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Almost all graphs and h-hypergraphs have small diameter

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Abstract

In this paper it is shown that all results previously deduced by the author concerning the asymptotic number of graphs, digraphs, h-hypergraphs or h-connected graphs and digraphs having diameter equal to k are also valid for finite diameter greater than or equal to k as the number of vertices tends to infinity. This implies that in the class of connected graphs, digraphs, h-hypergraphs or h-connected graphs and digraphs with diameter greater than or equal to k ($k \geq 2$) almost all have diameter equal to k and settles a conjecture proposed in *Discrete Math.* 235 (2001), 291–299.

1 Notation and preliminary results

All graphs or digraphs in this paper are finite, labeled (i.e., different means non-identical), without loops or parallel edges or arcs. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. A graph G is said to be h-connected if $\kappa(G) \geq h$. The strong connectivity, denoted $s\kappa(G)$ of a digraph G is the minimum number of vertices whose removal results in a digraph which is not strongly connected or trivial. A digraph G is said to be h-strongly connected if $s\kappa(G) \geq h$. A strongly connected digraph is also said to be 1-strongly connected. The distance $d(x, y)$ between vertices x and y of a connected graph G is the length of a shortest path between them. For a strongly connected digraph G the distance $d(x, y)$ from vertex x to vertex y is the length of a shortest path of the form $(x, \ldots, y)$. The eccentricity of a vertex x is $\mathrm{ecc}(x) = \max_{y \in V(G)} d(x, y)$. The diameter (resp. strong diameter) of G, denoted $d(G)$, is equal to $\max_{x \in V(G)} \mathrm{ecc}(x) = \max_{x,y \in V(G)} d(x, y)$ if G is connected (resp. strongly connected) and ∞ otherwise.

Consider $V(G) = \{1, \ldots, n\}$ and denote for every $h \geq 1$ by $G(n; d = k)$ and $G(n; d \geq k)$; $D(n; d = k)$ and $D(n; d \geq k)$; $G(n; h, d = k)$ and $G(n; h, d \geq k)$;

$D_s(n;h,d=k)$ and $D_s(n;h,d\geq k)$, the number of labeled graphs; digraphs; h-connected graphs; h-strongly connected digraphs G of order n and diameter $d(G)=k$ and $d(G)\geq k$, respectively.

A simple hypergraph $H=(X,\mathcal{E})$, of order $|X|=n$ and size $|\mathcal{E}|=m$, consists of a vertex-set $V(H)=X$ and an edge-set $E(H)=\mathcal{E}$, where for each edge $E\in\mathcal{E}$ one has $E\subseteq X$ and $|E|\geq 2$. H is h-uniform, or is an h-hypergraph, if $|E|=h$ for each E in $\mathcal{E}$. The degree of a vertex $x\in V(H)$ is denoted by $d_H(x)$. Two vertices u, v of H are in the same component if there are vertices $x_0=u,x_1,\ldots,x_k=v$ and edges $E_1,\ldots,E_k$ of H such that $x_{i-1},x_i\in E_i$ for each i $(1\leq i\leq k)$. If H has only one component then it is said to be connected. A path P of length k in H [1] is a subhypergraph comprising $k+1$ distinct vertices $x_1,\ldots,x_{k+1}$ and k distinct edges $E_1,\ldots,E_k$ of H such that $x_i,x_{i+1}\in E_i$ for each i, $1\leq i\leq k$.

For a connected hypergraph H the distance $d(x,y)$ between vertices x and y is the length of a shortest path between them. The eccentricity of a vertex x is $\mathrm{ecc}(x)=\max_{y\in V(H)}d(x,y)$. The diameter of H, denoted $d(H)$, is equal to $\max_{x\in V(H)}\mathrm{ecc}(x)=\max_{x,y\in V(H)}d(x,y)$ if H is connected and ∞ otherwise. By $H(n,h;d=k);H(n,h;d\geq k)$ and $CH(n,h;d=k);CH(n,h;d\geq k)$ we denote the number of labeled h-hypergraphs and connected h-hypergraphs H of order n and diameter $d(H)=k$ and $d(H)\geq k$, respectively.

If $\lim_{n\to\infty}f(n)/g(n)=1$, this is denoted by $f(n)\sim g(n)$, or $f(n)=g(n)(1+o(1))$.

It is well known [2] that almost all graphs and digraphs have diameter two and for every fixed integer $h\geq 1$ almost all graphs are h-connected. Also in [5] it was proved that for every fixed integer $h\geq 1$ almost all digraphs are h-strongly connected. In [6], [10] it was shown that for every $h\geq 3$ almost all h-hypergraphs H of order n have diameter one as $n\to\infty$. Hence for every $h\geq 1$ we have:

$$G(n;h,d=2)=2^{\binom{n}{2}}(1+o(1));$$

$$D_s(n;h,d=2)=4^{\binom{n}{2}}(1+o(1)).$$

Also, for every $h\geq 3$,

$$H(n,h;d=1)=2^{\binom{n}{h}}(1+o(1)).$$

Notice that $G(n;d=k)=G(n;1,d=k)$ and $D(n;d=k)=D_s(n;1,d=k)$ since all graphs and digraphs having diameter equal to k must be connected (strongly connected). This property is not longer true for $d\geq k$ since in this case disconnected graphs have diameter equal to ∞ which is greater than k.

2 The case of graphs and digraphs

Theorem 2.1 *We have*
$G(n;d\geq 3)=2^{\binom{n}{2}}(0.75+o(1))^n$ *and* $D(n;d\geq 3)=4^{\binom{n}{2}}(0.75+o(1))^n$*;*
$G(n;h,d\geq 3)=2^{\binom{n}{2}}(0.75+o(1))^n$ *and* $D_s(n;h,d\geq 3)=4^{\binom{n}{2}}(0.75+o(1))^n$
for every fixed $h\geq 1$*.*

Proof: These equalities appear explicitely or can be deduced easily from the proofs of Theorem 2.1 and Corollary 2.2 of [12], Lemma 1.3 of [9] and of Theorem 3.1 of [14]. □

We shall consider three arithmetic functions:

A. For $k \geq 4$ let $D_f = \{(n_1, \ldots, n_k) | n_1 + \cdots + n_k = n, n_i \geq h$ for every $1 \leq i \leq k-1$ and $n_k \geq 1\}$ and

$$f(n, h; n_1, \ldots, n_k) = \binom{n}{n_1, \ldots, n_k} 2^{\sum_{i=1}^{k} \binom{n_i}{2}} \prod_{i=1}^{k-1} (2^{n_i} - 1)^{n_{i+1}},$$

where $(n_1, \ldots, n_k) \in D_f$. Further, let

$$f(n, h, k) = \max_{D_f} f(n, h; n_1, \ldots, n_k)$$

[13], [14].

B. Let $k \geq 3$, $D(n, k, h) = \{(n_1, \ldots, n_k) | n_1, \ldots, n_k$ are positive integers such that $n_1 + \cdots + n_k = n; n_1 = 1$ and $n_i + n_{i+1} \geq h$ for every $i = 1, \ldots, k-1\}$ and

$$g(n, h; n_1, \ldots, n_k) = \binom{n}{n_1, \ldots, n_k} 2^{\sum_{i=1}^{k-1} \binom{n_i + n_{i+1}}{h} - \sum_{i=2}^{k-1} \binom{n_i}{h} - |\{i | n_i + n_{i+1} = h\}|},$$

where $(n_1, \ldots, n_k) \in D(n, k, h)$. We denote

$$g(n, k, h) = \max_{(n_1, \ldots, n_k) \in D(n,k,h)} g(n, h; n_1, \ldots, n_k).$$

The function $g(n, h; n_1, \ldots, n_k)$ counts the number of labeled h-hypergraphs with a certain layered structure, having n_i vertices on the level i for $1 \leq i \leq k$ [10].

C. Let $k \geq 4$ and

$$\varphi(n, h; n_1, \ldots, n_k) = \binom{n}{n_1, \ldots, n_k} 2^{\sum_{i=1}^{k-2} \binom{n_i}{2}} \prod_{i=1}^{k-2} (2^{n_i} - 1)^{n_{i+1}} 2^{\binom{n_{k-1} + n_k}{2}},$$

where $n_1 + \cdots + n_k = n, n_i \geq h$ for every $1 \leq i \leq k-1$ and $n_k \geq 1$. Let $\varphi(n, h, k) = \max_{D_\varphi} \varphi(n, h; n_1, \ldots, n_k)$, where D_φ is defined similarly to D_f. We recall the following facts:

Theorem 2.2 [8] *The equality*

$$f(n, 1, k) = 2^{\binom{n}{2}} (3 \cdot 2^{-k+2} + o(1))^n$$

holds for every $k \geq 3$.

Theorem 2.3 [13],[14] *We have*

$$f(n, h, 4) = 2^{\binom{n}{2}} (2^{-h-1} + 2^{-1} + o(1))^n$$

for every $h \geq 2$;

$$f(n, h, k) = 2^{\binom{n}{2}} ((2^{h+1} - 1) 2^{-kh+3h-1} + o(1))^n$$

for every $h \geq 2$ *and* $k \geq 5$.

Functions $f(n,h,k)$ and $\varphi(n,h,k)$ have the same asymptotic behavior, as can be seen below.

Theorem 2.4 *We have*

$$\varphi(n,1,k) = 2^{\binom{n}{2}}(3 \cdot 2^{-k+2} + o(1))^n$$

for every $k \geq 3$*;*

$$\varphi(n,h,4) = 2^{\binom{n}{2}}(2^{-h-1} + 2^{-1} + o(1))^n$$

for every $h \geq 2$*;*

$$\varphi(n,h,k) = 2^{\binom{n}{2}}((2^{h+1} - 1)2^{-kh+3h-1} + o(1))^n$$

for every $h \geq 2$ *and* $k \geq 5$*.*

Proof: If $n_k = \alpha + 1$ and $\alpha \geq 1$ then

$$\frac{\varphi(n,h;n_1,\ldots,n_{k-3},n_{k-2}+\alpha,n_{k-1},1)}{\varphi(n,h;n_1,\ldots,n_k)} > 1. \tag{1}$$

Indeed, the ratio in the left-hand side of (1) equals

$$\frac{\alpha+1}{\binom{n_{k-2}+\alpha}{\alpha}} \left(\frac{2^{n_{k-2}+\alpha} - 1}{2^{n_{k-2}} - 1}\right)^{n_{k-1}} (2^{n_{k-3}} - 1)^{\alpha} 2^{\alpha(n_{k-2}-n_{k-1})}.$$

We have $(2^{n_{k-3}} - 1)^{\alpha} \geq 1$; $\frac{2^{n_{k-2}+\alpha}-1}{2^{n_{k-2}}-1} > 2^{\alpha}$ and for every $\alpha, \beta \geq 1$ the following inequality holds:

$$\frac{\alpha+1}{\binom{\alpha+\beta+1}{\alpha}} > 2^{-\alpha\beta}$$

since

$$\frac{(\alpha+1)2^{\alpha\beta}}{\binom{\alpha+\beta+1}{\beta+1}} = \frac{(\beta+1)2^{\alpha}}{\beta+\alpha+1} \cdot \frac{\beta 2^{\alpha}}{\beta+\alpha} \cdots \frac{2 \cdot 2^{\alpha}}{2+\alpha} > 1,$$

hence $(\alpha+1)/\binom{n_{k-2}+\alpha}{\alpha} > 2^{-\alpha(n_{k-2}-1)}$, which imply (1).

It follows that all systems $(n_1,\ldots,n_k) \in D_{\varphi}$ which maximize $\varphi(n,h;n_1,\ldots,n_k)$ satisfy $n_k = 1$. The same property holds for all systems $(n_1,\ldots,n_k) \in D_f$ which maximize $f(n,1;n_1,\ldots,n_k)$ [8] and $f(n,h;n_1,\ldots,n_k)$ [14]. We have

$$\frac{\varphi(n,h;n_1,\ldots,n_k)}{f(n,h;n_1,\ldots,n_k)} = \left(\frac{2^{n_{k-1}}}{2^{n_{k-1}} - 1}\right)^{n_k},$$

which implies

$$1 < \frac{\varphi(n,h;n_1,\ldots,n_{k-1},1)}{f(n,h;n_1,\ldots,n_{k-1},1)} \leq 2.$$

Let $(\alpha_1,\ldots,\alpha_{k-1},1) \in D_f$ and $(\beta_1,\ldots,\beta_{k-1},1) \in D_{\varphi}$ be such that $f(n,h;\alpha_1,\ldots,\alpha_{k-1},1) = f(n,h,k)$ and $\varphi(n,h;\beta_1,\ldots,\beta_{k-1},1) = \varphi(n,h,k)$. We get

$\varphi(n,h,k) = \varphi(n,h;\beta_1,\ldots,\beta_{k-1},1) \le 2f(n,h;\beta_1,\ldots,\beta_{k-1},1) \le 2f(n,h;\alpha_1,\ldots,\alpha_{k-1},1) = 2f(n,h,k)$ and $\varphi(n,h,k) = \varphi(n,h;\beta_1,\ldots,\beta_{k-1},1) \ge \varphi(n,h;\alpha_1,\ldots,\alpha_{k-1},1) > f(n,h;\alpha_1,\ldots,\alpha_{k-1},1) = f(n,h,k)$. Hence $f(n,h,k) < \varphi(n,h,k) \le 2f(n,h,k)$, which concludes the proof by Theorems 2.2 and 2.3. □

Theorem 2.5 *The following equalities hold:*
a) For every fixed $k \ge 4$,
$G(n;1,d \ge k) = 2^{\binom{n}{2}}(3 \cdot 2^{-k+1} + o(1))^n$ *and*
$D_s(n;1,d \ge k) = 4^{\binom{n}{2}}(3 \cdot 2^{-k+1} + o(1))^n$;
b) For every fixed $h \ge 2$,
$G(n;h,d \ge 4) = 2^{\binom{n}{2}}(2^{-h-2} + 2^{-2} + o(1))^n$ *and*
$D_s(n;h,d \ge 4) = 4^{\binom{n}{2}}(2^{-h-2} + 2^{-2} + o(1))^n$;
c) For every fixed $h \ge 1$ *and* $k \ge 5$,
$G(n;h,d \ge k) = 2^{\binom{n}{2}}((2^{h+1}-1)2^{-kh+3h-2} + o(1))^n$ *and*
$D_s(n;h,d \ge k) = 4^{\binom{n}{2}}((2^{h+1}-1)2^{-kh+3h-2} + o(1))^n$.

Proof: Notice that in [8–14] it was proved that all asymptotic equalities in the statement of the theorem hold in the case when graphs have diameter equal to $k \ge 4$. The idea of the proof of these evaluations is the same as in the case when diameter is equal to k: We start with the lower bound construction by generating a large class of connected graphs and strong digraphs of order n and diameter greater than or equal to $k \ge 4$ and then prove that an upper bound for the number of these graphs and digraphs has the same asymptotic expression.
Notice that it is sufficient to use the lower bounds for the number of graphs and digraphs of order n and diameter equal to k proposed in [8-14] that have the asymptotic expressions appearing respectively in the right-hand sides of the equalities in the statement.
For the upper bound we shall consider only case a) since cases b) and c) are similar. If k is finite, $k \ge 4$, G is a connected graph of order n and $x \in V(G)$ has eccentricity $\mathrm{ecc}(x) \ge k$, then there exists a partition $V_1(x) \cup \cdots \cup V_{k-1}(x) \cup W_k(x)$ of $V(G)\backslash\{x\}$ such that $V_i(x) = \{y | y \in V(G) \text{ and } d(x,y) = i\}$ for $1 \le i \le k-1$ and $W_k(x) = \{y | y \in V(G) \text{ and } d(x,y) \ge k\}$. It follows that x is adjacent to all vertices of $V_1(x)$ and for every $2 \le i \le k-1$ any vertex $z \in V_i(x)$ is the extremity of an edge zt, where $t \in V_{i-1}(x)$. By denoting $|V_i(x)| = n_i$ for $1 \le i \le k-1$ and $|W_k(x)| = n_k$, the edges between $V_i(x)$ and $V_{i+1}(x)$ can be chosen in $(2^{n_i}-1)^{n_{i+1}}$ ways [8] and the edges between $V_{k-1}(x)$ and $W_k(x)$ in at most $2^{\binom{n_{k-1}+n_k}{2}}$ ways (the same argument was used in [3]). We can write
$|\{G | V(G) = \{1,\ldots,n\}, G \text{ is connected and } \mathrm{ecc}(x) \ge k\}| \le$
$\sum_{\substack{n_1+\ldots+n_k=n-1 \\ n_1,\ldots,n_k \ge 1}} \binom{n-1}{n_1,\ldots,n_k} 2^{\sum_{i=1}^{k-2}\binom{n_i}{2}} \prod_{i=1}^{k-2}(2^{n_i}-1)^{n_{i+1}} 2^{\binom{n_{k-1}+n_k}{2}} \le$
$\binom{n-2}{k-1}\max_{D_\varphi}\varphi(n-1,1;n_1,\ldots,n_k) = \binom{n-2}{k-1}\varphi(n-1,1,k)$,
since the number of compositions $n-1 = n_1+\ldots+n_k$ having k positive terms equals $\binom{n-2}{k-1}$.
We get $G(n;1,d \ge k) \le |\bigcup_{x\in\{1,\ldots,n\}}\{G | V(G) = \{1,\ldots,n\}$, G is connected and

$\mathrm{ecc}(x) \geq k\}| \leq n\binom{n-2}{k-1}\varphi(n-1,1,k) = 2^{\binom{n}{2}}(3\cdot 2^{-k+1}+o(1))^n$ by Theorem 2.4, since the multiplicative term $n\binom{n-2}{k-1}$ is estimated by $o(1)$ in the expression.
If G is a digraph, with the same notation as above we find
$|\{G|V(G)=\{1,\ldots,n\}$, G is strongly connected and $\mathrm{ecc}(x)\geq k\}| \leq$
$\sum_{\substack{n_1+\cdots+n_k=n-1\\ n_1,\ldots,n_k\geq 1}} \binom{n-1}{n_1,\ldots,n_k} 4^{\sum_{i=1}^{k-2}\binom{n_i}{2}} \prod_{i=1}^{k-2}(2^{n_i}-1)^{n_{i+1}} \prod_{i=1}^{k-1} 2^{n_i(n_{i-1}+\ldots+1)} \times$
$4^{\binom{n_{k-1}+n_k}{2}} 2^{n_k(n_{k-2}+\ldots+1)} = 2^{\binom{n}{2}} \sum_{\substack{n_1+\ldots+n_k=n-1\\ n_1,\ldots,n_k\geq 1}} \varphi(n-1,1;n_1,\ldots,n_k)$,
because
$2^{\binom{n_{k-1}+n_k}{2}} = 2^{\binom{n_{k-1}}{2}+\binom{n_k}{2}+n_{k-1}n_k}$ and $2^{\sum_{i=1}^{k}\binom{n_k}{2}} \prod_{i=1}^{k} 2^{n_i(n_{i-1}+\ldots+1)} = 2^{\binom{n}{2}}$.
Furthermore $\sum_{\substack{n_1+\ldots+n_k=n-1\\ n_1,\ldots,n_k\geq 1}} \varphi(n-1,1;n_1,\ldots,n_k) \leq \binom{n-2}{k-1}\varphi(n-1,1,k)$ and in the same way as above we get
$D(n;1,d\geq k) \leq |\bigcup_{x\in V(G)}\{G|V(G)=\{1,\ldots,n\}, G$ is strongly connected and $\mathrm{ecc}(x)\geq k\}| \leq n2^{\binom{n}{2}}\binom{n-2}{k-1}\varphi(n-1,1,k) = 4^{\binom{n}{2}}(3\cdot 2^{-k+1}+o(1))^n$ by Theorem 2.4.

□

Corollary 2.6 *We have*

$$\lim_{n\to\infty}\frac{G(n;h,d=k)}{G(n;h,d\geq k+1)} = \lim_{n\to\infty}\frac{D_s(n;h,d=k)}{D_s(n;h,d\geq k+1)} = \infty$$

for every fixed $h\geq 1$ and $k\geq 2$.

Corollary 2.7 *For $k=2$ or $k=3$ we have*

$$\lim_{n\to\infty}\frac{G(n;d=k)}{G(n;d\geq k+1)} = \lim_{n\to\infty}\frac{D(n;d=k)}{D(n;d\geq k+1)} = \infty,$$

but

$$\lim_{n\to\infty}\frac{G(n;d=k)}{G(n;d\geq k+1)} = \lim_{n\to\infty}\frac{D(n;d=k)}{D(n;d\geq k+1)} = 0$$

for every fixed $k\geq 4$.

Proof: The case $k=2$ follows from the property that almost all graphs and digraphs of order n have diameter two as $n\to\infty$ and for $k=3$ the property was shown in [7] (the case of graphs) and in [9] (the case of digraphs). If $k\geq 4$, $G(n;d=k)=G(n;1,d=k)$ and $D(n;d=k)=D_s(n;1,d=k)$ and they have asymptotic expressions that coincide with those given in Theorem 2.5 for the case of diameters $d\geq k$. On the other hand, $G(n;d\geq k+1)$ and $D(n;d\geq k+1)$ are greater than the number of disconnected graphs (resp. of digraphs that are not strong) (when the diameter equals infinity) and these numbers are asymptotically equal to $n2^{\binom{n-1}{2}} = 2^{\binom{n}{2}}(2^{-1}+o(1))^n$ and $n2^n4^{\binom{n-1}{2}} = 4^{\binom{n}{2}}(2^{-1}+o(1))^n$, respectively [4]. Since $3\cdot 2^{-k+1} < 2^{-1}$ for every $k\geq 4$ the conclusion follows. □

3 The case of hypergraphs

Theorem 3.1 [10] *We have* $g(n,3,h) = n(n-1)2^{\binom{n}{h}-\binom{n-2}{h-2}} = n(n-1)2^{\binom{n}{h}-[1/(h-2)!]n^{h-2}+O(n^{h-3})}$ *for every* $h \geq 3$ *and* $g(n,k,h) = 2^{\binom{n}{h}+n^{h-1}\beta(k,h)+o(n^{h-1})}$ *for every fixed* $h \geq 3$ *and* $k \geq 4$*, where* $\beta(k,h) = \frac{1}{2(h-1)!}(h(5-k)-4)$ *for odd* $k \geq 5$ *and* $\beta(k,h) = \frac{1}{2(h-1)!}(h(4-k)-2)$ *for even* $k \geq 4$.

Lemma 3.2 [10] *Let* A*,* B *be two disjoint sets,* $|A| = p$ *and* $|B| = q$*. The number of* h*-hypergraphs* H *with vertex set* $V(H) = A \cup B$*,* $E(H)$ *has no edge included in* A *or in* B *and* $d_H(x) \geq 1$ *for every vertex* $x \in B$ *equals* $a(p,q) = \alpha(p,q) - \binom{q}{1}\alpha(p,q-1) + \binom{q}{2}\alpha(p,q-2) - \ldots + (-1)^q$*, where* $\alpha(p,q) = 2^{\binom{p+q}{h}-\binom{p}{h}-\binom{q}{h}}$ *for every* $p,q \geq 0$.

Theorem 3.3 *We have*

$$2^{\binom{n}{h}-\binom{n-2}{h-2}}(1+o(1)) \leq CH(n,h;d \geq 2) \leq \binom{n}{2}2^{\binom{n}{h}-\binom{n-2}{h-2}},$$

$$\frac{1}{n-h}g(n,4,h)(1+o(1)) \leq CH(n,h;d \geq 3) \leq g(n,4,h)(1+o(1)),$$

$$\frac{1}{2(h-1)}g(n,5,h)(1+o(1)) \leq CH(n,h;d \geq 4) \leq g(n,5,h)(1+o(1)),$$

and for every $k \geq 5$,

$$CH(n,h;d \geq k) = g(n,k+1,h)(1+o(1)),$$

where $g(n,k,h)$ *is given by Theorem 3.1.*

Proof: Notice that all asymptotic equalities and inequalities in the statement of the theorem also hold in the case when hypergraphs have diameter equal to $k \geq 2$. We use the same method of proof by producing lower and upper bounds, which have the same asymptotic behavior for $k \geq 5$.
Since every h-hypergraph having diameter $k < \infty$ is connected, we may use the same lower bounds for the number of h-hypergraphs of order n and diameter equal to k [10], [11].
The upper bound for $CH(n,h;d \geq 2)$ follows from equality (1) of [10]. Let k be finite, $k \geq 2$, H be a connected h-hypergraph, $v \in V(H)$ and $\mathrm{ecc}(v) \geq k$. By denoting $V_{i+1}(v) = \{u | u \in V(H) \text{ and } d(u,v) = i\}$ for $0 \leq i \leq k-1$ and $W_{k+1}(v) = \{u | u \in V(H) \text{ and } d(u,v) \geq k\}$, it follows that $V_1(v) = \{v\}, V_2(v), \ldots, W_{k+1}(v)$ is an ordered partition of $V(H)$. We deduce that v is adjacent to all vertices of $V_2(v)$ and for every $2 \leq i \leq k-1$, any vertex $z \in V_{i+1}(v)$ is included in some edge $E \subseteq V_i(v) \cup V_{i+1}(v)$ which contains at least one vertex $t \in V_i(v)$. If we denote $|V_i(v)| = n_i$ for $1 \leq i \leq k$ and $|W_{k+1}(v)| = n_{k+1}$, since H is connected it follows that $n_1 = 1, n_2 \geq h-1$ and $n_i + n_{i+1} \geq h$ for every $2 \leq i \leq k$. By Lemma 3.2 we get

$|\{H|H$ is connected, $V(H)=\{1,\ldots,n\}$ and $\mathrm{ecc}(v)\geq k\}|\leq$
$\frac{1}{n}\sum_{(n_1,\ldots,n_{k+1})\in D(n,k+1,h)}\binom{n}{n_1,\ldots,n_{k+1}}2^{\sum_{i=2}^{k-1}\binom{n_i}{h}}\prod_{i=1}^{k-1}a(n_i,n_{i+1})2^{\binom{n_k+n_{k+1}}{h}}$.
We have [10]:

$$a(n_i,n_{i+1})\leq\alpha(n_i,n_{i+1})=2^{\binom{n_i+n_{i+1}}{h}-\binom{n_i}{h}-\binom{n_{i+1}}{h}}$$

and if $n_i+n_{i+1}=h$ then

$$a(n_i,n_{i+1})=1=2^{\binom{n_i+n_{i+1}}{h}-\binom{n_i}{h}-\binom{n_{i+1}}{h}-1}.$$

Hence $|\{H|H$ is connected, $V(H)=\{1,\ldots,n\}$ and $\mathrm{ecc}(v)\geq k\}|\leq$
$\frac{1}{n}\sum_{(n_1,\ldots,n_{k+1})\in D(n,k+1,h)}\binom{n}{n_1,\ldots,n_{k+1}}2^{\sum_{i=1}^{k}\binom{n_i+n_{i+1}}{h}-\sum_{i=2}^{k}\binom{n_i}{h}-|\{i|n_i+n_{i+1}=h\}|}=$
$\frac{1}{n}\sum_{(n_1,\ldots,n_{k+1})\in D(n,k+1,h)}g(n,h;n_1,\ldots,n_{k+1})$, the same upper bound as for $\mathrm{ecc}(v)=k$ [10]. We get $CH(n,h;d\geq k)\leq|\bigcup_{v\in\{1,\ldots,n\}}\{H|H$ is connected, $V(H)=\{1,\ldots,n\}$ and $\mathrm{ecc}(v)\geq k\}|\leq n|\{H|H$ is connected, $V(H)=\{1,\ldots,n\}$ and $\mathrm{ecc}(v)\geq k\}|\leq g(n,k+1,h)(1+o(1))$ for every $k\geq 3$, which concludes the proof in the same way as for $H(n,h;d=k)$. □

Since $g(n,k+1,h)=2^{\binom{n}{h}+n^{h-1}\beta(k+1,h)+o(n^{h-1})}$, where

$$\beta(k+1,h)=[1/(2(h-1)!)](3h-kh-2) \tag{2}$$

for every odd $k\geq 3$ and

$$\beta(k+1,h)=[1/(2(h-1)!)](4h-kh-4) \tag{3}$$

for every even $k\geq 4$, we deduce:

Corollary 3.4 *For every fixed $h\geq 3$ and $k\geq 1$ the following equality holds:*

$$\lim_{n\to\infty}\frac{CH(n,h;d=k)}{CH(n,h;d\geq k+1)}=\infty.$$

Corollary 3.5 *Let $h\geq 3$ be fixed. Then for $k=1$ or $k=2$ we get*

$$\lim_{n\to\infty}\frac{H(n,h;d=k)}{H(n,h;d\geq k+1)}=\infty,$$

but

$$\lim_{n\to\infty}\frac{H(n,h;d=k)}{H(n,h;d\geq k+1)}=0$$

for every fixed $k\geq 4$. For $k=3$ this limit equals ∞ for $h=3$ and 0 for every fixed $h\geq 4$.

Proof: For $k=1$ the property is obvious since almost all h-hypergraphs have diameter equal to one as $n\to\infty$. Let $DH(n,h)$ denote the number of disconnected h-hypergraphs of order n. In [6] it was shown that $DH(n,h)\sim n2^{\binom{n-1}{h}}=n2^{\binom{n}{h}-\binom{n-1}{h-1}}=n2^{\binom{n}{h}-n^{h-1}/(h-1)!+O(n^{h-2})}$. We have

$$H(n,h;d\geq k+1)=CH(n,h;d\geq k+1)+DH(n,h)$$

and

$$H(n,h;d=k)=CH(n,h;d=k)\le g(n,k+1,h)(1+o(1))$$

for every $k\ge 4$ by Theorem 3.3.

We have found that $g(n,k+1,h)=2^{\binom{n}{h}+n^{h-1}\beta(k+1,h)+o(n^{h-1})}$, where $\beta(k+1,h)$ is given by (2) and (3), hence $1/(h-1)!+\beta(k+1,h)<0$ for every $k\ge 4$. It follows that for every $k\ge 4$ we have $\lim_{n\to\infty}H(n,h;d=k)/DH(n,h)=0$, which implies $\lim_{n\to\infty}H(n,h;d=k)/H(n,h;d\ge k+1)=0$.
For $k=2$ we deduce, as above, $H(n,h;d\ge 3)=CH(n,h;d\ge 3)+DH(n,h)$ and $\lim_{n\to\infty}CH(n,h;d=2)/CH(n,h;d\ge 3)=\infty$.
We get $CH(n,h;d=2)=H(n,h;d=2)\ge 2^{\binom{n}{h}-\binom{n-2}{h-2}}(1+o(1))$, hence $\lim_{n\to\infty}H(n,h;d=2)/DH(n,h)=\infty$ and the case $k=2$ is proved.
For $k=3$ we use an explicit form of $g(n,4,h)$ given in [10] p. 289, namely:

$$g(n,4,h)=\frac{(n)_{n-h-1}}{(h-1)!}2^{\binom{n-2}{h}+\binom{n-h-1}{h-1}},$$

where $(x)_n=x(x-1)\cdots(x-n+1)$ for every $x\in\mathbb{R}$ and $n\in\mathbb{N}$ and double inequality

$$\frac{1}{n-h}g(n,4,h)(1+o(1))\le H(n,h;d=3)\le g(n,4,h)(1+o(1)).$$

Note that the coefficient

$$\frac{(n)_{n-h-1}}{(h-1)!}=\frac{n!}{(h+1)!(h-1)!}\sim\sqrt{2\pi n}(n/e)^n/[(h+1)!(h-1)!]=2^{n\log_2 n+O(n)}$$

by Stirling's formula. We deduce

$$\frac{H(n,h;d\ge 4)}{H(n,h;d=3)}=\frac{CH(n,h;d\ge 4)}{H(n,h;d=3)}+\frac{DH(n,h)}{H(n,h;d=3)},$$

where $\lim_{n\to\infty}CH(n,h;d\ge 4)/H(n,h;d=3)=0$ and $DH(n,h)/g(n,4,h)=2^{\binom{n-2}{h-1}-\binom{n-h-1}{h-1}-n\log_2 n+O(n)}$. Hence $\lim_{n\to\infty}DH(n,h)/H(n,h;d=3)$ equals 0 if $h=3$ and ∞ if $h\ge 4$, which concludes the proof. □

4 Concluding remarks

It is not difficult to prove, in the same way as above, that the results for strongly connected digraphs hold also for connected digraphs. In this case we must consider the weak diameter, when the distance from vertex x to vertex y is defined as the length of a shortest chain of the form $[x,\dots,y]$ (in a chain some pairs of arcs may have opposite orientations). Some asymptotic results about the number of digraphs of order n and weak diameter equal to k as $n\to\infty$ were deduced in [9], [12] and [13]. Also, in [11] it was conjectured that $\lim_{n\to\infty}G(n;d=k)/G(n;d\ge k+1)=\lim_{n\to\infty}D(n;d=k)/D(n;d\ge k+1)=\infty$ for every fixed $k\ge 2$ and

$\lim_{n\to\infty} H(n, h; d = k)/H(n, h; d \geq k + 1) = \infty$ for every fixed $h \geq 3$ and $k \geq 1$. By Corollaries 2.6 and 2.7 in the case of graphs or digraphs this is true only for $k = 2$ and $k = 3$, but the property is always valid if we restrict ourselfs to the case of connected graphs and of strongly connected digraphs, respectively. For h-hypergraphs this property is true for $k = 1$, $k = 2$ and $k = 3$ (only whenever $h = 3$) by Corollary 3.5, but it is always true in the class of connected h-hypergraphs by Corollary 3.4.

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