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Multidesigns of the λ-fold complete graph for graph-pairs of orders 4 and 5

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Abstract

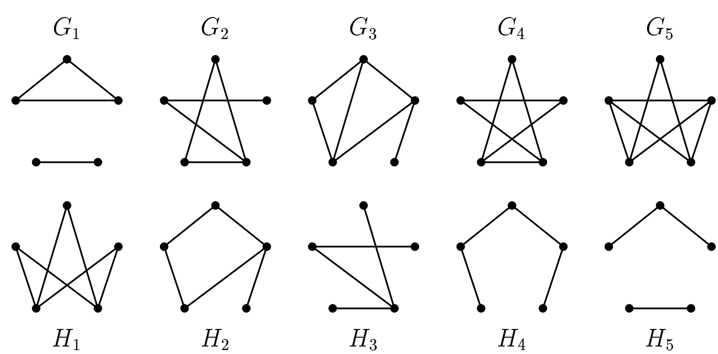
By a *graph-pair of order* t, we mean two non-isomorphic graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$ for some integer $t \geq 4$. Given a graph-pair (G, H), we say (G, H) divides λK_m if the edges of λK_m can be partitioned into copies of G and H with at least one copy of G and at least one copy of H. We will refer to this partition as a (G, H)-*multidecomposition*.

In this paper, we consider the existence of multidecompositions of λK_m for the graph-pairs of order 4 or 5. For those graph-pairs, we will also look for maximum multipackings and minimum multicoverings of λK_m. The existence problem for multidecompositions on K_m has been solved for all graph-pairs of order 4 or 5.

1 Introduction

The λ-fold complete graph λK_m is the graph with m vertices in which each pair of vertices is joined by exactly λ edges. A partition of the edges of λK_m into copies of G is called a *G-decomposition*. When a G-decomposition is not permissible, it is natural to ask how close can we get to a G-decomposition. This question can be answered either by looking at a *packing* of the complete graph λK_m having a *leave* with as few edges as possible, or by looking at a *covering* of the complete graph λK_m having a *padding* with as few edges as possible.

In this paper, we consider different ways of partitioning the edges of λK_m. In [1], the authors looked at decompositions involving two different graphs (specifically, the clique K_t and the star $K_{1,t}$). The main restriction was that the final decomposition have at least one copy of each of the different subgraphs. In [2], the authors define the following: a *graph-pair of order t* consists of two non-isomorphic graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$ for some integer $t \geq 4$. More generally, a *graph-n-tuple of order t* consists of n non-isomorphic graphs $G_1, G_2, \ldots, G_n$ on t non-isolated vertices for which $\cup_{i=1}^{n} G_i \cong K_t$ for some integer $t \geq 4$. The only graph-pair of order 4 is $(C_4, K_2 + K_2)$, and there are 5 graph-pairs of order 5, as follows:



Given a graph-pair (G, H), a partition of the edges of λK_m into copies of G and H with at least one copy of G and at least one copy of H is called a *(G, H)-multidecomposition*. When λK_m does not admit a (G, H)-multidecomposition, we seek a *(G, H)-multipacking* and a *(G, H)-multicovering*. In a maximum multipacking, the remaining edges form a graph, called the *leave*, having as few edges as possible. In a minimum multicovering, the extra edges form a graph, called the *padding*, having as few edges as possible.. A *multidesign* is a multidecomposition, a maximum multipacking, or a minimum multicovering.

In [2], the first two authors completely determined the values of m for which K_m admits a (G, H)-multidecomposition, when (G, H) is a graph-pair of order 4 or 5. The results they obtained may be summarized as follows:

Theorem 1.1 *There is a (G_i, H_i)-multidecomposition of K_m if and only if*

(a) *when* $G_i \cong C_4$ *and* $H_i \cong K_2 + K_2$, $m \equiv 0, 1 \mod 4$ $(m \neq 5)$*;*

(b) *when* $i \in \{1, 3, 4\}$, $m \equiv 0, 1 \mod 4$, $m \geq 5$ (*except for* $i = 1$ *and* $m = 8$)*;*

(c) *when* $i = 2$, $m \equiv 0, 1 \mod 5$*;*

(d) *when* $i = 5$, $m \notin \{6, 7\}$.

The authors also completed the corresponding multipacking and multicovering problems. The results are summarized in the following theorem:

Theorem 1.2 *Let* $L(K_m)$ *be the leave from a maximum* (G, H)*-multipacking, and let* $P(K_m)$ *be the padding from a minimum* (G, H)*-multicovering of* K_m*. The following are true:*

(a) *If* $(G_i, H_i) \in \{(C_4, K_2+K_2), (G_3, H_3), (G_4, H_4)\}$ *and* $m \equiv 2, 3 \mod 4$ $(m \geq 6)$, *then the leave and the padding consist of exactly one edge;*

(b) *If* $(G_i, H_i) \cong (G_1, H_1)$ *and* $m \equiv 2, 3 \mod 4$ $(m \geq 7)$, *then the leave and the padding consist of exactly one edge;*

(c) *If* $(G_i, H_i) \cong (G_2, H_2)$ *and* $m \equiv 2, 4 \mod 5$ $(m \geq 7)$, *then the leave consists of exactly one edge while the padding consists of 4 edges;*

(d) *If* $(G_i, H_i) \cong (G_2, H_2)$ *and* $m \equiv 3 \mod 5$ $(m \geq 8)$, *then the leave is equivalent to* K_3 *while the padding consists of exactly one edge;*

(e) *If* $(G_i, H_i) \cong (G_5, H_5)$, *then for* K_6 *the leave consists of 2 non-adjacent edges while the padding consists of exactly one edge, and for* K_7 *the leave and the padding consist of exactly one edge.*

In this paper, we solve the same problems for λK_m.

Let $V(\lambda K_m) = \mathbb{Z}_m$ and $V(\lambda K_{s,t}) = \mathbb{Z}_{s+t}$. If $S \subseteq \mathbb{Z}_m$, then $\lambda K_m[S]$ is the subgraph of λK_m induced by the vertices in S, and if $S \cup T \subseteq \mathbb{Z}_m$, then $\lambda K_m[S; T]$ is the bipartite subgraph of λK_m on the vertices $S \cup T$. When $s = |S|$ and $t = |T|$, it is clear that $\lambda K_m[S] \cong \lambda K_s$ and $\lambda K_m[S; T] \cong \lambda K_{s,t}$. Define $[a, b] = \{t \in \mathbb{Z}_m \mid a \leq t \leq b\}$. If $S = [a, b]$ and $T = [c, d]$, then we write $\lambda K_m[a, b]$ and $\lambda K_m[a, b; c, d]$ rather than $\lambda K_m[S]$ and $\lambda K_m[S; T]$.

For an integer j, define the permutation $\pi^j : \mathbb{Z}_n \to \mathbb{Z}_n$ by $\pi^j(t) = t + j \mod n$. We write $\pi(t)$ rather than $\pi^1(t)$. For integers i and j, define $\pi_i^j : \mathbb{Z}_n \to \mathbb{Z}_n$ by $\pi_i^j(i) = i$, $\pi_i^j(t) = t + j \mod n$ for $i + 1 \leq t \leq i + n - j - 1$, and $\pi_i^j(t) = t + j + 1 \mod n$ otherwise. We use $\pi^j(G)$ and $\pi_i^j(G)$ to denote the subgraphs obtained by applying these permutations to shift the labels of a given subgraph G. Given a set S of graphs, $\pi^j(S)$ and $\pi_i^j(S)$ indicate the sets of subgraphs obtained by applying the permutations defined above to the vertices of each graph in S.

2 The graph-pair of order 4

Note: in [2], the authors used the name "$2K_2$" for the graph consisting of two disjoint edges, while "E_2" was used in [4]. Here, we use "$K_2 + K_2$."

Note that for K_5, the best we may obtain is a maximum multipacking having a leave with 2 edges. In the Appendix, we list a multidecomposition of $2K_5$. We can improve upon part (a) of Theorem 1.1 and Theorem 1.2, as follows:

Theorem 2.1 *The following are true if $m \geq 4$:*

(a) If $m \equiv 0,1 \mod 4$ ($m \neq 5$), then there is a $(C_4, K_2 + K_2)$-multidecomposition of λK_m.

(b) If $m \equiv 2,3 \mod 4$ and λ is even, then there is a $(C_4, K_2 + K_2)$-multidecomposition of λK_m.

(c) If $m \equiv 2,3 \mod 4$ and λ is odd, then there is a maximum $(C_4, K_2 + K_2)$-multipacking (minimum $(C_4, K_2 + K_2)$-multicovering) with a single edge as the leave (padding).

Proof. Suppose $m \equiv 0,1 \mod 4$ ($m \neq 5$). Use a multidecomposition of K_m a total of λ times to obtain a $(C_4, K_2 + K_2)$-multidecomposition of λK_m.

Suppose $m \equiv 2,3 \mod 4$, $m \geq 6$, and λ is even. Let β be the set of subgraphs used in a multipacking of K_m; recall that a multipacking of K_m has a leave that consists of a single edge, say $\{0,1\}$. For $1 \leq i \leq \lambda$, $\pi^i(\beta)$ and $\pi^{i+2}(\beta)$ are each a set of subgraphs used in a multipacking of K_m, and taken together the leave edges $\pi^i(\{0,1\})$ and $\pi^{i+2}(\{0,1\})$ form an additional copy of $K_2 + K_2$. So the subgraphs in

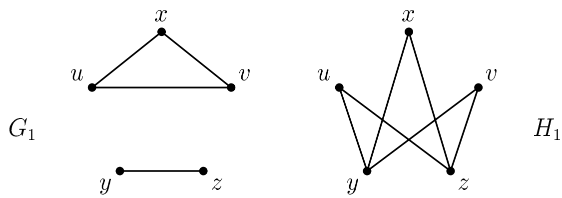
$$\bigcup_{i=0}^{\lambda/2-1} \left(\pi^i(\beta) \cup \pi^{i+2}(\beta) \cup \pi^i(\{0,1\}) \cup \pi^{i+2}(\{0,1\})\right)$$

form a $(C_4, K_2 + K_2)$-multidecomposition of λK_m.

Suppose $m \equiv 2,3 \mod 4$, $m \geq 6$, and λ is odd. In [2], the authors resolved the case $\lambda = 1$, so we will assume $\lambda \geq 3$. Let β^* be the set of subgraphs used in a multidecomposition of $(\lambda - 1)K_m$, and let β^- (β^+) be the set of subgraphs used in a maximum multipacking (minimum multicovering) of K_m with a single edge as the leave (padding). So the subgraphs in $\beta^* \cup \beta^-$ ($\beta^* \cup \beta^+$) form a maximum multipacking (minimum multicovering) of λK_m with a single edge as the leave (padding). □

3 The first graph-pair of order 5

We use a notation similar to that used in [3]. Given the labelling below, we denote G_1 by $[(u, x, v)(y, z)]$ and H_1 by $[(u, x, v)(y, z)]$.



Theorem 3.1 *There is a (G_1, H_1)-multidecomposition of λK_m if:*

(i) $m = 6$ *and* λ *is even, or*

(ii) $m = 8$ *and* $\lambda \geq 2$*, or*

(iii) $m \equiv 0, 1 \mod 4$*,* $m \geq 5$ *(*$m \neq 8$*), or*

(iv) $m \equiv 2, 3 \mod 4$*,* $m \geq 7$*, and* λ *is even.*

Proof. (i) A (G_1, H_1)-multidecomposition of $2K_6$ is given in the Appendix. If $\lambda \geq 4$ is even, one can find a multidecomposition of λK_6 using the multidecomposition for $2K_6$.

(ii) A computer search has shown that there is no (G_1, H_1)-multidecomposition of K_8; see [2]. In the Appendix, we list multidecompositions for $2K_8$ and $3K_8$. For $\lambda \geq 4$, one can find a multidecomposition of λK_8 using the multidecompositions for $2K_8$ and $3K_8$.

(iii) By a result in [2], there is a (G_1, H_1)-multidecomposition for $m \equiv 0, 1 \mod 4$ on K_m, $m \geq 5$ ($m \neq 8$). So there is an exact multidecomposition for any λK_m for those same values of m.

(iv) Let β be the set of subgraphs from a (G_1, H_1)-multidesign of K_m with leave (padding) consisting of the edge $\{0, 1\}$.

If $\lambda = 4x$, then use the multidecompositions $\pi_0^3(\beta), \pi(\beta), \pi^2(\beta)$, and $\pi_1(\pi(\beta))$ x times with leave (padding) consisting of the edges $\{0, 4\}, \{1, 2\}, \{2, 3\}, \{1, 3\}$. These edges form x extra copies of G_1.

If $\lambda = 4x + 2 = 4(x - 1) + 6$, then use a multidecomposition of $4(x - 1)K_m$ as described above. What remains are the edges of $6K_m$; use the following multidecompositions $\beta, \pi_1^3(\beta), \pi_0(\beta), \pi_0^2(\beta)$, $\pi^3(\beta)$, and $\pi_4^{-1}(\pi^3(\beta))$ with leave (padding) consisting of the edges $\{0, 1\}, \{1, 4\}, \{0, 2\}, \{0, 3\}$, $\{3, 4\}, \{4, 2\}$. These edges form another copy of H_1. □

Theorem 3.2 *The following are true:*

(i) For odd λ*, there is a maximum multipacking (minimum multicovering) of* λK_6 *with leave (padding)* $\cong K_2$*.*

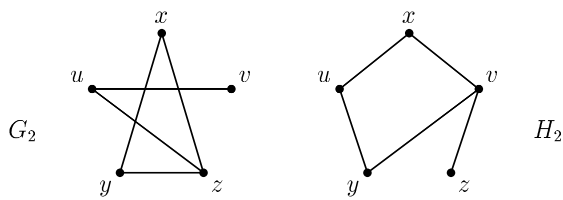
(ii) If $m \equiv 2, 3 \mod 4$*,* $m \geq 7$*, and* λ *is odd, then there is a multipacking (multicovering) of* λK_m *with a leave (padding)* $\cong K_2$*.*

Proof. (i) For $\lambda = 2x + 1 = 2(x - 1) + 3$, we merely find a multidecomposition $(2x - 1)K_6$ as above, and use the multipackings/multicoverings on $3K_6$ listed in the Appendix.

(ii) For $\lambda = 4x + 1$ ($\lambda = 4x + 3$) use a multidecomposition on $(\lambda - 1)K_m$ as described in the previous theorem together with a multipacking/multicovering on K_m for a leave (padding) consisting of exactly one edge. □

4 The second graph-pair of order 5

Given the labelling below, we denote both G_2 and H_2 by $[x, y, z, u, v]$.



Theorem 4.1 *Let $\lambda^* \in \mathbb{Z}_5$ such that $\lambda \equiv \lambda^* \mod 5$. If $m \equiv 2, 4 \mod 5$ and $m \geq 7$, then the leave of a maximum (G_2, H_2)-multipacking of λK_m consists of λ^* edges, and the padding of a minimum (G_2, H_2)-multicovering of λK_m consists of $5 - \lambda^*$ edges. If $m \equiv 3 \mod 5$, then the number of edges in the leave of a maximum (G_2, H_2)-multipacking of λK_m is congruent to $3\lambda^* \mod 5$ and the number of edges in the padding of a minimum (G_2, H_2)-multicovering of λK_m is congruent to $2\lambda^* \mod 5$.*

Proof. Since $e(G_2) = 5$ and $e(H_2) = 5$, it suffices to consider $1 \leq \lambda \leq 5$.

Suppose that $m \equiv 2, 4 \mod 5$ ($m \geq 7$). In [2], it was shown that a maximum (G_2, H_2)-multipacking of K_m has a leave consisting of exactly one edge. Let β be the set of subgraphs from a maximum multipacking of K_m, and we may assume that the leave is the edge $\{0, 1\}$.

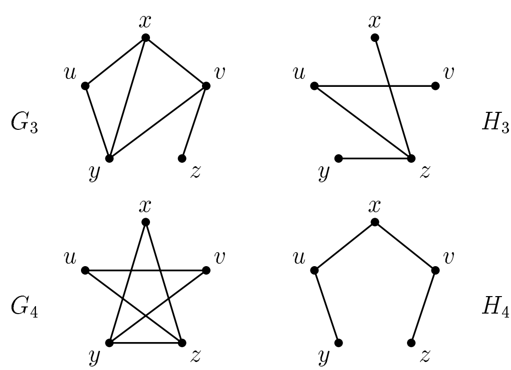
Let $p = \pi(\{0, 1, 2, 3\})$ and $p' = \pi(\{1, 4\})$ be permutations of the vertices of K_m (so $p(i) = i$ for $4 \leq i \leq m - 1$ and $p'(i) = i$ for $i \in \mathbb{Z}_m - \{1, 4\}$). With β defined above, it is clear that $p(\beta)$, $p^2(\beta)$, $p^3(\beta)$, and $p'(\beta)$ are also multipackings of K_m with the leaves $p(\{0, 1\}) = \{1, 2\}$, $p^2(\{0, 1\}) = \{2, 3\}$, $p^3(\{0, 1\}) = \{3, 0\}$, and $p'(\{0, 1\}) = \{0, 4\}$, respectively.

For $1 \leq \lambda \leq 4$, the subgraphs in $\cup_{j=0}^{\lambda-1} p^j(\beta)$ partition the edges in $\lambda K_m - \cup_{j=0}^{\lambda-1} p^j(\{0, 1\})$ and the leave consists of the λ edges in $\cup_{j=0}^{\lambda-1} p^j(\{0, 1\})$. For $\lambda = 5$, let $H = [1, 3, 4, 2, 0]$ be a subgraph isomorphic to H_2; then the subgraphs in $\left(\cup_{j=0}^{3} p^j(\beta)\right) \cup p'(\beta) \cup \{H\}$ completely partition the edges in $5K_m$. For each $1 \leq \lambda \leq 4$, it is clear that a (G_2, H_2)-multicovering of λK_m may be obtained from an optimal (G_2, H_2)-multipacking by including a single copy of H_2. Such a multicovering will have a padding with $5 - \lambda$ edges.

Suppose $m \equiv 3 \mod 5$ ($m \geq 8$). In [2], it was shown that a maximum (G_2, H_2)-multipacking of K_m has a leave isomorphic to K_3. Let β be the set of subgraphs from a maximum multipacking of K_m, and we may assume the leave is the subgraph $A = K_m[0,2] \cong K_3$. So $\pi^2(A) = K_m[2,4] \cong K_3$. Let G be a copy of G_2 formed from the edges $\{0,1\}$, $\{0,2\}$, $\{1,2\}$, $\{2,3\}$, and $\{3,4\}$. The subgraphs in $\beta \cup \pi^2(\beta) \cup \{G\}$ form a maximum (G_2, H_2)-multipacking of $2K_m$ whose leave is the single edge $\{2,4\}$. From this multipacking, we can form a minimum (G_2, H_2)-multicovering of $2K_m$ by adding a single copy of G_2 or H_2; this multicovering will have a padding with 4 edges. It is easy to use the multidesigns above to find maximum multipackings and minimum multicoverings of λK_m for $3 \le \lambda \le 4$, and for $\lambda = 5$, a multidecomposition can be found. □

5 The third and fourth graph-pairs of order 5

For $t = 3$ or 4, we use the notation $G_t = [x,y,z,u,v]$ and $H_t = [x,y,z,u,v]$; see the figure below.



The following is useful for recursive constructions:

Theorem 5.1 *Let $\alpha \in \{3,4\}$. For all $s,t \in \mathbb{N}$, H_α divides $K_{4s,t}$ ($t \ge 2$).*

Proof. It is sufficient to show that H_α divides both $K_{4,2}$ and $K_{4,3}$. Consider the following:

$$\text{for } K_{4,2}: \quad H_3 \cong [0,1,4,2,5],[0,1,5,3,4]\ ; \quad H_4 \cong [0,2,3,4,5],[1,2,3,5,4]\ ;$$

$$\text{for } K_{4,3}: \quad H_3 \cong [4,5,0,6,3],[4,6,1,5,3],[5,6,2,4,3]\ ; \quad H_4 \cong [0,2,3,4,5],[1,0,3,6,4],[2,1,3,5,6]\ .$$

□

In light of Theorem 1.1(b), the following is clear.

Theorem 5.2 *Let $m \ge 5$ and $t = 3$ or 4. There is a (G_t, H_t)-multidecomposition of λK_m for all $\lambda \ge 1$ and $m \equiv 0,1 \mod 4$.*

For $m \equiv 2, 3 \mod 4$, we have the following:

Theorem 5.3 *Let* $m \geq 6$ *and* $t = 3$ *or* 4, *and let* $m \equiv 2, 3 \mod 4$.

(a) If λ *is even, then there is a* (G_t, H_t)*-multidecomposition of* λK_m.

(b) If λ *is odd, then there is a maximum* (G_t, H_t)*-multipacking of* λK_m *with leave consisting of exactly one edge, and a minimum* (G_t, H_t)*-multicovering of* λK_m *with padding consisting of exactly one edge.*

Proof. First suppose $\lambda \geq 2$ is even; it suffices to show that there is a (G_t, H_t)-multidecomposition of $2K_m$. In the Appendix, we list (G_t, H_t)-multidecompositions for $2K_6$, $2K_7$, $2K_{10}$, and $2K_{11}$. For the remainder of the proof, assume $m \geq 14$.

By [2], we may assume that, β, the multipacking of K_m consists of a multipacking of $K_m[0, m-9]$, a multidecomposition of $K_m[m-8, m-1]$, and an H_t-design on $K_m[0, m-9; m-8, m-5]$ and $K_m[0, m-9; m-4, m-1]$. Without loss of generality, we may assume the leave from this multidesign is the edge $\{0, 1\}$. Let $S = [m-4, m-1]$, and $M = K_m[0, 2; S]$ if $m \equiv 2 \mod 4$ (or $M = K_m[0, 3; S]$ if $m \equiv 3 \mod 4$).

Let $j = m-4$, $\pi_j(0) = m-4$ and $\pi_j(1) = m-3$. Then $\pi_j(\beta)$ is a (G_t, H_t)-multipacking of K_m with leave $\pi_j(\{0, 1\}) = \{m-4, m-3\}$. Let $E^* = E(M) \cup \big\{\{0, 1\}, \{m-4, m-3\}\big\}$. It is clear that the subgraphs in $\big(\beta - E(M)\big) \cup \big(\pi_j(\beta)\big)$ partition the edges of $2K_m - E^*$. For even λ, it remains to partition the edges in E^*.

If $m \equiv 2 \mod 4$, use the following copies of G_t and H_t:

$$
\begin{aligned}
t = 3: \qquad & G_3 \cong [0, 1, m-4, m-2, m-3], \\
& H_3 \cong [0, 1, m-4, 2, m-3], [0, 1, m-1, 2, m-2]\ ; \\
\\
t = 4: \qquad & G_4 \cong [m-2, 0, 1, m-4, m-3], \\
& H_4 \cong [m-4, m-3, m-1, 2, 0], [m-1, m-3, m-2, 1, 2]\ .
\end{aligned}
$$

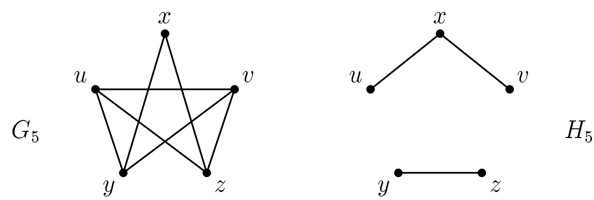
If $m \equiv 3 \mod 4$, use the following copies of G_t and H_t:

$$
\begin{aligned}
t = 3: \quad & G_3 \cong [m-4, m-3, m-2, 2, 3], \\
& H_3 \cong [m-4, m-3, 0, m-2, 2], [m-3, m-2, 1, 0, m-1], \\
& \qquad [2, 3, m-1, 1, m-4]; \\
\\
t = 4: \quad & G_4 \cong [2, m-4, m-3, 1, 0], \\
& H_4 \cong [0, 2, 3, m-2, m-1], [1, 2, 3, m-1, m-2], \\
& \qquad [3, 0, 1, m-3, m-4]\ .
\end{aligned}
$$

Suppose $\lambda \geq 3$ is odd. We have just shown that there is a (G_t, H_t)-multidecomposition of $(\lambda - 1)K_m$, so there is a maximum (G_t, H_t)-multipacking (minimum (G_t, H_t)-multicovering) of λK_m with a leave consisting of exactly one edge (padding consisting of exactly one edge). □

6 The final graph-pair of order 5

Given the labelling below, we denote G_5 by $[x, y, z, u, v]$ and H_5 by $[(u, x, v)(y, z)]$.



In [2], the authors laid the groundwork for the following:

Theorem 6.1 *There is a (G_5, H_5)-multidecomposition of λK_m for all $\lambda \geq 1$ and $m \geq 5$, except for $\lambda = 1$ and $m \in \{6, 7\}$.*

Proof. By Theorem 1.1(d), there is a multidecomposition of K_m for all $m \notin \{6, 7\}$. So if $m \notin \{6, 7\}$, then it is clear that there is a multidecomposition of λK_m for any λ.

Let $m \in \{6, 7\}$; it is clear that a (G_5, H_5)-multidecomposition cannot be obtained for K_m. In order to find a (G_5, H_5)-multidecomposition of λK_m for $\lambda \geq 2$, it suffices to show that there is a (G_5, H_5)-multidecomposition of $2K_m$ and $3K_m$. Multidecompositions for $2K_6$ and $2K_7$ are listed in the Appendix.

Recall that the (G_5, H_5)-multidesign for K_6 has a leave consisting of 2 edges, say $\{0, 5\}$ and $\{2, 3\}$. For $3K_6$, use the multidesigns for K_6, $\pi(K_6)$ and $\pi^2(K_6)$, together with the following copies of H_5: $[(0, 1, 2)(3, 4)]$, $[(0, 5, 4)(2, 3)]$. The (G_5, H_5)-multidesign for K_7 has a leave consisting of a single edge, say $\{0, 1\}$. For $3K_7$, use the multidesigns for K_7, $\pi(K_7)$ and $\pi^3(K_7)$, together with $H_5 \cong [(0, 1, 2)(3, 4)]$. □

7 Acknowledgements

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Appendix

- $(C_4, K_2 + K_2)$-multidecomposition of $2K_5$

  Let β be a maximum multipacking of K_5, with one copy of C_4, $[1,2,3,4]$, and two copies of $K_2 + K_2$, $\{0,1\}$,$\{2,4\}$ and $\{0,4\}$,$\{1,3\}$. The leave A consists of the edges $\{0,2\}$ and $\{0,3\}$. So $A \cup \pi(A)$ form two copies of $K_2 + K_2$. Thus the subgraphs in $\beta \cup \pi(\beta) \cup \big(A \cup \pi(A)\big)$ form a multidecomposition of $2K_5$.

- (G_1, H_1) multidecomposition of $2K_6$

$$\begin{aligned} G_1 &\cong [(0,1,5)(2,4)],[(0,2,3)(1,5)],[(1,2,4)(0,3)] \\ H_1 &\cong [(0,3,5)(2,4)],[(1,4,5)(0,3)],[(2,3,4)(1,5)] \end{aligned}$$

- (G_1, H_1)-multipacking of $3K_6$

$$\begin{aligned} G_1 &\cong [(0,1,3)(2,4)],[(0,1,5)(2,4)],[(0,2,3)(1,5)],[(0,2,4)(1,5)], \\ &\quad [(0,3,5)(1,4)] \\ H_1 &\cong [(0,3,5)(2,4)],[(1,3,5)(2,4)],[(1,4,5)(0,3)],[(2,3,4)(1,5)] \end{aligned}$$

  The leave is the edge $\{1,2\}$.

- (G_1, H_1)-multicovering of $3K_6$

$$\begin{aligned} G_1 &\cong [(0,1,3)(4,5)],[(0,1,5)(2,4)],[(0,2,3)(1,5)],[(0,2,4)(1,5)], \\ &\quad [(0,2,5)(3,4)],[(1,2,4)(0,3)],[(2,3,5)(0,4)] \\ H_1 &\cong [(1,3,5)(2,4)],[(1,4,5)(0,3)],[(2,3,4)(1,5)] \end{aligned}$$

  The padding is the edge $\{2,5\}$.

- (G_1, H_1) multidecomposition of $2K_8$

$$\begin{aligned} G_1 &\cong [(0,1,7)(3,5)],[(0,6,7)(1,2)],[(1,2,6)(3,5)],[(1,3,4)(0,5)], \\ &\quad [(1,3,4)(2,5)],[(1,5,6)(2,3)],[(1,5,7)(0,3)],[(2,6,7)(0,1)] \\ H_1 &\cong [(0,4,7)(2,5)],[(0,6,7)(3,4)],[(2,4,6)(0,5)],[(2,6,7)(3,4)] \end{aligned}$$

- (G_1, H_1) multidecomposition of $3K_8$

$$\begin{aligned} G_1 &\cong [(0,6,7)(1,2)],[(0,6,7)(1,2)],[(0,6,7)(1,2)],[(1,3,4)(2,5)],\\ &\quad [(1,3,5)(2,4)],[(1,4,5)(2,3)]\\ H_1 &\cong [(0,6,7)(1,2)],[(0,6,7)(1,2)],[(0,6,7)(1,2)],[(2,3,6)(4,5)],\\ &\quad [(2,4,6)(3,5)],[(2,5,6)(3,4)],[(3,4,5)(0,7)],[(3,4,5)(0,7)],\\ &\quad [(3,4,5)(0,7)],[(3,4,5)(1,6)] \end{aligned}$$

- (G_3, H_3)-multidecomposition of $2K_6$

$$\begin{aligned} G_3 &\cong [0,1,3,2,5],[3,4,5,0,2],[4,5,0,1,3]\\ H_3 &\cong [0,5,2,1,3],[1,5,4,2,3],[4,5,0,1,3] \end{aligned}$$

- (G_4, H_4)-multidecomposition of $2K_6$

$$\begin{aligned} G_4 &\cong [1,0,5,2,4],[2,1,3,4,0],[4,2,5,3,0]\\ H_4 &\cong [1,0,3,5,4],[3,1,4,2,5],[3,2,4,0,1] \end{aligned}$$

- (G_3, H_3)-multidecomposition of $2K_7$

$$\begin{aligned} G_3 &\cong [1,2,4,3,0],[3,5,0,1,4],[6,2,3,4,0]\\ H_3 &\cong [0,6,3,5,4],[1,3,4,6,5],[3,1,2,5,0],\\ &\quad [3,5,6,1,0],[4,5,1,6,0],[4,6,2,5,0] \end{aligned}$$

- (G_4, H_4)-multidecomposition of $2K_7$

$$\begin{aligned} G_4 &\cong [6,0,5,4,2]\\ H_4 &\cong [0,1,6,4,3],[0,5,6,4,3],[1,2,6,5,4],\\ &\quad [1,2,5,6,3],[1,3,6,2,0],[2,0,4,1,3],\\ &\quad [2,1,3,6,5],[2,3,5,4,0],[5,3,4,1,6] \end{aligned}$$

- (G_3, H_3)-multidecomposition of $2K_{10}$

  Use a multidecomposition of $2K_7$ on $2K_{10}[0,6]$, and use Theorem 5.1 to find an H_3 design on $2K_{10}[0,3;7,9]$ together with the following copies of G_3 and H_3:

$$\begin{aligned} G_3 &\cong [7,8,9,4,5],[8,9,7,5,6]\\ H_3 &\cong [4,6,8,9,7],[4,6,9,7,5],[6,8,7,4,9] \end{aligned}$$

- (G_3, H_3)-multidecomposition of $2K_{11}$

  Use a multidecomposition of $2K_8$ on $2K_{11}[0,7]$, and use Theorem 5.1 to find an H_3 design on $2K_{11}[0,3;8,10]$ together with the following copies of G_3 and H_3:

$$\begin{aligned} G_3 &\cong [8,9,10,4,5],[8,10,9,6,7],[9,10,8,6,5]\\ H_3 &\cong [4,8,10,7,9],[6,7,8,9,10],[8,10,4,9,6] \end{aligned}$$

- (G_4, H_4)-multidecomposition of $2K_{10}$

  Use a multidecomposition of $2K_7$ on $2K_{10}[0,6]$, and use Theorem 5.1 to find an H_4 design on $2K_{10}[0,3;7,9]$ together with the following copies of G_4 and H_4:

$$\begin{aligned} G_4 &\cong [7,5,8,6,9],[9,5,8,4,7] \\ H_4 &\cong [8,4,6,9,7],[9,4,8,7,6],[9,6,8,7,4] \end{aligned}$$

- (G_4, H_4)-multidecomposition of $2K_{11}$

  Use a multidecomposition of $2K_8$ on $2K_{11}[0,7]$, and use Theorem 5.1 to find an H_4 design on $2K_{11}[0,3;8,10]$ together with the following copies of G_4 and H_4:

$$\begin{aligned} G_4 &\cong [8,6,9,7,10],[10,5,9,4,8],[10,5,9,4,8] \\ H_4 &\cong [7,6,9,10,8],[8,4,9,10,6],[8,4,9,10,7] \end{aligned}$$

- (G_5, H_5)-multidecomposition of $2K_6$

$$\begin{aligned} G_5 &\cong [1,4,5,0,2],[2,4,5,1,3],[5,0,3,1,2] \\ H_5 &\cong [(0,3,4)(1,2)],[(0,4,5)(2,3)],[(1,0,3)(4,5)] \end{aligned}$$

- (G_5, H_5)-multidecomposition of $2K_7$

$$\begin{aligned} G_5 &\cong [1,5,6,0,2],[2,0,4,1,3],[3,5,6,2,4] \\ H_5 &\cong [(0,1,6)(4,5)],[(0,4,1)(3,6)],[(1,2,3)(5,6)],[(1,2,3)(5,6)], \\ &\quad [(3,0,5)(4,6)],[(3,1,5)(0,4)],[(4,3,5)(0,6)] \end{aligned}$$

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