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The class of $\{3K_1, C_4\}$-free graphs

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Abstract

The problem of finding an optimal upper bound for the chromatic number of $3K_1$-free graphs is open and quite hard. Approximate bounds are known. Here, we characterize $\{3K_1, C_4\}$-free graphs and deduce that for such a graph G, $\chi(G) \leq \left\lceil \frac{5\omega(G)}{4} \right\rceil$, where $\omega(G)$ is the clique number of G.

1 Introduction

It is well known that the problem of finding the vertex chromatic number $\chi(G)$ of a graph G is NP-complete, even when G belongs to a well-defined apparently small class of graphs. As explained by Brandt [1], the problem of finding an optimal upper bound, as a function of clique number, for the chromatic number of graphs with independence number at most two, is also hopelessly difficult; the best one can conclude is that for such a graph G, $\chi(G)$ is bounded on both sides by $\Theta\left(\frac{\omega(G)^2}{log\ \omega(G)}\right)$, where $\omega(G)$ is the clique number of G. In fact, to draw such a conclusion one requires hard mathematics involving Ramsey numbers.

We follow standard notation and terminology of West [7] and all our graphs are finite and simple. We also assume that the reader is familiar with standard results on vertex colourings; see for example [7]. Given a family $\mathcal{F}$ of graphs, G is said to be $\mathcal{F}$-free, if no graph in $\mathcal{F}$ is an induced subgraph of G. As in [1], we find it convenient to call a graph G with independence number at most two as a $3K_1$-free graph. If H is a subgraph (respectively induced subgraph) of G, we write $H \subseteq G$ ($H \sqsubseteq G$). The subgraph of G induced by a vertex subset S is denoted by $[S]$. If S and T are vertex disjoint subsets of G, then $[S, T]$ denotes the set of all edges in G with one end in S and another end in T. If G_1 and G_2 are two vertex disjoint graphs, then their union $G_1 \cup G_2$ is the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Similarly, the join $G_1 + G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{(x, y) : x \in V(G_1), y \in V(G_2)\}$. For any positive integer k, kG denotes the union of k graphs each isomorphic with

G. As usual $\chi(G)$, $\omega(G)$, $\alpha(G)$ respectively denote the chromatic number, clique number, independence number, and P_n, C_n, K_n respectively denote the path, cycle, complete graph on n vertices.

In this note, we characterize $\{3K_1, C_4\}$-free graphs and deduce that for such a graph G, $\chi(G) \leq \left\lceil \frac{5\omega(G)}{4} \right\rceil$. This bound is optimal in the sense that given any two integers w and k such that $1 \leq w \leq k \leq \left\lceil \frac{5w}{4} \right\rceil$, there exists a $\{3K_1, C_4\}$-free graph G with $\omega(G) = w$ and $\chi(G) = k$. Figure 1 shows an optimal chromatic upper bound for any $\{3K_1, H\}$-free graph G, where H is a graph on four vertices such that $3K_1$ is not an induced subgraph of H. The known upper bounds shown in Column 2 are consequences of stronger results cited.

H	Chromatic upper bound for any $\{3K_1, H\}$-free graph G
K_4 / $K_4 - e$ / $(K_2 \cup K_1) + K_1$	$\omega(G) + 1$, [2, 4, 5]
C_4	$\left\lceil \frac{5\omega(G)}{4} \right\rceil$, this paper
P_4	$\omega(G)$, [6]
$K_3 \cup K_1$	$\left\lceil \frac{3\omega(G)}{2} \right\rceil$, G^c is a union of paths and cycles
$2K_2$	?

Figure 1: A table of optimal chromatic upper bounds

2 A special graph $\mathbb{C}_5(m_1, m_2, m_3, m_4, m_5)$

Let $C_5 = [v_1, v_2, v_3, v_4, v_5, v_1]$ be a 5-cycle and $m_1, m_2, \ldots, m_5$ be non-negative integers. We denote by $\mathbb{C}_5(m_1, m_2, m_3, m_4, m_5)$, the graph obtained from C_5 by

(i) replacing each v_i by K_{m_i}, $1 \leq i \leq 5$, and

(ii) joining every pair of vertices $x \in K_{m_i}$, $y \in K_{m_{i+1}}$, $1 \leq i \leq 5$, i mod 5.

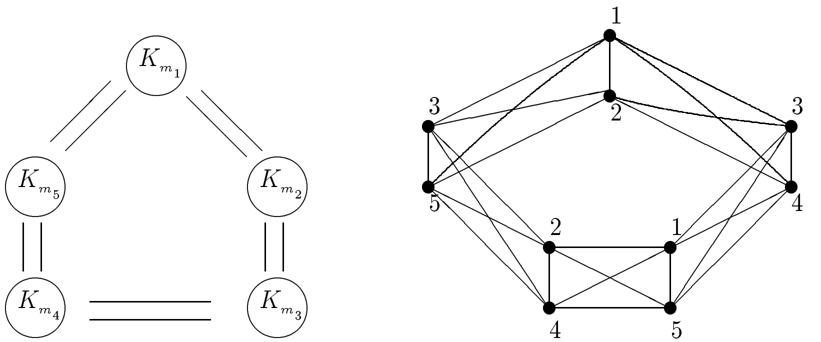
We drop the parameters m_i's in the notation of $\mathbb{C}_5(m_1, m_2, m_3, m_4, m_5)$ if they are clear from the context. A schematic representation of $\mathbb{C}_5$ is shown in Figure 2. Throughout the paper, the subscripts of vertices v_i in C_5 are modulo 5.

Lemma 1 *(i)* $\mathbb{C}_5(m_1, m_2, m_3, m_4, m_5)$ *is* $\{3K_1, C_4\}$*-free, for every integer* $m_i \geq 0$.

(ii) $\omega(\mathbb{C}_5) = max\,\{m_i + m_{i+1} : 1 \leq i \leq 5,\ i\ mod\ 5\}$.

(iii) $\omega(\mathbb{C}_5(m_1 - p, m_2 - p, m_3 - p, m_4 - p, m_5 - p)) = \omega(\mathbb{C}_5(m_1, m_2, m_3, m_4, m_5)) - 2p$, *where* $0 \leq p \leq min\,\{m_i :\ 1 \leq i \leq 5\}$

(iv) $\chi(\mathbb{C}_5(p, p, p, p, p)) = \left\lceil \frac{5p}{2} \right\rceil$, *for any integer* $p \geq 1$.



$\mathbb{C}_5(m_1, m_2, m_3, m_4, m_5)$ An optimal 5-coloring of $\mathbb{C}_5(2,2,2,2,2)$

Figure 2

Proof The statements (i), (ii) and (iii) are obvious.
(iv) : For any graph G with n vertices, $\chi(G) \geq \frac{n}{\alpha(G)}$. Hence, $\chi(\mathbb{C}_5(p,p,p,p,p)) \geq \lceil \frac{5p}{2} \rceil$.

To prove the upper bound consider the partition $(V_1, V_2, \ldots, V_t)$ of $V(\mathbb{C}_5(p,p,p,p,p))$, where $t = \lceil \frac{p}{2} \rceil$, $[V_i] = \mathbb{C}_5(2,2,2,2,2)$, for $1 \leq i \leq t-1$ and

$$[V_t] = \begin{cases} \mathbb{C}_5(2,2,2,2,2), & \text{if p is even} \\ C_5, & \text{if p is odd} \end{cases}$$

Since $\chi(\mathbb{C}_5(2,2,2,2,2)) = 5$ (see Figure 2), we have

$$\chi(\mathbb{C}_5(p,p,p,p,p)) \leq \begin{cases} 5t, & \text{if p is even} \\ 5(t-1)+3, & \text{if p is odd} \end{cases}$$

$$= \left\lceil \frac{5p}{2} \right\rceil . \qquad \blacksquare$$

3 $\{3K_1, C_4\}$-free graphs

A *universal* vertex of a graph G is a vertex which is adjacent to all other vertices in G.

Lemma 2 *If G is $\{3K_1, C_4\}$-free and contains an induced C_5, then any vertex $x \in G - V(C_5)$ is either (i) universal in G or (ii) it is adjacent with exactly three consecutive vertices of C_5.*

Proof Let $C_5 = [v_1, v_2, v_3, v_4, v_5, v_1]$ be a 5-cycle in G. If x is adjacent with at most one vertex or exactly two adjacent vertices of C_5, then one can choose two appropriate non-adjacent vertices of C_5, which together with x induce a $3K_1$ in G, a contradiction. If x is adjacent with exactly two non-adjacent vertices of C_5, say v_1, v_3, then $[x, v_1, v_2, v_3, x] \cong C_4 \sqsubseteq G$, a contradiction. So, we conclude that x is adjacent with at least three vertices of C_5. If x is adjacent with exactly three vertices of C_5, then these must be consecutive on C_5; else there exists an i, $1 \leq i \leq 5$, such that x is adjacent with v_{i-1}, v_{i+1} and it is not adjacent with v_i. But then $[x, v_{i-1}, v_i, v_{i+1}, x] \cong C_4 \sqsubseteq G$. If x is adjacent with exactly four vertices of C_5, then again $C_4 \sqsubseteq G$, as above. Next, if x is adjacent with all the vertices of C_5, we claim that x is universal: else, there exists a vertex $y \in G - V(C_5)$ $(y \neq x)$ such that (x,y) is not an edge in G. By the above analysis, y is adjacent with exactly three consecutive vertices of C_5, say v_1, v_2, v_3 or it is adjacent with all the vertices of C_5. In either case, $[x, v_1, y, v_3, x] \cong C_4 \sqsubseteq G$, a contradiction. ■

Lemma 3 *If G is a $\{3K_1, C_4\}$-free graph containing an induced C_5 but containing no universal vertex, then $G \cong \mathbb{C}_5(m_1, m_2, m_3, m_4, m_5)$, for some integers $m_i \geq 1$.*

Proof Let $[v_1, v_2, v_3, v_4, v_5, v_1]$ be a C_5 in G. For each i, $1 \leq i \leq 5$, define

$$V_i = \{x \in G - V(C_5)\colon x \text{ is adjacent with } v_{i-1},\ v_i,\ v_{i+1}\}.$$

By Lemma 2, V_i's are disjoint and $V(G - V(C_5)) = \bigcup_{i=1}^{5} V_i$. We now make two more claims on V_i which will imply the lemma. First, $[V_i \cup V_{i+1}]$ is complete in G, $1 \leq i \leq 5$, i mod 5; on the contrary, if $x, y \in [V_i \cup V_{i+1}]$ are two non-adjacent vertices, then $[x, y, v_{i+3}] \cong 3K_1$. Next, $[V_{i-1}, V_{i+1}] = \phi$, for $1 \leq i \leq 5$, i mod 5; on the contrary, if $[V_1, V_3] \neq \phi$ (say), and (x, y) is an edge in $[V_1, V_3]$, then $[x, y, v_4, v_5, x] \cong C_4$. ■

Theorem 1 *If G is a $\{3K_1, C_4\}$-free graph, then either (i) G is chordal or (ii) $G \cong \mathbb{C}_5(m_1, m_2, m_3, m_4, m_5) + K_t$, for some integers $m_i \geq 1$ and $t \geq 0$.*

Proof Since $\alpha(G) \leq 2$, every cycle C_n $(n \geq 6)$ in G contains a chord. So, if G is C_5-free too, then G is chordal. Next suppose that G contains an induced C_5. Clearly, no universal vertex of G belongs to C_5, since it is chordless. So, if W is the set of all universal vertices in G, then by Lemma 3, $G - W \cong \mathbb{C}_5(m_1, m_2, m_3, m_4, m_5)$, for some $m_i \geq 1$. Hence, $G \cong \mathbb{C}_5 + K_t$, where $V(K_t) = W$. ■

Lemma 4 *Let G be a $\{3K_1, C_4\}$-free graph with no universal vertex. Suppose $G \sqsupseteq \mathbb{C}_5(m_1, m_2, m_3, m_4, m_5)$, for some $m_i \geq 1$. If $G - V(\mathbb{C}_5)$ contains an induced C_5, then $G \sqsupseteq \mathbb{C}_5(m_1 + 1, m_2 + 1, m_3 + 1, m_4 + 1, m_5 + 1)$.*

Proof Let the vertex set of $\mathbb{C}_5(m_1, m_2, m_3, m_4, m_5)$ be $\bigcup_{i=1}^{5} V_i$, where $V_i = V(K_{m_i})$, $1 \leq i \leq 5$; see Figure 2. Let $[v_1, v_2, v_3, v_4, v_5, v_1]$ be an induced 5-cycle C_5, where $v_i \in V_i$, $1 \leq i \leq 5$. Let $[x_1, x_2, x_3, x_4, x_5, x_1]$ be a 5-cycle in $G - \mathbb{C}_5(m_1, m_2, m_3, m_4, m_5)$. For each i, $1 \leq i \leq 5$, define $W_i = \{x \in V(G) : x \neq v_i$ and x is adjacent with v_{i-1}, v_i, $v_{i+1}\}$.

By Lemma 3, W_i's are pairwise disjoint, $\bigcup_{i=1}^{5} W_i \cup \{v_i\} = V(G)$, $[W_i \cup W_{i+1}]$ is complete and $[W_{i-1}, W_{i+1}] = \phi$, $1 \leq i \leq 5$, i mod 5. Clearly, $V_i - v_i \subseteq W_i$, $1 \leq i \leq 5$. Without loss of generality(w.l.g.) suppose $x_1 \in W_1$. Since $(x_2, x_1) \in E(G)$, $[W_1, W_3] = \phi$ and $[W_1, W_4] = \phi$, it follows that $x_2 \notin W_3 \cup W_4$; so $x_2 \in W_1 \cup W_2 \cup W_5$. If $x_2 \in W_1$, we arrive at a contradiction. Since $(x_3, x_1) \notin E(G)$, $x_3 \in W_3 \cup W_4$. On the other hand since $(x_3, x_2) \in E(G)$, $x_3 \in W_1 \cup W_2 \cup W_5$. It is a contradiction, since W_i's are disjoint. So, $x_2 \in W_2 \cup W_5$. W.l.g. suppose that $x_2 \in W_2$. By using similar arguments, we can show that $x_i \in W_i$, $3 \leq i \leq 5$. So, $W_i \cup \{v_i\} \supseteq V_i \cup \{x_i\}$, $1 \leq i \leq 5$, and hence the lemma. ■

Theorem 2 *Let G be a $\{3K_1, C_4\}$-free graph. Then*

(i) $\chi(G) = \omega(G)$, or

(ii) there exists a maximum integer $p \geq 1$ such that $G \sqsupseteq \mathbb{C}_5(p,p,p,p,p)$ and $\chi(G) \leq \lceil \frac{p}{2} \rceil + \omega(G) \leq \left\lceil \frac{5\omega(G)}{4} \right\rceil$

Proof We apply Theorem 1. If G is chordal, then (i) holds. Next suppose $G \cong \mathbb{C}_5(m_1, m_2, m_3, m_4, m_5) + K_t$, for some integers $m_i \geq 1$ and $t \geq 0$. Let $p = \min\{m_1, m_2, \ldots, m_5\}$. Then by Lemma 4, p is the maximum integer such that $G \sqsupseteq \mathbb{C}_5(p,p,p,p,p)$. Let $G' = G - \mathbb{C}_5(p,p,p,p,p) = \mathbb{C}_5(m_1 - p, m_2 - p, m_3 - p, m_4 - p, m_5 - p) + K_t$. Again, by Lemma 4, G' is C_5-free and so it is chordal. We then have,

$$
\begin{aligned}
\chi(G) &\leq \chi(\mathbb{C}_5(p,p,p,p,p)) + \chi(G') \\
&= \left\lceil \frac{5p}{2} \right\rceil + \omega(G'), \text{ (by Lemma 1 and the chordal property of } G') \\
&= \left\lceil \frac{5p}{2} \right\rceil + \omega(G) - 2p, \text{(by Lemma 1)} \\
&= \omega(G) + \left\lceil \frac{p}{2} \right\rceil \\
&\leq \left\lceil \frac{5\omega(G)}{4} \right\rceil \text{ (since } 2p \leq \omega(G)\text{).}
\end{aligned}
$$

■

Corollary 1 *If G is $\{3K_1, C_4, \mathbb{C}_5(3,3,3,3,3)\}$-free, then $\chi(G) \leq \omega(G) + 1$.*

Proof Apply Theorem 2, with $p \leq 2$. ■

4 Remark

For every pair of integers (w, k) (except one pair) such that $1 \leq w \leq k \leq \lceil \frac{5w}{4} \rceil$, there exists a $\{3K_1, C_4\}$-free graph G with $\omega(G) = w$ and $\chi(G) = k$. The exceptional pair is $(w = 4t + 1,\ k = 5t + 2)$. The required graphs are shown in Figure 3.

w	$k = w + s$, where $0 \le s \le \lceil \frac{w}{4} \rceil$	A graph G with $\omega(G) = w$ and $\chi(G) = k$
$4t, \quad t \ge 1$	$0 \le s \le t$	$\mathbb{C}_5\,(2t, 2t, 2s, 2t, 2t)$
$4t+2, \quad t \ge 0$	$0 \le s \le t+1$	$\mathbb{C}_5\,(2t+1, 2t+1, 2s-1, 2t+1, 2t+1)$
$4t+1, \quad t \ge 0$	$0 \le s \le t$	$\mathbb{C}_5\,(2t+1, 2t, 2s-1, 2t+1, 2t)$
$4t+3, \quad t \ge 0$	$0 \le s \le t$	$\mathbb{C}_5\,(2t+2, 2t+1, 2s, 2t+2, 2t+1)$
$4t+3, \quad t \ge 0$	$s = t+1$	$\mathbb{C}_5\,(2t+1, 2t+2, 2t+1, 2t+1, 2t+2)$

Figure 3: A table of extremal graphs

Clearly, for each of these graphs G, we have $\omega(G) = w$, since $0 \le s \le \lceil \frac{w}{4} \rceil$. In each case one can show that $\chi(G) \le k$, by using Theorem 2, and that $\chi(G) \ge k$ by using the general upper bound $\chi(F) \ge \frac{n(F)}{\alpha(F)}$, for any graph F. The last inequalities in the proof of Theorem 2 imply that there is no $\{3K_1, C_4\}$-free graph G with $\omega(G) = 4t+1$ and $\chi(G) = 5t+2$, $(t \ge 0)$.

5 Conclusion

$K_{1,3}$-free graphs have received much attention as they form a superclass of line graphs and they are amenable to polynomial time algorithms to find many graph theoretical parameters; see [3]. The results in this paper suggest that if G is $\{K_{1,3}, C_4\}$-free (or more generally $\{K_{1,3}, K_1 + C_4\}$-free), then $\chi(G)$ is bounded above by a constant multiple of $\omega(G)$. We are unable to obtain such a bound. Note that the neighbourhood of any vertex in a $\{K_{1,3}, K_1 + C_4\}$-free graph induces a $\{3K_1, C_4\}$-free graph.

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