move) *Suppose A is a matrix of 1s and 0s as above. A* rook move *of length n is a vector of positions $(x_0, x_1, \ldots, x_n) : x_i \in A$ such that:*

1. *Every position x_i contains a 1.*
2. *x_i and x_{i+1} belong in either the same column or the same row, and this alternates, depending only on the parity of i. If x_0 and x_1 belong in the same column then the rook move is said to begin vertically. If x_0 and x_1 belong in the same row then the rook move is said to begin horizontally.*

We say that x_n can be reached from x_0 by a rook move of length n. If $x_i = x_{i+1}$, for some $i \in [0, n-1]$ then the rook move is called degenerate. Otherwise, it is called non-degenerate. Note that a degenerate rook move may begin both horizontally and vertically.

3 Proof of Theorem 1

I will prove Theorem 1 in the following equivalent matrix formulation:

Theorem 2 *Let A be a matrix with dimensions $M \times N$ whose entries are either 1's or 0's and suppose that $N \geq M$. If A does not contain any cycles of length $2l$, for every $l \in [2, 2k]$ then the number of entries in A is no more than*

$$M^{\frac{1}{2}} N^{\frac{k+1}{2k}} + D(N+M),$$

for some constant D depending only on k.

I will prove the theorem by induction in N and M. The base case is clearly true with $N = 1$ or $M = 1$ and $D = 1$.

Suppose then that there is some constant $D \geq 1$, for which our theorem is true for all $n \leq N, m \leq M$ such that $(n, m) \neq (N, M)$ and $n \geq m$.

Suppose first that $N > M$. Let A be a matrix of 1s and 0s with M rows and N columns and number of non-zero entries $X > M^{\frac{1}{2}} N^{\frac{k+1}{2k}} + D(N+M)$.

Throughout the proof, C will represent any positive constant which only depends on k but not N or M.

The first step is the following lemma.

Lemma 3 *No row contains less than $CN^{\frac{k+1}{2k}} M^{-\frac{1}{2}} + 1$ entries. No column contains less than $CM^{\frac{1}{2}} N^{\frac{1-k}{2k}} + 1$ entries.*

Proof Pick a row R with r entries. Then removing this row and applying the inductive hypothesis to the resulting matrix gives:

$$X - r \leq N^{\frac{k+1}{2k}} (M-1)^{\frac{1}{2}} + D(N+M-1)$$

$$M^{\frac{1}{2}} N^{\frac{k+1}{2k}} + D(N+M) - r < N^{\frac{k+1}{2k}} (M-1)^{\frac{1}{2}} + D(N+M-1).$$

expanding the right hand side using the Taylor series $f(x) = x^{\frac{1}{2}}$ we get the weaker inequality:

$$-r < -CN^{\frac{k+1}{2k}} M^{-\frac{1}{2}} - D$$

from which our first desired result immediately follows as $D \geq 1$. The proof for columns is essentially the same. ■

Lemma 4 *No row or column contains more than* $CN^{\frac{1-k}{2k}}M^{\frac{1}{2}}$ *entries.*

Proof Pick a row R with say r entries. We want to count the number of destination entries which can be reached from any entry of R by non-degenerate rook moves of length $2k-1$, starting vertically.

Pick any entry of R, x_0. The number of entries that can be reached from x_0 by a non-degenerate rook move of length 1, starting vertically, is at least $CM^{\frac{1}{2}}N^{\frac{1-k}{2k}}$, using Lemma 3. Repeating this, we see that the number of non-degenerate rook moves of length $2k-1$ starting at x_0 is at least

$$(CN^{\frac{k+1}{2k}}M^{\frac{-1}{2}})^{k-1}(CM^{\frac{1}{2}}N^{\frac{1-k}{2k}})^k = CM^{\frac{1}{2}}N^{\frac{k-1}{2k}}$$

Now note that all these non-degenerate rook moves must end on a different row (and in particular, end on different entries). For if we have two rook moves $(x_0, x_1, \ldots x_{2k-1})$ and $(x_0, y_1, \ldots y_{2k-1})$ such that x_{2k-1} and y_{2k-1} are entries of the same row, then

$$(x_1, \ldots x_{2k-1}, y_{2k-1}, y_{2k-2}, \ldots, y_1, x_1)$$

will contain a cycle of even length $\leq 4k-2$ (x_1 and y_1 are both on the same column as x_0 so are on the same column as each other).

Now consider the set of all such non-degenerate rook moves as x_0 varies over all the entries of R. Again I claim no two such rook moves say $(x_0, x_1, \ldots x_{2k-1})$ and $(y_0, y_1, \ldots y_{2k-1})$ where $x_0 \neq y_0$ end on the same row. For otherwise,

$$(x_0, x_1, \ldots x_{2k-1}, y_{2k-1}, y_{2k-2}, \ldots, y_1, y_0, x_0)$$

will contain a cycle of length $\leq 4k$.

Thus we get the inequality:

$$r \times CM^{\frac{1}{2}}N^{\frac{k-1}{2k}} < M$$

Or:

$$r < CN^{\frac{1-k}{2k}}M^{\frac{1}{2}}.$$

Similarly, we obtain

$$c \times (CN^{\frac{k+1}{2k}}M^{-\frac{1}{2}})^k(CM^{\frac{1}{2}}N^{\frac{1-k}{2k}})^{k-1} < N,$$

for a column with c entries, which gives the corresponding bound for columns. ∎

At this point we require a combinatorial lemma from [10].

Lemma 5 *Let* S *and* $A_1, \ldots, A_n$ *be finite sets for some* $n \geq 0$, *and for each* $1 \leq i \leq n$ *let* $f_i : S \to A_i$ *be a function. Then:*

$$\#\{(s_0, \ldots, s_n) \in S^{n+1} : f_i(s_{i-1}) = f_i(s_i), 1 \leq i \leq n\} \geq \frac{(\#S)^{n+1}}{\prod_{i=1}^n \#A_i}. \qquad (2)$$

Let S be the set of entries in A, and let A_i be the set of rows for i odd and the set of columns when i is even. Thus $\#S = X$. Apply Lemma 5 with $n = 2k-1$.

The left hand side of the (2) corresponds to (possibly degenerate) rook moves of length $2k-1$ which start horizontally. We now use Lemma 4 to prove:

Lemma 6 *The number of degenerate rook moves of length* $2k-1$ *starting horizontally is no more than* $CN(N^{\frac{1-k}{2k}}M^{\frac{1}{2}})^{2k-1}$.

Proof A degenerate rook move $(x_0, x_1, \ldots, x_{2k-1})$ must have some i such that $x_i = x_{i+1}$. Let the row with the maximum number of entries have C_r entries, and correspondingly C_c for the columns. Fixing i, the maximum number of such rook moves is no more than $XC_r^{k-1}C_c^{k-1}$ when i is even or $XC_r^kC_c^{k-2}$ when i is odd. We now use Lemma 4 to give $C_r \leq CN^{\frac{1-k}{2k}}M^{\frac{1}{2}}$ and $C_c \leq CN^{\frac{1-k}{2k}}M^{\frac{1}{2}}$ and we note that $X \leq NC_c$ and $X \leq MC_r$. Summing over $0 \leq i \leq 2k-1$ (this summation just contributes to the constant as it doesn't depend on N or M) , we see that the number of degenerate rook moves is no more than N or M multiplied by :

$$C(N^{\frac{1-k}{2k}}M^{\frac{1}{2}})^{2k-1}.$$

■

Now, observe that no two non-degenerate rook moves of length $2k-1$ starting horizontally can start on the same column and end on the same column, for otherwise we would immediately have a cycle of length $4k$ or less.

Thus combining Lemma 5 and Lemma 6 we now have:

$$\frac{X^{2k}}{M^kN^{k-1}} - CN(N^{\frac{1-k}{2k}}M^{\frac{1}{2}})^{2k-1} \leq N^2$$

or

$$X^{2k} \leq N^{k+1}M^k + CN^kM^k(N^{\frac{1-k}{2k}}M^{\frac{1}{2}})^{2k-1}$$

Using the first two terms of the Taylor series for $f(x) = x^{\frac{1}{2k}}$, this implies that

$$X \leq M^{\frac{1}{2}}N^{\frac{k+1}{2k}} + CM^kN^{1-k}.$$

Thus, for the region $N > M$,

$$X \leq M^{\frac{1}{2}}N^{\frac{k+1}{2k}} + C(N+M).$$

(Note that for the case $k = 1$, the bound is true for $N < M$ as well).

This constant C does not depend on the original constant in the inductive hypothesis, nor does it depend on N or M. Thus, we have proved the inductive step for the case $N > M$.

For the case $N = M$, all the steps are identical except that to prove Lemma 3 we end up exiting the region $N \geq M$ (we get $N = M - 1$). However the lower bound of the Lemma can still be established easily by symmetry by swapping N and M and noting that $N^{\frac{k+1}{2k}}M^{-\frac{1}{2}} = M^{\frac{k+1}{2k}}N^{-\frac{1}{2}}$ when $N = M$.

This proves Theorem 2 and thus Theorem 1. ■

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References

[1] C. Benson, *Minimal Regular Graphs of Girths Eight and Twelve*, Canad. J. Math., **26** (1966), 1091–1094.

[2] J.A. Bondy and M. Simonovits, *Cycles of even lengths in graphs*, J. Combin. Theory Ser. B, **16** (1974) 97–105.

[3] D. de Caen and L.A. Székely, *On dense bipartite graphs of girth eight and upper bounds for certain configurations in planar point-line systems*, J. Combin. Theory Ser. A, **77** (1997), 268–278.

[4] F.M. Dong and K.M. Koh, *The sizes of Graphs with Small Girth*, Bulletin of the ICA, **18** (1996), 33–44.

[5] R.D. Dutton and R.C. Brigham, *Edges in Graphs with Large Girth*, Graphs and Combinatroics, **7** (1991), 315–321.

[6] P. Erdős, *Extremal problems in graph theory*, 'Theory of Graphs and Its Applications' (M.Fiedler, Ed.), Academic Press, New York, 1965.

[7] P. Erdős and M. Simonovits, *Compactness results in extremal graph theory*, Combinatorica, **2**(3) (1982), 275–288.

[8] E. Győri, *C_6-free bipartite graphs and product representation of squares*, Discrete Math., **165/166** (1997), 371–375.

[9] S. Hoory, *The size of bipartite graphs with a given girth*, J. Combin. Theory Ser. B, **86**(2) (2002), 215–220.

[10] N. Katz and T. Tao, *Bounds on arithmetic progressions and applications to the Kakeya conjecture*, Math. Res. Letters, **6** (1999), 625–630.

[11] T. Lam, *Graphs without cycles of even length*, Bull. Austral. Math. Soc., **63** (2001), 435–440.

[12] F. Lazebnik, V.A. Ustimenko and A.J. Woldar, *New constructions of bipartite graphs on m, n vertices with many edges and without small cycles*, J. Combin. Theory Ser. B, **61** (1994), 111–117.

[13] F. Lazebnik, V.A. Ustimenko and A.J. Woldar, *A new series of dense graphs of high girth*, Bull. Amer. Math. Soc., **32** (1995), 73–79.

[14] G.N. Sárközy, *Cycles in bipartite graphs and an application in number theory*, J. Graph Theory, **19** (1993), 323–331.

[15] J. Verstraëte, *On Arithmetic Progressions of Cycle Lengths in Graphs*, Combinatorics, Probability and Computing, **9** (2000), 369–373.

[16] R. Wenger, *Extremal Graphs with no C^4's, C^6's, or C^{10}'s*, J. Combin Theory Series B, **52** (1991), 113–116.

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