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Triangular Cayley maps of $K_{n,n,n}$

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Abstract

A Cayley map is an embedding of a Cayley graph G in an orientable surface, such that the group of orientation-preserving automorphisms of the embedding contains a subgroup acting regularly on the vertex set of G. We investigate the problem of determining for which n there exists a Cayley map with underlying graph $K_{n,n,n}$ such that all faces of the map are triangular.

1 Introduction

Group actions on graphs and maps have received considerable attention in recent years. A prominent role in the study of group actions is played by regular actions. The weakest "degree of symmetry" occurs when a group acts regularly on vertices of a graph or a map; this is the case of Cayley graphs and maps. On the opposite end of the spectrum, in the case of maps the highest symmetry is achieved when a group acts regularly on flags, giving rise to regular maps.

In this contribution we will study the above phenomena on triangular embeddings of the complete tripartite graph $K_{n,n,n}$. In particular, we will focus on conditions under which $K_{n,n,n}$ admits a triangular Cayley map. It turns out that among triangular embeddings of $K_{n,n,n}$ there is a unique one which is regular; we will characterize the values of n for which this regular map of $K_{n,n,n}$ is a Cayley map.

Our choice to investigate triangular Cayley maps of $K_{n,n,n}$ is motivated by the following facts. Triangular embeddings are automatically genus embeddings, and these are of primary interest in topological graph theory. Historically the most important were orientable triangular embeddings of complete graphs K_n for $n \equiv 0, 3, 4$ or $7 \pmod{12}$. Their construction constitutes one third of a complete solution of the famous Heawood Map Colouring Problem which took about 70 years to solve [8]. In the case $n \equiv 0, 4$ and $7 \pmod{12}$, the solution of [8] was based on constructing

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a triangular Cayley map of K_n. Very recent result of [1, 4] show that the number of triangular embeddings of K_n exceeds 2^{cn^2} for n in certain residue classes mod 36. From our point of view it is important to note that some of these constructions require a large number of triangular embeddings of $K_{n,n,n}$ as ingredients. It is therefore of interest to ask which of these embeddings are Cayley maps.

In general, Cayley maps and regular Cayley maps have been subject of a thorough investigation in [7]. For complete tripartite graphs it was shown in [5, 6] that for each n there is a unique regular triangular map with underlying graph $K_{n,n,n}$. The question of characterization of all values of n for which this map is a Cayley map is therefore natural in the above context.

The paper is organized as follows. In Section 2 we present the necessary background on Cayley graphs and Cayley maps. Section 3 contains results for small orders n and provides an infinite family of values for which there is no triangular Cayley map of $K_{n,n,n}$. In Section 4, regular triangular Cayley maps with underlying graphs $K_{n,n,n}$ are characterized. In Section 5 we study conditions for a triangular Cayley map to extend (in a certain way) to a larger triangular Cayley map. All results are summarized in Section 6.

2 Preliminaries

Let Γ be a finite group and let X be a finite set of elements of Γ closed under taking inverses (i. e. $x \in X$ implies $x^{-1} \in X$) and such that $1 \notin X$. Then the *Cayley graph* $G = Cay(\Gamma, X)$ for Γ and X is a graph with vertex set $V(G) = \Gamma$ and arc set $D(G) = \{(g, gx) | g \in \Gamma, x \in X\}$. It is obvious that such a graph is vertex transitive and the degree of every vertex is $|X|$. If $h = gx$, then the arc (h, hx^{-1}) is the *reverse* arc to (g, gx), and the pair of the two arcs forms an undirected edge. Our Cayley graphs are therefore undirected. Moreover, they do not contain multiple edges, loops and semi-edges. Clearly, $Cay(\Gamma, X)$ is connected if and only if X generates Γ. Let us now state and prove a necessary and sufficient condition for $Cay(\Gamma, X)$ to be isomorphic to $K_{n,n,\ldots,n}$ (a complete $k-$partite graph with k parts of order n).

Proposition 1: *Let $k > 1$ be a natural number. A Cayley graph $Cay(\Gamma, X)$ is isomorphic to a complete k-partite graph $K_{n,n,\ldots,n}$ if and only if $\Gamma \setminus X$ is a subgroup of Γ of index k.*

Proof: Let $Cay(\Gamma, X) \cong K_{n,n,\ldots,n}$. Then Γ splits into k disjoint sets $X_i, 1 \le i \le k$, with $|X_i| = n$, such that there is no edge $\{x, y\}$ if the vertices x, y are in the same set X_i. Let $1 \in X_1$. Then there are $(k-1)n$ edges from the unit element 1 to the sets $X_2, \ldots, X_k$. Let $\bigcup_{i=2}^{k} X_i = X$, so that $X_1 = \Gamma \setminus X$. Furthermore, let $x, y \in X_1$ and $xy = z$. If $z \notin X_1$ then z is one of the vertices for which there exists the edge $\{x, z\}$, and this holds only if $y \in X$, a contradiction. Summing up we obtain the following: X_1 is a subset of Γ and for every $x, y \in X_1$ we have $xy \in X_1$. Therefore X_1 is a subgroup of Γ. Because $|X_1| = \ldots = |X_k| = n$, X_1 has index k in Γ.

For the converse, let $X_1 = \Gamma \setminus X$ be a subgroup of Γ of index k. Then we can decompose the group Γ into the right cosets $X_1 = 1X_1$, $X_2 = x_2X_1$ where $x_2 \notin X_1$,

$X_3 = x_3X_1, x_3 \notin X_1 \bigcup X_2, \ldots$ and $X_k = x_kX_1, x_k \notin \bigcup_{i=1}^{k-1} X_i$. It is obvious that in $Cay(\Gamma, X_1 \setminus \{1\})$ there are only edges of type $\{x, y\}$, where x, y are from the same coset X_i. It follows that its complement $Cay(\Gamma, X)$ has to be isomorphic to $K_{n,n,\ldots,n}$. □

Corollary 1: *A Cayley graph $Cay(\Gamma, X)$ is isomorphic to $K_{n,n,n}$ if and only if $\Gamma \setminus X$ is a subgroup of Γ of index 3.*

For the purpose of this article, a *surface* will be a connected, oriented, 2-dimensional manifold without boundary. A *map* M is a *cellular* embedding of a graph G in a surface $\mathcal{S}$, which mens that each component of the complement of the graph in the surface is homeomorphic to an open disc; the graph G is the *underlying graph* of M. Every map can be described by a *rotation* P acting on the set $D(G)$ of all arcs (edges with preassigned direction) of G defined as follows: for each vertex v the orientation of $\mathcal{S}$ induces a cyclic permutation P_v of the set $D(v)$ of arcs emanating from v. We call P_v the *local rotation* of the embedding at the vertex v. Then the product $P = \prod_{v \in V} P_v$ is a permutation of $D(G)$, called *rotation* of G. The map M is completely described by the pair $(D(G), P)$; we formally set $M = (D(G), P)$ or just $M = (G, P)$. Faces of M can be recovered with the help of the rotation P and the involution L that sends each arc of $D(G)$ onto its reverse. Then, oriented face boundaries of M are in a 1-1 correspodence with cycles of the permutation PL.

Let now $M = (G, P)$ be a map where the underlying graph G is a Cayley graph for a group Γ and a generating set X. Let us arrange the generating set to form a cyclic sequence $X = (x_0, \ldots, x_{d-1})$ where $d = |X|$. We say that M is a *Cayley map* for Γ and the cyclic sequence X, denoted by $CM(\Gamma, X)$ if the rotation P is given by $P(g, gx_i) = (g, gx_{i+1})$, where i is taken mod d. It is clear that different cyclic sequences of the generating set may yield different Cayley maps. In what follows, whenever we use the symbol X in the context of a Cayley map, we will automatically assume that X denotes a cyclic sequence of generators. The arc-reversing involution L is now given by $L(g, x_i) = (gx_i, x_i^{-1})$. Cayley maps, like Cayley graphs, are vertex transitive. To reconstruct faces of a Cayley map $CM(\Gamma, X)$ it is therefore sufficient to repeatedly apply the permutation PL to an arc of the form $(1, x_i)$. Observe that $PL(1, x_i) = P(x_i, x_i^{-1}) = (x_i, y)$ where y is the successor of x_i^{-1} in the cyclic generating sequence X. It follows that a sequence of arcs $(g_0, y_0), \ldots, (g_{k-1}, y_{k-1})$ forms the boundary of a face if and only if $g_i = g_{i-1}y_{i-1}$ and y_i is the immediate successor of y_{i-1}^{-1} in the cyclic generating sequence X.

Now we will present a necessary and sufficient condition for a Cayley map to be triangular, that is, to have all faces bounded by triangles.

Let $M = CM(\Gamma, X)$ be a Cayley map with the generating sequence $X = (x_0, x_1, \ldots, x_{d-1})$. We say that (x, y) is a *consecutive pair* of X if $x = x_i$ and $y = x_{i+1}$ for some $i \in [d] = \{0, 1, \ldots, d-1\}$, i being taken mod d.

Proposition 2: *Let $M = CM(\Gamma, X)$ be a Cayley map. A necessary and sufficient condition for M to be a triangulation is the following: If (x, y) is a consecutive pair of X then $(y^{-1}, y^{-1}x)$ and $(x^{-1}y, x^{-1})$ are consecutive pairs of X.*

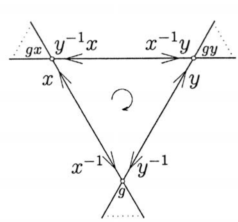

Figure 1: Part of a triangular Cayley map

Proof: Let M be a Cayley triangulation and let one of its triangular faces be as in Figure 1. Then the existence of the consecutive pair (x, y) in Figure 1 implies that the pairs $(y^{-1}, y^{-1}x)$ and $(x^{-1}y, x^{-1})$ are also consecutive.

Conversely, let M be a Cayley map, such that whenever (x, y) is a consecutive pair of X then $(y^{-1}, y^{-1}x)$ and $(x^{-1}y, x^{-1})$ are consecutive pairs of X. By the face reconstruction procedure it follows that the arcs $(g, y), (gy, y^{-1}x), (gx, x^{-1})$ form a triangular face. Since this applies to every consecutive pair (x, y), each face of the map is a triangle.□

Observe that the three consecutive pairs $(x, y), (y^{-1}, y^{-1}x)$ and $(x^{-1}y, x^{-1})$ of X in a Cayley triangulation are all distinct except when $x^3 = 1$ and $y = x^{-1}$ in which case they yield a single consecutive pair of the form (x, x^{-1}). Accordingly, faces of the map come in two varieties: Type 1 faces correspond to triples of consecutive pairs where $x^3 \neq 1$ or $y \neq x^{-1}$, and type 2 faces correspond to a single consecutive pair of the form (x, x^{-1}) where $x^3 = 1$. It is known [3] that each oriented triangulation of $K_{n,n,n}$ can be face 2-coloured, say, black and white. Therefore, in a Cayley triangulation $M = CM(\Gamma, X)$ of $K_{n,n,n}$ we obtain n black and n white triangular faces at every vertex of our map M. Let t_{b1} be the number of black triangles of type 1 and let t_{b2} be the number of black triangles of type 2 at a vertex. Similarly, we define t_{w1} and t_{w2} as the numbers of white triangles of type 1 and 2 at a vertex. We then have:

Corollary 2: $n = 3t_{b1} + t_{b2} = 3t_{w1} + t_{w2}$.

As a consequence of Corollary 2 we obtain the following result.

Corollary 3: *Let n be not divisible by 3, let Γ be a group of order $3n$ and let $M = CM(\Gamma, X)$ be a Cayley triangulation of $K_{n,n,n}$. Then the generating sequence X contains at least two consecutive pairs of the form (x, x^{-1}).*

3 Triangular Cayley maps of $K_{n,n,n}$: small orders and non-existence results

We begin with discussing Cayley triangulations of $K_{n,n,n}$ for small orders n. The spherical map of $K_{1,1,1} \cong K_3$ is clearly a triangular Cayley map for the group Z_3. The octahedron represents a triangular embedding of $K_{2,2,2}$ which is a Cayley map for the dihedral group $D_3 = \langle a, b | a^3 = b^2 = (ab)^2 = 1\rangle$ with cyclic generating sequence $X = (a, a^2, ab, a^2b)$. A triangular embedding of $K_{3,3,3}$ can be obtained from the unique (up to isomorphism) hexagonal embedding ψ of $K_{3,3}$ in a torus by inserting a vertex in the centre of each of the three hexagonal faces of ψ and joining each such vertex to the six vertices on the boundary of corresponding face [9] (see Section 4). This is again a Cayley triangulation of $K_{3,3,3}$, for the group $Z_3 \times Z_3 = \langle a, b | a^3 = b^3 = 1, ba = ab\rangle$ and the generating sequence $X = (a, a^2, ab, a^2b, ab^2, a^2b^2)$. The case $n = 4$ is slightly more complicated and here we need to recall metacyclic groups, which are semidirect products of two cyclic groups and have the standard presentation

$$Z_n \rtimes_k Z_m = \langle a, b | a^n = b^m = 1, ba = a^k b\rangle, \tag{1}$$

where $\gcd(n, k) = 1$, $1 < k < n$ and $k^m \equiv 1 \pmod n$. If there exists a Cayley triangulation $CM(\Gamma, X)$ of $K_{4,4,4}$, we have $|\Gamma| = 12, |X| = 8$ and using Corollary 2 we obtain $4 = 3t_{b1} + t_{b2}$ and $4 = 3t_{w1} + t_{w2}$. Therefore, the generating sequence X contains 2, 5, or 8 consecutive pairs of the form (x, x^{-1}). It is a matter of routine to check that the number of consecutive pairs of the form (x, x^{-1}) has to be equal to 2. Using Proposition 2 we obtain the following three possibilities for the distribution of elements of X:

i) $X = (a, a^{-1}, b, b^{-1}, c, d, e, c^{-1})$

ii) $X = (a, a^{-1}, c, b, b^{-1}, d, c^{-1}, e)$

iii) $X = (a, a^{-1}, c, d, b, b^{-1}, c^{-1}, e)$.

In all cases we need two elements of order two and six elements of order at least three in the sequence X. There are five pairwise non-isomorphic groups of order 12, and in what follows we exclude each of these (one at a time).

The cyclic group $Z_{12} = \langle a | a^{12} = 1\rangle$ can be excluded since there do not exist two generators of order two. If $\Gamma = Z_6 \times Z_2 = \langle a, b | a^6 = b^2 = 1, ba = ab\rangle$, the only subgroup of index 3 in Γ is $X_1 = \langle a^3, b\rangle$. Then the generating set for $CM(\Gamma, X)$ is $X = \Gamma \backslash X_1$ and there do not exist two generators of order two in X. In the case of the group $Z_6 \rtimes_5 Z_2 = \langle a, b | a^6 = b^2 = 1, ba = a^5 b\rangle$ it is easy to see that there do not exist six generators of order at least three. Similarly, in $Z_3 \rtimes_2 Z_4 = \langle a, b | a^3 = b^4 = 1, ba = a^2 b\rangle$ there do not exist two generators of order two. Finally, let $\Gamma = A_4 = \langle (12)(34), (123)\rangle$. Then $X_1 = \langle (12)(34), (13)(24)\rangle$ is the only subgroup of index 3 in Γ. Furthermore, $X = \Gamma \setminus X_1$ and there do not exist two generators of order two in X. We conclude that there is no Cayley triangulation of $K_{4,4,4}$.

If $n = 5$ there is no triangular Cayley map of $K_{5,5,5}$; this will follow from Theorem 1.

It can be checked that in the case $n = 6$ there is a unique (up to isomorphism) triangular Cayley map of $K_{6,6,6}$. It is a Cayley map for the group $\Gamma = S_3 \times Z_3 =$

$\langle a,b,c|a^3=b^2=c^3=1, ba=a^2b, ca=ac, cb=bc\rangle$ and for the generating sequence $X=(a, a^2c, abc^2, a^2bc, ac^2, a^2c^2, abc, a^2bc^2, ac, a^2, ab, a^2b)$. Later we will see that this generating sequence can be obtained by applying Lemma 1 to the generating sequence for triangular Cayley map of $K_{2,2,2}$ mentioned above.

In the remaining part of this section, on the basis of Proposition 2 we will identify an infinite family of values of n for which there does not exist a Cayley triangulation of $K_{n,n,n}$. Let n be such that $(3n, \varphi(3n))=1$, where φ is the Euler function (the number of positive integers less than n and coprime with n). We note that if $(3n,\varphi(3n))=1$, then n is odd and not divisible by 3. We show that in this case there exists no triangular Cayley map $M=CM(\Gamma,X)$ of order $3n$ with underlying graph $K_{n,n,n}$. In the proof we use the following result of Burnside [2]: If $(n,\varphi(n))=1$ then the cyclic group Z_n is the unique (up to isomorphism) group of order n.

Theorem 1: *Let $n>1$ be such that $(3n,\varphi(3n))=1$. Then there does not exist a Cayley triangulation of $K_{n,n,n}$.*

Proof: Assume to the contrary that there exists a Cayley triangulation $M=CM(\Gamma,X)$ whose underlying graph is isomorphic to $K_{n,n,n}$. By Burnside Theorem, $\Gamma\cong Z_{3n}$. The number of elements of the generating sequence X for M is then $|X|=2n$. From Corollary 3 it follows that the generating sequence X has to contain at least two consecutive pairs of the form (x,x^{-1}) and every such x has to have order 3. But the group Z_{3n} contains only two elements of order 3 (since 3 does not divide n), a contradiction. □

Note that the condition $(3n,\varphi(3n))=1$ is satisfied e. g. for prime numbers n such that $n\equiv -1 \pmod 6$. This gives:

Corollary 4: *There does not exist a Cayley triangulation of $K_{n,n,n}$, if n is a prime number of the form $n=6r-1$.*

If we restrict ourselves to Cayley maps on cyclic groups, Theorem 1 combined with Corollary 3 yields:

Corollary 5: *A triangular Cayley map $M=CM(Z_{3n},X)$ of $K_{n,n,n}$ can only exist if $n=1$ or if 3 divides n.*

4 Regular Cayley triangulations

In Section 1 we stated that for each n there exists a unique regular triangular map with underlying graph $K_{n,n,n}$ [5, 6]. We recall that regularity in this context means that the automorphism group of the map acts regularly on flags. A construction of the embedding is as follows [9]. We begin with an n-pole embedded in the 2-sphere. Then we lift this n-pole with the help of a voltage assignment in the group Z_n as indicated in Figure 2.

We obtain an embedding M'_n of the complete bipartite graph $K_{n,n}$ such that every face of the embedding is a hamiltonian cycle of length $2n$ (see Figure 3).

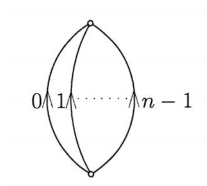

Figure 2: An n-pole and a voltage assignment in the group Z_n

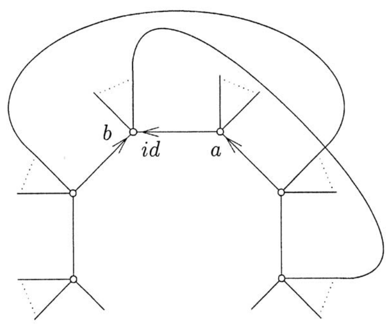

Figure 3: The embedding of $K_{n,n}$

It is not hard to check that the orientation-preserving automorphism group of the map is $Aut^+(M'_n) = \langle a, b | a^{2n} = b^n = (ab)^2 = 1, ba^2 = a^2 b\rangle$, where the generator a corresponds to a counterclockwise rotation of M'_n about the centre of a face f by the angle of π/n and the generator b corresponds to a counterclockwise rotation of M'_n about a vertex incident with the face f by the angle of $2\pi/n$. Equivalently, $Aut^+(M'_n) = \langle A, B, C | A^n = B^n = C^2 = 1, BA = AB, CA = AC, CB = A^{-1}B^{-1}C\rangle$, where $A = a^2$, $B = b$, $C = ab$, $a = ABC$. Therefore the orientation-preserving automorphism group of M'_n is isomorphic to $(Z_n \times Z_n) \rtimes Z_2$.

Now, insert a new vertex in the centre of each face of the embedding and join this vertex to every vertex on the boundary of the face as indicated in Figure 4. We obtain a map M_n which is a regular triangulation of $K_{n,n,n}$. The orientation-preserving automorphism group $Aut^+(M_n)$ of M_n can be presented in the form

$$\begin{aligned} Aut^+(M_n) = \langle A, B, C, D | A^n = B^n = C^2 = D^3 = 1, BA = AB, \\ CA = AC, DA = BD, CB = A^{-1}B^{-1}C, \\ DB = A^{-1}B^{-1}D, CD = B^{-1}D^{-1}C\rangle. \end{aligned} \tag{2}$$

Here the generator A corresponds to a counterclockwise rotation of M_n about the

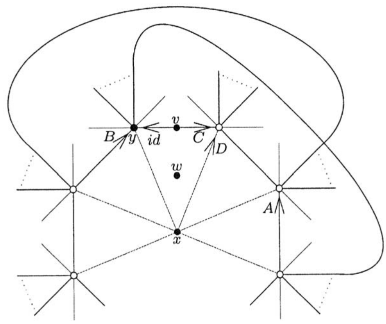

Figure 4: The embedding of $K_{n,n,n}$

vertex x by the angle $2\pi/n$, the generator B corresponds to a counterclockwise rotation of M_n about the vertex y by the angle $2\pi/n$, C corresponds to a rotation of M_n about the vertex v by the angle π, and the generator D corresponds a counterclockwise rotation of M_n about the vertex w by the angle $2\pi/3$. Therefore $Aut^+(M_n)$ is isomorphic to the group $((Z_n \times Z_n) \rtimes Z_3) \rtimes Z_2$. Equivalently, the group may be written in the form $Aut^+(M_n) \cong \langle C, D | C^2 = D^3 = (CD)^{2n} = ([C,D])^3 = id \rangle$, where $[C,D] = C^{-1}D^{-1}CD$ is the commutator of the elements C and D.

To determine for which n the map M_n is a Cayley map we will use a theorem which characterizes Cayley maps in terms of subgroups of the orientation-preserving automorphism group of a given map [7].

Theorem 2: *A map M is a Cayley map if and only if its orientation-preserving automorphism group contains a subgroup that is regular on vertices of M.*

Because the graph $K_{n,n,n}$ has $3n$ vertices, we want to look for subgroups of $Aut^+(M_n)$ of order $3n$ acting regularly on vertices of M_n. Any such subgroup Θ has to have the following two properties:
I) Θ is not a subgroup of $\langle A, B, C \rangle$ (that is, Θ contains an element of the form $A^iB^jD^lC^m$, $l \neq 0$) and
II) Θ must not contain any element of the form $(CD)^l, B^l = (CD)^{2l}$ and A^l, $l \neq 0$.
The requirement I) follows from the fact that the subgroup $\langle A, B, C \rangle$ is not transitive on vertices of M_n. As regards II), if x is an arc of M_n then x and $x(CD)^l$ are incident with the same vertex of M_n, violating the regularity of the action of Θ. The same applies to B^l. Excluding A^l from Θ is based on the following four identities, valid for each $l \in [n] = \{0, 1, \ldots, n-1\}$:
1) $A^iB^jDC \cdot B^l = A^iB^jDCBB^{l-1} = A^iB^jDA^{-1}B^{-1}CB^{l-1} = \ldots$
$= A^iB^jDA^{-l}B^{-l}C = A^iB^jB^{-l}DB^{-l}C = A^iB^jB^{-l}A^{-(-l)}B^{-(-l)}DC$
$= A^iB^jA^lDC = A^l \cdot A^iB^jDC$. Similarly,

2) $A^l \cdot A^iB^jD^2C \cdot A^l = A^iB^jD^2C \cdot B^l$,
3) $A^l \cdot A^iB^jD \cdot A^l = A^iB^jD \cdot B^{-l}$,
4) $A^l \cdot A^iB^jD^2 = A^iB^jD^2 \cdot B^l$.
Therefore, if a subgroup Θ contains an element A^l, $l \neq 0$, then Θ has a non-trivial vertex stabilizer.

Furthermore, if Θ is a subgroup of $Aut^+(M_n)$ with the required properties, elements of Θ have to be from different cosets of $Aut^+(M_n)\backslash\langle CD\rangle$. If this is the case, Θ can be generated by an element from $D\langle B^{-1}D^{-1}C\rangle = \{A^iB^iD, A^iB^iC\}$ and by another element from the set $D^2\langle B^{-1}D^{-1}C\rangle = \{A^iD^2, A^iDC\}$. We set
$X_1 = \langle A^iB^iD\rangle = \{1, A^iB^iD, B^iD^2\}$,
$X_2 = \langle A^iB^iC\rangle = \{1, A^iB^iC, A^i, A^{2i}B^iC, A^{2i}, \ldots\}$,
$Y_1 = \langle A^jD^2\rangle = \{1, A^jD^2, B^{-j}D\}$ and
$Y_2 = \langle A^jDC\rangle = \{1, A^jDC, A^{j+1}B^{j+1}, A^{2j+1}B^{j+1}DC, A^{2j+2}B^{2j+2}, \ldots\}$. It follows that we must have $\Theta = \langle x, y\rangle$ for some x from X_1 or X_2 and for some y from Y_1 or Y_2. We are now in position to prove the following result.

Theorem 3: *The map M_n is a Cayley map if and only if $n = 1, 2, 3$ or if n is odd and there is a k not dividing n such that $k^2 + k + 1 \equiv 0 \pmod{n}$.*

Proof: By the analysis preceding the statement of the theorem, it is sufficient to investigate the existence of a subgroup Θ of order $3n$ of the group $Aut^+(M_n)$ with the presentation (2) such that $\Theta = \langle x, y\rangle$ where $x \in X_1 \bigcup X_2$ and $y \in Y_1 \bigcup Y_2$. We will show that such a subgroup exists if and only if $n \leq 3$ or if n is odd and there is a k such that $(k, n) = 1$ and $k^2 + k + 1 \equiv 0 \pmod{n}$.

We divide the argument into two main cases. In the first case we will have $x \in X_1$ and $y \in Y_1$ and in the second case $x \notin X_1$ or $y \notin Y_1$.

Let $x \in X_1$ and $y \in Y_1$. Then, $A^iB^iD \cdot A^jD^2 = A^iB^{i+j}$ and $A^jD^2 \cdot A^iB^iD = A^jB^{-i}$. The element A^jB^{-i} has to be from the group $\langle A^iB^{i+j}\rangle$; otherwise there exists an element $A^l \neq 1$ or $B^l \neq 1$ in the group Θ. Because A and B commute, $(A^iB^{i+j})^j = (A^jB^{-i})^i$. It follows that $A^{ij}B^{ij+j^2} = A^{ij}B^{-i^2}$ and hence $i^2 + ij + j^2 \equiv 0 \pmod{n}$. We obtain a group Θ with elements:
$1, A^iB^{i+j}, A^{2i}B^{2i+2j}, \ldots, A^iB^iD, A^{2i}B^{2i+j}D, A^{3i}B^{3i+2j}D, \ldots,$
$A^jD^2, A^{i+j}B^{i+j}D^2, A^{2i+j}B^{2i+2j}D^2, \ldots$. This group has order $3n$ if and only if i and j are coprime to n. If n is even, then i, j have to be odd, but then $i^2 + ij + j^2 \not\equiv 0 \pmod{n}$. Furthermore, because $(i, n) = 1$ and $(j, n) = 1$, there exists a number k such that $(k, n) = 1$ and $j \equiv ik \pmod{n}$. Therefore for such a k we have $k^2 + k + 1 \equiv 0 \pmod{n}$.

Now, let $x \notin X_1$ or $y \notin Y_1$. We present a detailed proof for the case $x \in X_1, y \in Y_2$. Proofs for the cases $x \in X_2, y \in Y_1$ and $x \in X_2, y \in Y_2$ are similar and in both these cases we obtain as a result a group isomorphic to S_3. So, let $\Theta = \langle x, y\rangle$ and let $x \in X_1, y \in Y_2$. If the element y is from the set Y_2 and $j \neq -1$, the set Y_2 contains an element $A^{j+1}B^{j+1}$. Because $A^lB^l \cdot A^iB^iD = A^iB^iD \cdot B^{-l}$ for all $l \in [n]$, the number j has to be equal to -1. This gives the sets $X_1 = \langle A^iB^iD\rangle$

and $Y_2 = \langle A^{-1}DC \rangle$. Moreover it is easy to show that $(A^{-1}DC \cdot A^iB^iD)^2 = A^{i-1}$. Therefore, if $x \in X_1$, the number i has to be equal to 1. We thus obtain the group $\Theta = \langle ABD, A^{-1}DC \rangle = \{1, ABD, A^{-1}DC, AD^2C, AB^2C, BD^2\}$ of order $6 = 3 \cdot 2$. Hence $n = 2$ and we have $\Theta = \langle ABD, ADC \rangle = \{1, ABD, ADC, AD^2C, AC, BD^2\}$. This group is regular on vertices of M_2 so M_2 is a Cayley map (in fact, an octahedron). $\square$

5 Extendability of triangular Cayley maps

In this Section we will give constructions of infinite families of triangular Cayley maps with underlying graphs $K_{n,n,n}$.

Let $M = CM(\Gamma, X)$ be a Cayley triangulation with the generating sequence $X = (x_0, x_1, \ldots, x_{2n-1})$. We say that $CM(\Gamma, X)$ is *extendable* if the generating sequence X contains a consecutive pair (x, y) such that $y = x^{-1} = x^2$ and $CM(\Gamma, X)$ is r*-extendable* if it contains at least $r \geq 2$ consecutive pairs. Equivalently, $CM(\Gamma, X)$ is extendable if there exists a face of type 2 at each vertex of the map, and it is r-extendable if there are at least r faces of type 2 at each vertex of the map. Consecutive pairs (x, y) such that $y = x^{-1} = x^2$ will be called *consecutive pairs of insertion.*

Now, let $M = CM(\Gamma, X)$ be an extendable Cayley triangulation and let (x_0, x_1) be a consecutive pair of insertion (recall that X is a cyclic sequence). Next, let $\Gamma' = \Gamma \times Z_3$ and let X' consist of all elements of the form (x, i), where x is in X and $i \in \{0, 1, 2\}$. We say that a sequence X' is an *insertion extension* of X if:

$$X' = ((x_0,0),(x_1,1),(x_2,2),(x_3,1),\ldots,(x_{2n-1},1),$$
$$(x_0,2),(x_1,2),(x_2,1),(x_3,2),\ldots,(x_{2n-1},2),$$
$$(x_0,1),(x_1,0),(x_2,0),(x_3,0),\ldots,(x_{2n-1},0)).$$

Lemma 1: *Let $M = CM(\Gamma, X)$ be an extendable Cayley triangulation. Further, let $M' = CM(\Gamma', X')$ be a Cayley map with $\Gamma' = \Gamma \times Z_3$ and with the generating sequence X' which is an insertion extension of X. Then M' is a Cayley triangulation.*

Proof: There are two kinds of consecutive pairs in X':
1) $((x_0, j), (x_1, 1-j)),\ j \in \{0, 1, 2\}$ and
2) $((x_i, j), (x_{i+1}, -j)),\ i \in \{1, 2, \ldots 2n-1\},\ j \in \{0, 1, 2\}$. We need to show that in the generating sequence X', the following pairs are consecutive:
i) $((x_1, 1-j)^{-1}, (x_1, 1-j)^{-1} \cdot (x_0, j)),\ j \in \{0, 1, 2\}$
ii) $((x_0, j)^{-1} \cdot (x_1, 1-j), (x_0, j)^{-1}),\ j \in \{0, 1, 2\}$
iii) $((x_{i+1}, -j)^{-1}, (x_{i+1}, -j)^{-1} \cdot (x_i, j)),\ i \in \{1, 2, \ldots 2n-1\},\ j \in \{0, 1, 2\}$ and
iv) $((x_i, j)^{-1} \cdot (x_{i+1}, -j), (x_i, j)^{-1}),\ i \in \{1, 2, \ldots 2n-1\},\ j \in \{0, 1, 2\}$.
We prove the case i) and iii); proofs for cases ii) and iv) are similar. The calculations are as follows:
Case i): $((x_1, 1-j)^{-1}, (x_1, 1-j)^{-1}(x_0, j)) = ((x_1^{-1}, j-1), (x_1^{-1}, j-1)(x_0, j)) = ((x_0, j-1), (x_0, j-1)(x_0, j)) = ((x_0, j-1), (x_0^2, j-1+j)) = ((x_0, j-1), (x_1, 2j-1)) = ((x_0, j-1), (x_1, 1-(j-1)))$; this later pair is easily seen to be in X'.
Case iii): $((x_{i+1}, -j)^{-1}, (x_{i+1}, -j)^{-1}(x_i, j)) = ((x_{i+1}^{-1}, j), (x_{i+1}^{-1}, j)(x_i, j)) =$

$((x_{i+1}^{-1}, j), (x_{i+1}^{-1}x_i, 2j)) = ((x_{i+1}^{-1}, j), (x_{i+1}^{-1}x_i, -j))$.
Because M is a Cayley triangulation and there is a consecutive pair (x_i, x_{i+1}) in X, the consecutive pair $(x_{i+1}^{-1}, x_{i+1}^{-1}x_i)$ is in X and therefore $((x_{i+1}^{-1}, j), (x_{i+1}^{-1}x_i, -j))$ is a consecutive pair in X'. □

Lemma 2: *Let $M = CM(\Gamma, X)$ be a 2-extendable Cayley triangulation. Further, let $M' = CM(\Gamma', X')$ be a Cayley map with $\Gamma' = \Gamma \times Z_3$ and with the generating sequence X' which is an insertion extension of X. Then M' is 2-extendable.*

Proof: Let $X = (x, x^{-1}, \ldots, y, y^{-1}, \ldots)$. Then X' (an insertion extension of X) has the following form:

$$X' = ((x,0),(x^{-1},1),\ldots,(y,2),(y^{-1},1),\ldots,\\ (x,2),(x^{-1},2),\ldots,(y,1),(y^{-1},2),\ldots,\\ (x,1),(x^{-1},0),\ldots,(y,0),(y^{-1},0),\ldots) \text{ or}$$

$$X' = ((x,0),(x^{-1},1),\ldots,(y,1),(y^{-1},2),\ldots,\\ (x,2),(x^{-1},2),\ldots,(y,2),(y^{-1},1),\ldots,\\ (x,1),(x^{-1},0),\ldots,(y,0),(y^{-1},0),\ldots).$$

We see that in both cases there are at least three consecutive pairs of insertion, namely $((y,0),(y^{-1},0)), ((y,1),(y^{-1},2))$ and $((y,2),(y^{-1},1))$. Therefore M' is 2-extendable. □

In the previous part of this section we presented a way to extend a Cayley triangulation M to a Cayley triangulation M' if M had some additional properties. Now we show that there are Cayley triangulations of $K_{n,n,n}$ with such properties.

Theorem 4: *Let $\Gamma = Z_n \rtimes_k Z_3 = \langle a^n = b^3 = 1 | ba = a^k b \rangle$, where $k^3 \equiv 1 \pmod n$ and $k^2 + k + 1 \equiv 0 \pmod n$. Furthermore, let the generating sequence have the form*
$X = (a^i b, (a^{ik^2} b)^{-1}, a^{i+1} b, (a^{(i+1)k^2} b)^{-1}, a^{i+2} b, (a^{(i+2)k^2} b)^{-1}, \ldots)$. Then,
$CM(\Gamma, X)$ is a 2-extendable Cayley triangulation of $K_{n,n,n}$.

Proof: The proof is divided into two parts. In the first part we show that M is a Cayley triangulation of $K_{n,n,n}$ and in the second part we prove that the map is 2-extendable.

We begin by showing, that i) M is a Cayley triangulation and that ii) the underlying graph of M is $K_{n,n,n}$.
i) We see, that if $(x, y) = (a^i b, (a^{ik^2} b)^{-1}), i \in [n]$, is a consecutive pair of X then $(y^{-1}, y^{-1}x) = (a^{ik^2}, (a^{-i(k^2+1)} b)^{-1}) = (a^{ik^2}, (a^{ik^2 k^2} b)^{-1})$ and $(x^{-1}y, x^{-1}) = (a^{-i(k^2+1)} b, (a^i b)^{-1}) = (a^{-i(k^2+1)} b, (a^{-i(k^2+1)k^2} b)^{-1})$ are consecutive pairs of X. Similarly, if $(x, y) = ((a^{ik^2} b)^{-1}, a^{i+1} b), i \in [n]$, is a consecutive pair of X, then $(y^{-1}, y^{-1}x)$ and $(x^{-1}y, x^{-1})$ are consecutive pairs of X. Therefore, by Proposition 2, M is a triangular Cayley map.
ii) It is easy to see that $\Lambda = \Gamma \setminus X$ is a subgroup of index three of the group Γ. By Corollary 1, the underlying graph of M is $K_{n,n,n}$. Hence M is a triangular Cayley map with underlying graph $K_{n,n,n}$.

Now we prove that there are at least two consecutive pairs of insertion in the generating sequence X.
i) Let $i=0$. We see that (b,b^{-1}) is a consecutive pair of insertion.
ii) Let $(k-1)/n$ and let $i=\frac{n}{k-1}$. Then $(a^ib)(a^{ik^2}b)^{-1}=a^ibb^{-1}a^{-ik^2}=a^{i(1-k^2)}=a^{\frac{n}{k-1}(1-k^2)}=a^{-n(1+k)}=1$ and therefore $x_{i+1}=(x_i)^{-1}$ for $i=\frac{n}{k-1}$. It follows that $(a^{\frac{n}{k-1}},(a^{\frac{n}{k-1}k^2})^{-1})$ is another consecutive pair of insertion X.
iii) Let n be not divisible by $(k-1)$. Then the equation $i(k-1)+1\equiv 0 \pmod n$ has a solution for some $i\in\{1,2,\dots n-1\}$. Let i be a solution of the equation $i(k-1)+1\equiv 0 \pmod n$. Then $(a^ib)^{-1}(a^{ik+1}b)=b^{-1}a^{-i}a^{ik+1}b=b^{-1}a^{i(k-1)+1}b=b^{-1}b=1$ and therefore $x_{i+1}=(x_i)^{-1}$ for i a solution of the equation $i(k-1)+1\equiv 0 \pmod n$. □

6 Conclusion

The main results of this paper are now obtained by summing up the facts obtained in Sections 2, 3, 4 and 5.

Theorem 5: *A triangular Cayley map with underlying graph $K_{n,n,n}$ exists in the following cases:*

i) $n=1,2,3,6$ (Section 3),

ii) $n=3^rm$ where $r\geq 0$ and $m>1$ is an odd number such that $m|(k^2+k+1)$ for some k, $1<k<m$ (Theorem 4).

Further, a Cayley triangulatin of $K_{n,n,n}$ is regular if and only if $n=1,2,3$ or if n is an odd number such that there exists a number k with properties $(n,k)=1$ and $k^2+k+1\equiv 0 \pmod n$ (Theorem 3).

On the other hand, there is no triangular Cayley map with underlying graph $K_{n,n,n}$ if n is an odd number such that $(3n,\varphi(3n))=1$ (Theorem 1).

We conclude the paper with a remark. Let p be a prime number of the form $p=6r+1$. By Fermat's little theorem, the congruence $a^{6r}\equiv 1 \pmod p$ holds for all integers a which are not multiples of p. In particular, $(2^{2r})^3\equiv 1 \pmod p$ and it is easy to see that $2^{2r}\not\equiv 1 \pmod p$. Let k be the number between 1 and p which satisfies the congruence $k\equiv 2^{2r} \pmod p$. Then we have $k^3\equiv 1 \pmod p$. Because $k^3-1=(k-1)(k^2+k+1)$ and $(k-1)$ is not a divisor of p, we have $k^2+k+1\equiv 0 \pmod p$ and we obtain the following result.

Corollary 6: *Let p be a prime number of the form $p=6r+1$. Then there exists a regular triangular Cayley map of $K_{p,p,p}$.*

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