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On $\mathbb{Z}_4$ codes satisfying the chain condition

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Abstract

The concept of the chain condition was introduced by Wei and Yang (*IEEE Trans. Inform. Theory*, 39(5) (1991), 1709–1713) for binary linear codes and later studied by many others for linear codes over finite fields (see Encheva and Cohen, *IEICE T. Fund. Electr. E80A:* 11 (1997), 2256–2259, and Encheva, *IEEE Trans. Inform. Theory* 42(3) (1996), 1038–1047, and the references therein). The chain condition is important in applications as more can be said about the trellis description of codes that satisfy the chain condition and the trellis description leads to simple soft-decision decoding algorithm for the code. Thus the results on chain condition are of interest in implementing the respective code. In a paper (to appear in Des. Codes and Cryptogr.) by Gupta, Bhandari and Lal, the concept of chain condition was investigated for codes over the ring of integers modulo p^s and it was shown that $\mathbb{Z}_4$-simplex codes of both type (α and β) and the quaternary Reed-Muller code $ZRM(1,m)$ satisfy the chain condition. In this note it is shown that various known self-dual codes over $\mathbb{Z}_4$ satisfy the chain condition. In particular we have shown that all self-dual codes of length up to 9, Klemm codes and lifted Golay code QR_{24} satisfy the chain condition. In this process we determine the complete weight hierarchy of several codes over $\mathbb{Z}_4$.

1 Introduction

The concept of chain condition for codes over finite fields, especially binary and ternary fields including their relationship to trellis description, soft-decision decoding

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and efficient coordinate ordering has been studied very well (see [7, 9] etc.). Recently, codes over rings have increased in importance, generating much interest, for example see [4, 6] etc. The concept of chain condition for linear codes over $GF(q)$ was also found useful in expressing weight hierarchies of a product code in terms of the weight hierarchies of its component codes. In [4] this concept has been investigated for codes over $\mathbb{Z}_{p^s}$. In this note, various classes of self-dual codes over $\mathbb{Z}_4$ that satisfy the chain condition have been identified. As it is difficult, in general, to show that a code over $\mathbb{Z}_4$ satisfy the chain condition (see, for example conjecture 1) we are unable to consider more general classes of the codes.

A *linear code* $\mathcal{C}$, of length n, over $\mathbb{Z}_4$ is an additive subgroup of $\mathbb{Z}_4^n$. An element of $\mathcal{C}$ is called a *codeword of* $\mathcal{C}$ and a *generator matrix* of $\mathcal{C}$ is a matrix whose rows generate $\mathcal{C}$. The *Hamming weight* $w_H(x)$ of a vector x in $\mathbb{Z}_4^n$ is the number of non-zero components. The *Lee weight* $w_L(x)$ of a vector $x = (x_1, x_2, \ldots, x_n)$ is $\sum_{i=1}^n \min\{|x_i|, |4-x_i|\}$. The *Euclidean weight* $w_E(x)$ of a vector x is $\sum_{i=1}^n \min\{x_i^2, (4-x_i)^2\}$. The Euclidean weight is useful in connection with lattice constructions. The Hamming, Lee and Euclidean distances $d_H(x,y)$, $d_L(x,y)$ and $d_E(x,y)$ between two vectors x and y are $w_H(x-y)$, $w_L(x-y)$ and $w_E(x-y)$, respectively. The minimum Hamming, Lee and Euclidean weights, d_H, d_L and d_E, of $\mathcal{C}$ are the smallest Hamming, Lee and Euclidean weights among all non-zero codewords of $\mathcal{C}$, respectively.

A linear code $\mathcal{C}$ over $\mathbb{Z}_4$, is said to be a code of *type* $\alpha(\beta)$ if $d_H = \lceil \frac{d_L}{2} \rceil$ $(d_H > \lceil \frac{d_L}{2} \rceil)$[5]. The *Gray map* ϕ : $\mathbb{Z}_4^n \to \mathbb{Z}_2^{2n}$ is the coordinate-wise extension of the function from $\mathbb{Z}_4$ to $\mathbb{Z}_2^2$ defined by $0 \to (0,0), 1 \to (0,1), 2 \to (1,1)$ and $3 \to (1,0)$. Thus $\phi(\mathcal{C})$, the image of a linear code $\mathcal{C}$ over $\mathbb{Z}_4$ of length n by the Gray map is a binary code of length $2n$.

The *dual code* $\mathcal{C}^\perp$ of $\mathcal{C}$ is defined as $\{x \in \mathbb{Z}_6^n \mid x \cdot y = 0$ for all $y \in \mathcal{C}\}$ where $x \cdot y$ is the standard inner product of x and y. $\mathcal{C}$ is *self-orthogonal* if $\mathcal{C} \subseteq \mathcal{C}^\perp$ and $\mathcal{C}$ is *self-dual* if $\mathcal{C} = \mathcal{C}^\perp$.

Two codes are said to be *equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates. Codes differing by only a permutation of coordinates are called *permutation-equivalent.*

In this paper we investigate the concept of chain condition for various type α and β codes over $\mathbb{Z}_4$. Section 2 contains some preliminaries and notations. Main results are given in Section 3. Section 4 concludes with an important conjecture.

2 Preliminaries and Notations

Any linear code $\mathcal{C}$ over $\mathbb{Z}_4$ is permutation-equivalent to a code with generator matrix G of the form

$$G = \begin{bmatrix} I_{k_0} & A & B_1 + 2B_2 \\ 0 & 2I_{k_1} & 2C \end{bmatrix} \tag{1}$$

where A, B_1, B_2 and C are matrices with entries 0 or 1 and I_k is the identity matrix of order k. One can associate two binary linear codes with $\mathcal{C}$ viz. the *residue code*

$$\mathcal{C}^{(1)} = \{\mathbf{c} \pmod 2 \mid \mathbf{c} \in \mathcal{C}\}$$

and the *torsion code*

$$\mathcal{C}^{(2)} = \{\mathbf{c} \in \mathbb{Z}_2^n \mid 2\mathbf{c} \in \mathcal{C}\}.$$

A vector $\mathbf{v}$ is a *2-linear combination* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if $\mathbf{v} = \lambda_1\mathbf{v}_1 + \ldots + \lambda_k\mathbf{v}_k$ with $\lambda_i \in \mathbb{Z}_2$ for $1 \le i \le k$. A subset $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ of $\mathcal{C}$ is called a *2-basis* for $\mathcal{C}$ if for each $i = 1, 2, ..., k-1$, $2\mathbf{v}_i$ is a 2−linear combination of $\mathbf{v}_{i+1}, ..., \mathbf{v}_k$, $2\mathbf{v}_k = 0$, $\mathcal{C}$ is the 2-linear span of S and S is 2-linearly independent [5]. The number of elements in a 2-basis for $\mathcal{C}$ is called the *2-dimension* of $\mathcal{C}$. It is easy to verify that the rows of the matrix

$$\mathcal{B} = \begin{bmatrix} I_{k_0} & A & B_1 + 2B_2 \\ 2I_{k_0} & 2A & 2B_1 \\ 0 & 2I_{k_1} & 2C \end{bmatrix} \tag{2}$$

form a 2-basis for the code $\mathcal{C}$ generated by G given in (1).

A linear code $\mathcal{C}$ over $\mathbb{Z}_4$ (over $\mathbb{Z}_2$) of length n, 2-dimension k, minimum Hamming distance d_H and minimum Lee distance d_L is called an $[n, k, d_H, d_L]$ ($[n, k, d_H]$) or simply an $[n, k]$ code. For $1 \le r \le k$, *the r-th Generalized Hamming weight* of $\mathcal{C}$ is defined by

$$d_r(\mathcal{C}) = \min\{w_S(D_r) \mid D_r \text{ is an } [n, r] \text{ subcode of } \mathcal{C}\},$$

where $w_S(D)$, called *support size* of D, is the number of coordinates in which some codeword of D has a nonzero entry. The set $\{d_1(\mathcal{C}), d_2(\mathcal{C}), \ldots, d_k(\mathcal{C})\}$ is called the *weight hierarchy* of $\mathcal{C}$. $\mathcal{C}$ is said to satisfy the *chain condition* if there exists a chain

$$D_1 \subseteq D_2 \subseteq \cdots \subseteq D_k,$$

of subcodes of $\mathcal{C}$ satisfying $w_S(D_r) = d_r(\mathcal{C})$, $1 \le r \le k$.

A relation between $d_r(\mathcal{C})$ and $d_r(\mathcal{C}^\perp)$ is given by the following theorem.

Theorem 1 ([1]) *Let $\mathcal{C}$ be an $[n, k]$ linear code over $\mathbb{Z}_4$. Then*

$$\{d_r(\mathcal{C}) : 1 \le r \le k\} = \{1, 1, 2, 2, \ldots, n, n\} \backslash \{n + 1 - d_r(\mathcal{C}^\perp) : 1 \le r \le 2n - k\}.$$

3 Self-Dual and Self-Orthogonal Codes

Self-dual and self-orthogonal codes over $\mathbb{Z}_4$ were recently studied by several researchers like Bonnecaze, Conway, Harada, Pless, Quian, Rains and Sloane etc. (see [2, 10, 11] etc.). They have been classified by Conway and Sloane up to lengths 9 in [2]. At length $n = 1$ the smallest self-dual code $\mathcal{A}_1 = \{0, 2\}$ trivially satisfies the chain condition. Also for each length n we have the trivial self-dual codes over $\mathbb{Z}_4$ with generator matrix $G = [2I_n]$. These codes satisfy the chain condition as they are direct sum of n copies of $\mathcal{A}_1$. Any code having $\mathcal{A}_1$ as a direct summand is called *trivially decomposable*. Let $\mathcal{C}$ be a linear code over $\mathbb{Z}_4$. A codeword $\mathbf{c} \in \mathcal{C}$ is said to be a *tetrad* if it has exactly four coordinates congruent to 1 *or* 3 $\pmod 4$ and the rest congruent to 0 $\pmod 4$.

Let $m \geq 2$ be a positive integer. Let $\mathcal{D}_{2m}$ be the $[2m, 2m-2, 4, 4]$ type β code generated by the $(m-1) \times 2m$ matrix

$$(3) \qquad \begin{bmatrix} 1 & 1 & 1 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 3 \end{bmatrix},$$

and let $\mathcal{D}^0_{2m}$ be the $[2m, 2m-1]$ code generated by $\mathcal{D}_{2m}$ and the tetrad $1300\ldots0011$. Equivalently $\mathcal{D}^0_{2m}$ is generated by the matrix (see [2])

$$(4) \qquad \begin{bmatrix} 1 & 1 & 1 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 3 \\ 2 & 0 & 2 & 0 & 2 & 0 & \cdots & 0 & 2 & 0 & 2 & 0 \end{bmatrix}.$$

Let $\mathcal{D}^+_{2m}$ be the $[2m, 2m-1]$ code generated by $\mathcal{D}_{2m}$ and $00\ldots0022$ and let $\mathcal{D}^\oplus_{2m}$ be the code generated by $\mathcal{D}^0_{2m}$ and $\mathcal{D}^+_{2m}$. Note that $\mathcal{D}^\oplus_{2m}$ is a $[2m, 2m]$ self-dual code([2]).

Let $\mathcal{E}_7$ be the $[7, 6, 4, 4]$ type β code generated by the matrix

$$(5) \qquad \begin{bmatrix} 1 & 0 & 0 & 3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 3 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 3 \end{bmatrix}$$

and let $\mathcal{E}^+_7$ be the $[7, 7, 4, 4]$ type β code generated by $\mathcal{E}_7$ and 2222222. It was observed in [2] that $\mathcal{E}^+_7$ is a self-dual code and the reduction code of both $\mathcal{E}_7$ and $\mathcal{E}^+_7$ is the Hamming code of length 7. They also show that the code $\mathcal{E}_8$ generated by the matrix

$$(6) \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 3 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 3 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 & 1 & 3 & 0 \end{bmatrix}$$

is a $[8, 8, 4, 4]$ self dual code. Note that it is a type β code.

In [2] Conway and Sloane have shown that any self-orthogonal code over $\mathbb{Z}_4$ generated by *'tetrads'* is equivalent to a direct sum of codes $\mathcal{D}_{2m}, \mathcal{D}^0_{2m}, \mathcal{D}^+_{2m}, \mathcal{D}^\oplus_{2m} (m = 1, 2, \ldots), \mathcal{E}_7, \mathcal{E}^+_7, \mathcal{E}_8$. The following Theorem shows that $\mathcal{D}_{2m}, \mathcal{E}_7, \mathcal{E}^+_7$ and $\mathcal{E}_8$ satisfy the chain condition.

Theorem 2 *$\mathcal{D}_{2m}, \mathcal{E}_7, \mathcal{E}_7^+$ and $\mathcal{E}_8$ satisfy the chain conditions.*

Proof. It is easy to verify that the weight hierarchy of $\mathcal{E}_7$ is $\{4, 4, 6, 6, 7, 7\}$. Consider the codewords $\mathbf{x}_1 = (1101003), \mathbf{x}_2 = (2111030)$ and $\mathbf{x}_3 = (3023132)$ of $\mathcal{E}_7$. Let $D_1 =< 2\mathbf{x}_1 >$, $D_2 =< \mathbf{x}_1, 2\mathbf{x}_1 >$, $D_3 =< \mathbf{x}_1, 2\mathbf{x}_1, 2\mathbf{x}_2 >$, $D_4 =< \mathbf{x}_1, 2\mathbf{x}_1, \mathbf{x}_2, 2\mathbf{x}_2 >$, $D_5 =< \mathbf{x}_1, 2\mathbf{x}_1, \mathbf{x}_2, 2\mathbf{x}_2, 2\mathbf{x}_3 >$, and $D_6 =< \mathbf{x}_1, 2\mathbf{x}_1, \mathbf{x}_2, 2\mathbf{x}_2, \mathbf{x}_3, 2\mathbf{x}_3 >$. It is easy to verify that $D_1 \subseteq D_2 \ldots \subseteq D_6$ is the required chain of subcodes.

Therefore for $\mathcal{E}_7^+$ the required chain of subcodes can be taken as $D_1 \subseteq D_2 \ldots \subseteq D_6 \subseteq D_7$ where D_i for $1 \leq i \leq 6$ are the subcodes defined for $\mathcal{E}_7$ and $D_7 =< \mathbf{x}_1, 2\mathbf{x}_1, \mathbf{x}_2, 2\mathbf{x}_2, \mathbf{x}_3, 2\mathbf{x}_3, 2222222 >$. Hence $\mathcal{E}_7^+$ satisfies the chain condition and its weight hierarchy is $\{4, 4, 6, 6, 7, 7, 8\}$. Clearly, the weight hierarchy of $\mathcal{E}_8$ is $\{4, 4, 6, 6, 7, 7, 8, 8\}$. If $R_i (1 \leq i \leq 4)$ denote the first four rows of the matrix given in (6) and if $D_1 =< 2R_4 >$, $D_2 =< R_4, 2R_4 >$, $D_3 =< 2R_3, R_4, 2R_4 >$, $D_4 =< R_3, 2R_3, R_4, 2R_4 >, \ldots$, $D_8 =< R_1, 2R_1, \ldots, R_3, 2R_3, R_4, 2R_4 >$ then $D_1 \subseteq D_2 \ldots \subseteq D_8$ and $w_s(D_r) = d_r(\mathcal{E}_8)$.

It is easy to see that the code $\mathcal{D}_{2m}$ has weight hierarchy $\{4, 4, 6, 6, 8, 8, \ldots, 2m-2, 2m-2, 2m, 2m\}$. Let $R_1, \ldots R_{m-1}$ be the first $m-1$ rows of the matrix given in (3) and let $D_1 =< 2R_1 >$, $D_2 =< R_1, 2R_1 >, \ldots$, $D_{2m-3} =< R_1, 2R_1, \ldots, R_{m-2}, 2R_{m-2}, 2R_{m-1} >$, $D_{2m-2} =< R_1, 2R_1, \ldots, R_{m-2}, 2R_{m-2}, R_{m-1}, 2R_{m-1} >$. Then $D_1 \subseteq D_2 \ldots \subseteq D_{2m-2}$ and $w_s(D_r) = d_r(\mathcal{D}_{2m}), 1 \leq r \leq 2m-2$. □

At length $n = 4$ there is a type α code $\mathcal{D}_4^{\oplus} : [4, 4, 2, 4]$ generated by the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$

and at length $n = 6$ we have $\mathcal{D}_6^{\oplus} : [6, 6, 2, 4]$ type α code generated by the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & 0 & 0 \end{bmatrix}.$$

Finally at length $n = 8$ we have $\mathcal{D}_8^{\oplus} : [8, 8, 2, 4]$ type α code generated by the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 2 & 2 & 1 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & 2 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem 3 *$\mathcal{D}_4^{\oplus}, \mathcal{D}_6^{\oplus}, \mathcal{D}_4^{\oplus} \oplus \mathcal{D}_4^{\oplus}, \mathcal{D}_8^{\oplus}$ satisfy the chain condition.*

Proof. The weight hierarchies of all the codes are given in the Table 1. It is straightforward to find the possible subcodes of the first three codes. We give the subcodes of $\mathcal{D}_8^{\oplus}$. If $R_i,\ 1 \le i \le 5$ denote the first 5 rows then the subcodes are given by $\mathcal{D}_1 =< R_4 >$, $\mathcal{D}_2 =< 2R_1, R_4 >$, $\mathcal{D}_3 =< R_1, 2R_1, R_4 >$, $\mathcal{D}_4 =< 2R_2, R_1, 2R_1, R_4 >$, $\mathcal{D}_5 =< R_2, 2R_2, R_1, 2R_1, R_4 >$, $\mathcal{D}_6 =< R_5, R_2, 2R_2, R_1, 2R_1, R_4 >$, $\mathcal{D}_7 =< 2R_3, R_5, R_2, 2R_2, R_1, 2R_1, R_4 >$, and $\mathcal{D}_8 =< R_3, 2R_3, R_5, R_2, 2R_2, R_1, 2R_1, R_4 >$. Then $D_1 \subseteq D_2 \ldots \subseteq D_8$ and $w_s(D_r) = d_r(QR_8), 1 \le r \le 8$. □

Table 1

Code	Weight hierarchy	Code	Weight hierarchy
$\mathcal{D}_4^{\oplus}$	$\{2,3,4,4\}$	$\mathcal{D}_4^{\oplus} \oplus \mathcal{D}_4^{\oplus}$	$\{2,3,4,4,6,7,8,8\}$
$\mathcal{D}_6^{\oplus}$	$\{2,4,5,5,6,6\}$	$\mathcal{D}_8^{\oplus}$	$\{2,4,4,6,6,7,8,8\}$

There is another self-dual code $\mathcal{L}_8$ of length 8 defined in [2]. $\mathcal{L}_8$ is $[8,8,2,4]$ type α code generated by the matrix

$$\begin{bmatrix} 00 & 11 & 02 & 13 \\ 00 & 02 & 13 & 11 \\ 11 & 02 & 00 & 13 \\ 02 & 02 & 02 & 02 \\ 00 & 00 & 00 & 22 \end{bmatrix}. \tag{7}$$

Proposition 1 *$\mathcal{L}_8$ satisfies the chain condition.*

Proof. It is easy to see that the weight hierarchy of $\mathcal{L}_8$ is $\{2,4,5,6,7,7,8,8\}$. Let $\mathbf{x}_1 = (00000022), \mathbf{x}_2 = (11020013), \mathbf{x}_3 = (13000211), \mathbf{x}_4 = (02001131)$ and $\mathbf{x}_5 = (13021122)$ be the five codewords of $\mathcal{L}_8$. Let $\mathcal{D}_1 =< \mathbf{x}_1 >$, $\mathcal{D}_2 =< 2x_2, \mathbf{x}_1 >$, $\mathcal{D}_3 =< \mathbf{x}_2, 2\mathbf{x}_2, \mathbf{x}_1 >$, $\mathcal{D}_4 =< \mathbf{x}_3, \mathbf{x}_2, 2\mathbf{x}_2, \mathbf{x}_1 >$, $\mathcal{D}_5 =< 2\mathbf{x}_4, \mathbf{x}_3, \mathbf{x}_2, 2\mathbf{x}_2, \mathbf{x}_1 >$, $\mathcal{D}_6 =< \mathbf{x}_4, 2\mathbf{x}_4, \mathbf{x}_3, \mathbf{x}_2, 2\mathbf{x}_2, \mathbf{x}_1 >$, $\mathcal{D}_7 =< 2\mathbf{x}_5, \mathbf{x}_4, 2\mathbf{x}_4, \mathbf{x}_3, \mathbf{x}_2, 2\mathbf{x}_2, \mathbf{x}_1 >$, and $\mathcal{D}_8 =< \mathbf{x}_5, 2\mathbf{x}_5, \mathbf{x}_4, 2\mathbf{x}_4, \mathbf{x}_3, \mathbf{x}_2, 2\mathbf{x}_2, \mathbf{x}_1 >$. Then $D_1 \subseteq D_2 \ldots \subseteq D_8$ and $w_s(D_r) = d_r(\mathcal{L}_8),\ 1 \le r \le 8$. □

Let $m \ge 1$. Let $\mathcal{K}_{4m}$ be the $[4m, 4m, 2, 4]$ type α code generated by the $(4m-1) \times 4m$ matrix

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 2 & 0 & \cdots & 0 & 2 \\ 0 & 0 & 2 & \cdots & 0 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 2 \end{bmatrix} \tag{8}$$

and let $\mathcal{K}_8^{'}$ be the $[8,8,2,4]$ type α code generated by the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix}. \tag{9}$$

Both of these self dual codes can also be obtained from a labelled graph [2].

Theorem 4 *The Klemm Code $\mathcal{K}_{4m}(m \geq 1)$ and the code $\mathcal{K}_8^{'}$ satisfy the chain condition and the weight hierarchy of $\mathcal{K}_{4m}$ is given by*

$$d_r(\mathcal{K}_{4m}) = \begin{cases} r+1, & 1 \leq r \leq 4m-2, \\ 4m, & r = 4m-1 \text{ or } 4m. \end{cases} \tag{10}$$

Proof. It is easy to see that the weight hierarchy of the Klemm code is given by (10). Let $R_i (1 \leq i \leq 4m-1)$ be the first $4m-1$ rows of (8). For $1 \leq r \leq 4m-2$, let $D_r =< R_{4m-i} : 1 \leq i \leq r >$ also let $\mathcal{D}_{4m-1} =< 2R_1, R_2, R_3, \cdots, R_{4m-2}, R_{4m-1} >$, $\mathcal{D}_{4m} =< R_1, 2R_1, R_2, R_3, \cdots, R_{4m-2}, R_{4m-1} >$. Then $D_1 \subseteq D_2 \ldots \subseteq D_{4m}$ and $w_s(D_r) = d_r(\mathcal{K}_{4m}), 1 \leq r \leq 4m$. For the code $\mathcal{K}_8^{'}$ it can be easily checked that its weight hierarchy is $\{2,3,4,5,6,7,8,8\}$. Let $\mathbf{x}_1 = (00000022), \mathbf{x}_2 = (00000202)$, $\mathbf{x}_3 = (00021111), \mathbf{x}_4 = (00201111), \mathbf{x}_5 = (02001111), \mathbf{x}_6 = (20001111)$ and $\mathbf{x}_7 = (20001133)$ be the seven codewords of $\mathcal{K}_8^{'}$. Then $D_1 =< \mathbf{x}_1 >$, $D_2 =< \mathbf{x}_2, \mathbf{x}_1 >$, $D_3 = < 2\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1 >$, $D_4 =< \mathbf{x}_3, 2\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1 >$, $D_5 =< \mathbf{x}_4, \mathbf{x}_3, 2\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1 >$, $D_6 = < \mathbf{x}_5, \mathbf{x}_4, \mathbf{x}_3, 2\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1 >$, $D_7 =< \mathbf{x}_6, \mathbf{x}_5, \mathbf{x}_4, \mathbf{x}_3, 2\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1 >$ and $D_8 =< \mathbf{x}_7, \mathbf{x}_6, \mathbf{x}_5, \mathbf{x}_4, \mathbf{x}_3, 2\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1 >$ form the required chain of subcodes. □

Some of the self-dual $\mathbb{Z}_4$-codes have the property that all Euclidean weights are multiple of 8 and they contain the all-one vector. These codes are called *Type-II* codes over $\mathbb{Z}_4$ ([2]). The key motivation to study these codes is that one can associate a Type-II even unimodular lattice via the construction $A \pmod 4$ [13].

3.1 Quadratic Residue Codes QR_n

The Quadratic residue codes over $\mathbb{Z}_4$ form a well known family of Type-II codes. These codes are obtained by the Hensel uplifting of the binary quadratic residue codes [12]. If $n = q+1$ and $q \equiv -1 \pmod 8$ is a prime power then Pless and Quian have shown that an extended quadratic residue code QR_n of length n is a Type II code. These have been widely studied by Pless et al for $n = 8, 24, 32, 48$ etc. [12]. QR_8 is the well known *Octacode* generated by the matrix

$$\begin{bmatrix} 3 & 3 & 2 & 3 & 1 & 0 & 0 & 0 \\ 3 & 0 & 3 & 2 & 3 & 1 & 0 & 0 \\ 3 & 0 & 0 & 3 & 2 & 3 & 1 & 0 \\ 3 & 0 & 0 & 0 & 3 & 2 & 3 & 1 \end{bmatrix}. \tag{11}$$

It is an $[8,8,4,6]$ code of type β. The code QR_{24} is the well known *lifted Golay Code.* It is a $[24,24,8,12]$ code of type β generated by the matrix

$$\left[\begin{array}{cccccccccccccccccccccccc}
3&3&1&0&0&2&3&3&3&0&3&2&1&0&0&0&0&0&0&0&0&0&0&0\\
3&0&3&1&0&0&2&3&3&3&0&3&2&1&0&0&0&0&0&0&0&0&0&0\\
3&0&0&3&1&0&0&2&3&3&3&0&3&2&1&0&0&0&0&0&0&0&0&0\\
3&0&0&0&3&1&0&0&2&3&3&3&0&3&2&1&0&0&0&0&0&0&0&0\\
3&0&0&0&0&3&1&0&0&2&3&3&3&0&3&2&1&0&0&0&0&0&0&0\\
3&0&0&0&0&0&3&1&0&0&2&3&3&3&0&3&2&1&0&0&0&0&0&0\\
3&0&0&0&0&0&0&3&1&0&0&2&3&3&3&0&3&2&1&0&0&0&0&0\\
3&0&0&0&0&0&0&0&3&1&0&0&2&3&3&3&0&3&2&1&0&0&0&0\\
3&0&0&0&0&0&0&0&0&3&1&0&0&2&3&3&3&0&3&2&1&0&0&0\\
3&0&0&0&0&0&0&0&0&0&3&1&0&0&2&3&3&3&0&3&2&1&0&0\\
3&0&0&0&0&0&0&0&0&0&0&3&1&0&0&2&3&3&3&0&3&2&1&0\\
3&0&0&0&0&0&0&0&0&0&0&0&3&1&0&0&2&3&3&3&0&3&2&1
\end{array}\right]. \tag{12}$$

The following theorem shows that these codes satisfy the chain condition.

Theorem 5 *Extended quadratic residue codes QR_8 and QR_{24} over $\mathbb{Z}_4$ satisfy the chain condition.*

Proof. It is easy to see that the weight hierarchy of the Octacode is $\{4,5,6,6,7,7,8,8\}$ (see also [1]). Let $R_i, 1 \leq i \leq 4$, be the last four rows of the matrix (11). Let $\mathcal{D}_1 =< 2R_1 >$, $\mathcal{D}_2 =< R_1, 2R_1 >$, $\mathcal{D}_3 =< 2R_2, R_1, 2R_1 >$, $\mathcal{D}_4 = < R_2, 2R_2, R_1, 2R_1 >$, $\mathcal{D}_5 =< 2R_3, R_2, 2R_2, R_1, 2R_1 >$, $\mathcal{D}_6 =< R_3, 2R_3, R_2, 2R_2, R_1, 2R_1 >$, $\mathcal{D}_7 =< 2R_4, R_3, 2R_3, R_2, 2R_2, R_1, 2R_1 >$, and $\mathcal{D}_8 =< R_4, 2R_4, R_3, 2R_3, R_2, 2R_2, R_1, 2R_1 >$. Then it is easy to verify that $D_1 \subseteq D_2 \ldots \subseteq D_8$ and $w_s(D_r) = d_r(QR_8), 1 \leq r \leq 8$. It can be seen easily that
$\{8,10,12,13,14,15,16,16,17,17,18,18,19,19,20,20,21,21,22,22,23,23,24,24\}$
is the weight hierarchy of QR_{24}. Let $R_i,\ 1 \leq i \leq 12$ be the first 12 rows of (12). Let $\mathcal{D}_1 =< 2R_{12} >$, $\mathcal{D}_2 =< R_{12}, 2R_{12} >$, $\mathcal{D}_3 =< R_{12}, 2R_{12}, 2R_{11} >, \ldots, \mathcal{D}_{24} =< R_1, 2R_1, \ldots, R_{12}, 2R_{12} >$. Then $D_1 \subseteq D_2 \ldots \subseteq D_{24}$ and $w_s(D_r) = d_r(QR_{24}), 1 \leq r \leq 24$. □

In [5] the authors have shown that $\mathbb{Z}_4$-simplex codes S_k^{α} and S_k^{β} satisfy the chain condition. The following theorem shows that their dual codes also satisfy the chain condition.

Theorem 6 *The $\mathbb{Z}_4$ Hamming codes of both types satisfy the chain condition.*

Proof. We prove it for $S_k^{\beta\perp}$ as the proof for the other Hamming code is similar. By Theorem 1 (see also [5]) the weight hierarchy of $S_k^{\beta\perp}$ is given by

$$\left\{d_r(S_k^{\beta\perp})\right\} = \{1,1,2,2,\ldots,n,n\} \setminus \left\{1 + \left(2^{i-1} - 2^{i-1-\lfloor \frac{i}{2} \rfloor}\right) \mid 0 \leq i < 2k\right\},$$

$$1 \leq r \leq 2^{2k} - 2^k - 2k.$$

Let H be the standard parity check matrix of S_k^β (see [5]) and let R_i be the last $2^{2k-1} - 2^{k-1} - k = n - k$ rows of H. Then it is easy to see that the required subcodes of ${S_k^\beta}^\perp$ are given by $\mathcal{D}_1 =< 2R_{n-k} >$, $\mathcal{D}_2 =< 2R_{n-k}, R_{n-k} >, \ldots$, $\mathcal{D}_{2n-2k} =< 2R_{n-k}, R_{n-k}, \ldots, 2R_1, R_1 >$. □

4 Conclusion and Further Work

In this article we have shown that all self-dual codes of length up to 9 satisfy the chain condition. This includes all indecomposable codes: $\mathcal{A}_1, \mathcal{D}_4^\oplus, \mathcal{D}_6^\oplus, \mathcal{E}_7^+, \mathcal{D}_8^\oplus, \mathcal{E}_8, \mathcal{K}_8, \mathcal{K}_8'$, Octacode QR_8 and $\mathcal{L}_8$ and all decomposable codes: $\mathcal{D}_4^\oplus \oplus \mathcal{D}_4^\oplus$. All trivial codes of length n satisfy the chain condition. Thus our study is complete up to length 9 for codes classified in [2]. We have shown that the class of codes $\mathcal{D}_{2m}, (m = 1, 2, \ldots)$, Klemm Code $\mathcal{K}_{4m} (m \geq 1)$ and Lifted Golay Code QR_{24} also satisfy the chain condition. It is clear from our proof techniques that showing a code satisfy the chain condition is a difficult problem. It would be interesting to come up with an algorithm to decide whether a given code satisfy the chain condition ? However, at present we do not know any such algorithm. Also, it would be interesting to extend our results to other self-dual codes in the classification of [10, 11] and to show that some other general classes of codes satisfy the chain condition. The following result could be helpful in this direction which is obtained in view of Theorem 6.

Conjecture 1 *If $\mathcal{C}$ satisfies the chain condition so does its dual $\mathcal{C}^\perp$.*

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References

[1] A. Ashikhmin, On generalized hamming weights for Galois ring linear codes, Des. Codes Cryptogr. **14** (1998), 107–126.

[2] J.H. Conway and N. J. A. Sloane, Self-dual codes over the integers modulo 4, *J. Combin. Theory Ser. A* **62** (1993), 30–45.

[3] M.K. Gupta, M.C. Bhandari and A.K. Lal, On linear codes over $\mathbb{Z}_{2^s}$, *Des. Codes and Cryptogr.*, to appear.

[4] M.K. Gupta, *On some linear codes over $\mathbb{Z}_{2^s}$*. PhD Thesis, IIT Kanpur, India (July 2000).

[5] M.C. Bhandari, M.K. Gupta and A.K. Lal, On $\mathbb{Z}_4$ simplex codes and their gray images, *Applied Algebra, Algebraic Algorthims and Error-Correcting Codes*, AAECC-13, *Lec. Notes Comp. Sci.* **vol 1719** (1999), 170–180.

[6] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane and P. Solé, The $\mathbb{Z}_4$-linearity of kerdock, preparata, goethals, and related codes, *IEEE Trans. Inform. Theory*, **40** (2) (1994), 301–319.

[7] S. B. Encheva and G. D. Cohen, Linear codes and their coordinate ordering, Des. Codes Cryptogr. **20** (3) (2000), 229–250.

[8] S. Encheva and G. Cohen, Self-orthogonal codes and their coordinate ordering, *IEICE T. Fund. Electr. E80A:* **(11)** (1997), 2256–2259.

[9] S. B. Encheva, On binary linear codes which satisfy the two-way chain condition, *IEEE Trans. Inform. Theory* **42** (3) (1996), 1038–1047.

[10] J. Fields, P. Gaborit, J. S. Leon and V. Pless, All self-dual codes of length 15 or less are known, *IEEE Trans. Inform. Theory*, **44** (1) (1998), 311–322.

[11] V. Pless, J. S. Leon and J. Fields, All $\mathbb{Z}_4$ codes of type II and length 16 are known, *J. Combin. Theory*, Series A **78** (1997), 32–50.

[12] V. Pless, P. Solé and Z. Qian, Cyclic self-dual $\mathbb{Z}_4$-codes, *Finite Fields and Their Applications*, **3** (1997), 48–69.

[13] E. M. Rains and N. J. A. Sloane, *Self-Dual Codes:* The Handbook of Coding Theory. North-Holland, New York, 1998.

[14] A. Vardy, *Trellis structure of codes:* The Handbook of Coding Theory North-Holland, New York, 1998.

[15] V. K. Wei and K. Yang, On the generalized hamming weights of product codes, *IEEE Trans. Inform. Theory*, **39** (5) (1991), 1709–1713.

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