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Indices of convergence on four digraph operators*

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Abstract

Let D be a digraph. We denote by $L(D), M(D), M'(D)$ and $T(D)$ the line digraph, the middle digraph, the special middle digraph and the total digraph of D, by $k(D)$, $p(D)$ and $Diam(D)$ the index of convergence, the period and the diameter of D, respectively. Zuo (in Acta. Mathematica Applicatae Sinica, 21(1) (1998), 144–147) proved that $k(D)-1 \leq k(L(D)) \leq k(D)+1$. In this paper, we prove that:

1. $\max\{k(M(D)), k(M'(D)), k(T(D))+1\} \leq \max\{2p(D), 2k(D)+2\}$.

2. $k(T(D)) \leq k(M(D)) \leq k(T(D))+1$;
$k(T(D)) \leq k(M'(D)) \leq k(T(D))+1$.

3. If there do not exist both sources and sinks in D, then
$k(M(D)) \leq k(M'(D)) \leq k(M(D))+1$.

4. If D is a strongly connected digraph, then
$\min\{k(M(D)), k(M'(D))-1, k(T(D))\} \geq Diam(D)+1$.

5. If D is a primitive digraph, then
$\max\{k(M(D)), k(M'(D))-1, k(T(D))\} \leq k(D)+1$.

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1 Introduction

Throughout this paper, let D be a digraph with the vertex-set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and the arc-set $A(D) = \{x_1, x_2, \ldots, x_m\}$. Digraphs in this paper will be allowed to have loops but not multiple arcs (arcs of the form (v_i, v_j) and (v_j, v_i) are allowed). Let C_n denote a directed cycle of length n. The directed cycle of length 1 is a loop. A **source** is a vertex with in-valency 0, and a **sink** is a vertex with out-valency 0. We suppose that the operations of matrices in this paper are Boolean operations. We use Hemminger and Beineke [1] for terminologies and notations not defined here.

In 1960, Harary and Norman [2] introduced the concept of the line digraph. For a digraph D, the **line digraph**, denoted by $L(D)$, has as its vertex-set the arc-set of D, (a, b) is an arc of $L(D)$ if and only if there are vertices u, v and w in D with $a = (u, v)$ and $b = (v, w)$. Line digraphs have been discussed in [1–8].

In 1966, Chartrand [9] introduced the concept of the total digraph. for a digraph D, the **total digraph**, denoted by $T(D)$, has its vertex-set $V(T(D)) = V(D) \cup A(D)$, there is an arc $(a, b) \in A(T(D))$ from vertex a to vertex b in $V(T(D))$ if and only if one of the following four cases holds: 1. If $a \in V(D)$ and $b \in V(D)$, then $(a, b) \in A(D)$. 2. If $a \in V(D)$ and $b \in A(D)$, then a is the tail of arc b in D. 3. If $a \in A(D)$ and $b \in V(D)$, then b is the head of arc a in D. 4. If $a \in A(D)$ and $b \in A(D)$, then the head of arc a in D is the tail of arc b in D. The total digraph has been discussed in [9–11].

In 1977, Zamfirescu [10] introduced the concept of the middle digraph. For a digraph D, the **middle digraph**, denoted by $M(D)$, has its vertex-set $V(M(D)) = V(D) \cup A(D)$, there is an arc $(a, b) \in A(M(D))$ from vertex a to vertex b in $V(M(D))$ if and only if one of the following three cases holds: 1. If $a \in V(D)$ and $b \in A(D)$, then a is the tail of arc b in D. 2. If $a \in A(D)$ and $b \in V(D)$, then b is the head of arc a in D. 3. If $a \in A(D)$ and $b \in A(D)$, then the head of arc a in D is the tail of arc b in D. The middle digraph has been discussed in [10,11].

Similarly, we can define the special middle digraph of D as follows.

Definition 1.1 For a digraph D, the **special middle digraph**, denoted by $M'(D)$, has its vertex $V(M'(D)) = V(D) \cup A(D)$. there is an arc $(a, b) \in A(M'(D))$ from vertex a to vertex b in $V(M'(D))$ if and only if one of the following three cases holds: 1. If $a \in V(D)$ and $b \in V(D)$, then $(a, b) \in A(D)$. 2. If $a \in V(D)$ and $b \in A(D)$, then a is the tail of arc b in D. 3. If $a \in A(D)$ and $b \in V(D)$, then b is the head of arc a in D.

Suppose that A is the adjacency matrix of D, whose entries are 0 or 1. Then the index of convergence and the period of D equal the index of convergence and the period of Boolean matrix A, respectively, defined as follows (see [12–14]):

Suppose that A is a Boolean matrix. Its Boolean sequence of powers is denoted by $(A^j) = I, A, A^2, \ldots, A^k, \cdots$. The index of convergence (say $k(A)$) and the period (say $p(A)$) of A are the least non-negative integer k and the least positive integer p such that $A^k = A^{k+p}$, respectively. By this definition, the Boolean sequence of powers of A is as follows:

$$(A^j) = I, A, A^2, \ldots, A^{k-1}, A^k, \ldots, A^{k+p-1}, A^k, \ldots, A^{k+p-1}, \ldots$$

where $A^i \neq A^j$ if $max(i,j) < k+p$ and $i \neq j$.

It is well known [12–14] that the digraph D is primitive if and only if D is strongly connected and the greatest common divisor of the lengths of all (elementary) directed cycles of D is 1.

In paper [7], Zuo considered relations between $k(D)$ and $k(L(D))$ and between $p(D)$ and $p(L(D))$, and obtained the following proposition.

Proposition 1.2 [7] Let D be a digraph; then:
1. $p(L(D)) = p(D)$.
2. $k(D) - 1 \leq k(L(D)) \leq k(D) + 1$.
3. If D is a primitive digraph, then $k(L(D)) = k(D) + 1$.
4. $k(L(D)) = k(D) - 1 = l$ if there exist no directed cycles in D, where l is the length of the longest directed path in D.

The above results have been proved by Zhou [15] by using a simpler method—algebraic method. In particular, by using the algebraic method, Yan and Zhang [8] obtained a stronger result than that of Proposition 1.1 as follows: If a digraph D has no sources or sinks, then $k(D) \leq k(L(D)) \leq k(D) + 1$. Moreover, if D has no sources and no sinks, then $k(L(D)) = k(D) + 1$ if there is at least one connected component of D which is not a directed cycle, and $k(L(D)) = k(D) = 0$ if every connected component of D is a directed cycle.

The following results are useful.

Proposition 1.3 [12] Let D be a digraph.
1. If D is strongly connected, then: (1). $p(D)$ equals the greatest common divisor of lengths of all directed cycles of D. (2). $Diam(L(D)) = Diam(D) + 1$ unless D is a directed cycle, where $Diam(D)$ denotes the diameter of D.
2. If D is weakly connected, then $p(D)$ equals the least common multiple of the periods of strongly connected components of D.
3. If $D_i (1 \leq i \leq c)$ are all of weakly connected components of D, then $k(D) = \max_{i=1}^{c}(k(D_i))$, and $p(D)$ equals the least common multiple of $p(D_1), p(D_2), \ldots$, and $p(D_c)$.

The following propositions are due to Hemminger and Beineke [1] and Lin and Zhang [4–6], respectively.

Proposition 1.4 [1] Suppose that D is a digraph with no isolated vertices. Then
(1). $L(D)$ is strongly connected if and only if D is strongly connected.
(2). $L(D)$ is a directed cycle if and only if D is a directed cycle.

Proposition 1.5 [4–6] Let D be a digraph, with vertex-set $V(D) = \{v_1, v_2, \ldots, v_n\}$, and arc-set $A(D) = \{x_1, x_2, \ldots, x_m\}$, B_0 and B_1 be the following two $n \times m$ matrices: $B_0 = (b^0_{ij}), B_1 = (b^1_{ij})$, respectively, where

$$b^0_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the tail of arc } x_j \text{ in } D; \\ 0 & \text{otherwise.} \end{cases}$$

$$b_{ij}^1 = \begin{cases} 1 & \text{if } v_i \text{ is the head of arc } x_j \text{ in } D; \\ 0 & \text{otherwise.} \end{cases}$$

Then $A = B_0 B_1^T, A_L = B_1^T B_0, A_T = \begin{pmatrix} A & B_0 \\ B_1^T & A_L \end{pmatrix}$, where A, A_L and A_T are the adjacency matrices of D, $L(D)$ and $T(D)$, respectively, and B_1^T denotes the transpose of B_1.

Definition 1.6 [4,12] We say that $\mathbf{B_0}$ and $\mathbf{B_1}$ in Proposition 1.5 are the **out-incidence matrix** and the **in-incidence matrix** of D, respectively.

2 Some lemmas

Similarly to Proposition 1.5, we can prove the following lemma.

Lemma 2.1 Let D be a digraph with n vertices and m arcs and let A_M and $A_{M'}$ be the adjacency matrices of $M(D)$ and $M'(D)$, respectively. Then

$$A_M = \begin{pmatrix} 0_n & B_0 \\ B_1^T & A_L \end{pmatrix}, \quad A_{M'} = \begin{pmatrix} A & B_0 \\ B_1^T & 0_m \end{pmatrix}$$

where 0_n is the $n \times n$ matrix with all entries equal zero, and B_0 and B_1 are the out-incidence matrix and the in-incidence matrix of D, respectively.

Lemma 2.2 Let D be a digraph. Then
1. $A_L^k = B_1^T A^{k-1} B_0, A^k = B_0 A_L^{k-1} B_1^T$, for $k \geq 1$.
2. If k is odd $(k > 1)$, then

$$A_M^k = \begin{pmatrix} A^{k-1} + A^{k-2} + \cdots + A^{\frac{k+1}{2}} & (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}})B_0 \\ B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}}) & A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix};$$

If k is even $(k \geq 2)$, then

$$A_M^k = \begin{pmatrix} A^{k-1} + A^{k-2} + \cdots + A^{\frac{k}{2}} & (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k}{2}})B_0 \\ B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k}{2}}) & A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k}{2}} \end{pmatrix}.$$

Proof By Proposition 1.5, the first assertion can be easily proved.

We prove the second assertion by induction on k.

When $k = 2$ or 3,

$$A_M^2 = \begin{pmatrix} A & AB_0 \\ B_1^T A & A_L^2 + A_L \end{pmatrix},$$

$$A_M^3 = A_M A_M^2 = \begin{pmatrix} 0_n & B_0 \\ B_1^T & A_L \end{pmatrix} \begin{pmatrix} A & AB_0 \\ B_1^T A & A_L^2 + A_L \end{pmatrix} = \begin{pmatrix} A^2 & (A^2 + A)B_0 \\ B_1^T(A^2 + A) & A_L^3 + A_L^2 \end{pmatrix}.$$

Hence, when $k = 2$ or 3, the second assertion holds. We assume the second assertion holds for k. First, we suppose that k is odd. Then

$$A_M^k = \begin{pmatrix} A^{k-1} + A^{k-2} + \cdots + A^{\frac{k+1}{2}} & (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}})B_0 \\ B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}}) & A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix}.$$

By the first assertion, then

$$B_0(A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}}) = (A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}})B_0.$$

Hence, we have

$$A_M^{k+1} = A_M A_M^k =$$

$$\begin{pmatrix} 0_n & B_0 \\ B_1^T & A_L \end{pmatrix} \begin{pmatrix} A^{k-1} + A^{k-2} + \cdots + A^{\frac{k+1}{2}} & (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}})B_0 \\ B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}}) & A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix} =$$

$$\begin{pmatrix} A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}} & (A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}})B_0 \\ B_1^T(A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}}) & A_L^{k+1} + A_L^k + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix}.$$

Similarly, we can prove that when k is even the second assertion holds. Hence our proof follows.

Similarly, we can prove the following two lemmas.

Lemma 2.3 Let D be a digraph. Then
1. If k is odd $(k > 1)$, then

$$A_{M'}^k = \begin{pmatrix} A^k + A^{k-2} + \cdots + A^{\frac{k+1}{2}} & (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}})B_0 \\ B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}}) & A_L^{k-1} + A_L^{k-2} + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix};$$

2. If k is even $(k \geq 2)$, then

$$A_{M'}^k = \begin{pmatrix} A^k + A^{k-2} + \cdots + A^{\frac{k}{2}} & (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k}{2}})B_0 \\ B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k}{2}}) & A_L^{k-1} + A_L^{k-2} + \cdots + A_L^{\frac{k}{2}} \end{pmatrix}.$$

Lemma 2.4 Let D be a digraph. Then
1. If k is odd $(k > 1)$, then

$$A_T^k = \begin{pmatrix} A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}} & (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}})B_0 \\ B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}}) & A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix};$$

2. If k is even $(k \geq 2)$, then

$$A_T^k = \begin{pmatrix} A^k + A^{k-1} + \cdots + A^{\frac{k}{2}} & (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k}{2}})B_0 \\ B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k}{2}}) & A_L^k + A_L^k + \cdots + A_L^{\frac{k}{2}} \end{pmatrix}.$$

Lemma 2.4 was proved by Yan and Zhang [16] and You and Liu et al. [17]. The following lemma is obvious.

Lemma 2.5 Suppose that there do not exist directed cycles in D. Then
1. $p(M(D)) = p(M'(D)) = p(T(D)) = 1$.
2. If we denote by $l(D)$ the length of the longest directed path of D, then $l(M(D)) = l(M'(D)) = l(T(D)) = 2l(D)$.

Partial results on $T(D)$ in Lemma 2.5 were proved by Yan and Zhang [16] and You and Liu et al. [17].

Lemma 2.6 Let D be a strongly connected digraph. Then all of $M(D), M'(D)$ and $T(D)$ are primitive digraphs.

Proof By the definition of $M(D)$, it is obvious that $M(D)$ is strongly connected. Note that there exists at least a directed cycle in D. If we denote by l the length of this directed cycle. Then there at least exist two directed cycles in $M(D)$ whose lengths are l and $l+1$, respectively. By Proposition 1.3, then $p(M(D)) = 1$. Hence $M(D)$ is a primitive digraph. Similarly, we can prove that both of $M'(D)$ and $T(D)$ are primitive digraphs. Thus our proof follows.

Corollary 2.7 Let D be a digraph. Then $p(M(D)) = p(M'(D)) = p(T(D)) = 1$.

Proof If there are not directed cycles in D, then, by Lemma 2.5, $p(M(D)) = p(M'(D)) = p(T(D)) = 1$. If there is at least a directed cycle in D, denoted by l its length, then there at least exist two directed cycles in $M(D)$ ($M'(D)$ and $T(D)$) whose lengths are l and $l+1$, respectively. we distinguish the following three cases:

Case 1 Let D be a strongly connected digraph.

By Lemma 2.6, all of $M(D), M'(D)$, and $T(D)$ are primitive digraphs, Hence $p(M(D)) = p(M'(D)) = p(T(D)) = 1$.

Case 2 Let D be a weakly connected digraph.

It is easy to see that $M(D)$ ($M'(D)$ and $T(D)$) is weakly connected. By Case 1 and Proposition 1.3, then $p(M(D)) = p(M'(D)) = p(T(D)) = 1$.

Case 3 Let $D_i (1 \leq i \leq c)$ be all of weakly connected components of D.

By Case 2 and Lemma 2.5, then $p(M(D_i)) = p(M'(D_i)) = p(T(D_i)) = 1$. Hence $p(M(D)) = p(M'(D)) = p(T(D)) = 1$.

Thus we have completed the proof of Corollary 2.7.

Partial results on $T(D)$ in Lemma 2.6 and Corollary 2.7 were proved by Yan and Zhang [16] and You and Liu et al. [17].

Lemma 2.8 Let D be a digraph with no sources, and let C_1 and C_2 be two $n \times n$ matrices whose entries are 0 or 1. If $B_1^T C_1 = B_1^T C_2$, then $C_1 = C_2$, where B_1 is the in-incidence matrix of D.

Proof Let $C_1 = (c_{ij}^1)$ and $C_2 = (c_{ij}^2)$. We need to prove that $c_{ij}^1 = c_{ij}^2$ for $1 \leq i, j \leq n$. By the definition of B_1, for vertex v_i of D, then there is an arc x_m in D such that v_i is the head of arc x_m. Hence $b_{im}^1 = 1$. So the $mj-$entry $(B_1^T C_1)_{mj}$ of matrix $B_1^T C_1$ equals $\sum_{k=1}^n b_{km}^1 c_{kj}^1 = b_{im}^1 c_{ij}^1 = c_{ij}^1$, since there only exists one entry b_{im}^1 which is not zero in the $m-th$ column of B_1. Similarly, the $mj-$entry $(B_1^T C_2)_{mj}$ of matrix $B_1^T C_2$ equals c_{ij}^2. Noting that $B_1^T C_1 = B_1^T C_2$, hence $c_{ij}^1 = c_{ij}^2$. This shows that $C_1 = C_2$. Our proof thus follows.

Lemma 2.9 Let D be a digraph, $k = k(D)$ and $p = p(D)$. Then $\sum_{i=s}^t A^i = A^k + A^{k+1} + \cdots + A^{k+p-1}$, for $s \geq k$ and $t \geq s+p-1$.

By the definitions of $k(D)$ and $p(D)$, this is clear.

3 Main results

Theorem 3.1 Let D be a digraph. Then
1. $k(M(D)) \leq \max\{2p(D), 2k(D)+2\}$.
2. $k(M'(D)) \leq \max\{2p(D), 2k(D)+2\}$.
3. $k(T(D)) \leq \max\{2p(D)-1, 2k(D)+1\}$.

Proof First, we prove that $k(M(D)) \leq \max\{2p(D), 2k(D)+2\}$.

Let $k = k(D), p = p(D)$. We distinguish the following two cases:

Case 1 If $2p(D) \geq 2k(D)+2$, then $p \geq k+1$. By Corollary 2.7, we only need to prove that $A_M^{2p} = A_M^{2p+1}$. By Lemma 2.2,

$$A_M^{2p} = \begin{pmatrix} A^{2p-1} + A^{2p-2} + \cdots + A^p & (A^{2p-1} + A^{2p-2} + \cdots + A^p)B_0 \\ B_1^T(A^{2p-1} + A^{2p-2} + \cdots + A^p) & A_L^{2p} + A_L^{2p-1} + \cdots + A_L^p \end{pmatrix};$$

$$A_M^{2p+1} = \begin{pmatrix} A^{2p} + A^{2p-1} + \cdots + A^{p+1} & (A^{2p} + A^{2p-1} + \cdots + A^p)B_0 \\ B_1^T(A^{2p} + A^{2p-1} + \cdots + A^p) & A_L^{2p+1} + A_L^{2p} + \cdots + A_L^{p+1} \end{pmatrix}.$$

Noting that $p \geq k+1$, by Lemma 2.9, then

$$A^{2p-1} + A^{2p-2} + \cdots + A^p = A^k + A^{k+1} + \cdots + A^{k+p-1}.$$

Similarly, we have

$$A^{2p} + A^{2p-1} + \cdots + A^{p+1} = A^k + A^{k+1} + \cdots + A^{k+p-1}.$$

Hence

$$A^{2p-1} + A^{2p-2} + \cdots + A^p = A^{2p} + A^{2p-1} + \cdots + A^{p+1}.$$

By Proposition 1.2, $k(L(D)) \leq k(D)+1$. Similarly, we can prove that

$$(A^{2p-1} + A^{2p-2} + \cdots + A^p)B_0 = (A^{2p} + A^{2p-1} + \cdots + A^p)B_0;$$

$$B_1^T(A^{2p-1} + A^{2p-2} + \cdots + A^p) = B_1^T(A^{2p} + A^{2p-1} + \cdots + A^p);$$

$$A_L^{2p} + A_L^{2p-1} + \cdots + A_L^p = A_L^{2p+1} + A_L^{2p} + \cdots + A_L^{p+1}.$$

Hence $A_M^{2p} = A_M^{2p+1}$. This shows that $k(M(D)) \leq 2p = \max\{2p, 2k+2\}$.

Case 2 If $2p(D) \leq 2k(D)+2$, then $p \leq k+1$. By Lemma 2.9, we only need to prove that $A_M^{2k+2} = A_M^{2k+3}$. By Lemma 2.2, we have

$$A_M^{2k+2} = \begin{pmatrix} A^{2k+1} + A^{2k} + \cdots + A^{k+1} & (A^{2k+1} + A^{2k} + \cdots + A^{k+1})B_0 \\ B_1^T(A^{2k+1} + A^{2k} + \cdots + A^{k+1}) & A_L^{2k+2} + A_L^{2k+1} + \cdots + A_L^{k+1} \end{pmatrix};$$

$$A_M^{2k+3} = \begin{pmatrix} A^{2k+2} + A^{2k+1} + \cdots + A^{k+2} & (A^{2k+2} + A^{2k+1} + \cdots + A^{k+1})B_0 \\ B_1^T(A^{2k+2} + A^{2k+1} + \cdots + A^{k+1}) & A_L^{2k+3} + A_L^{2k+2} + \cdots + A_L^{k+2} \end{pmatrix}.$$

Similarly to the proof of Case 1, we can see that

$$A^{2k+1} + A^{2k} + \cdots + A^{k+1} = A^{2k+2} + A^{2k+1} + \cdots + A^{k+2};$$

$$(A^{2k+1}+A^{2k}+\cdots+A^{k+1})B_0=(A^{2k+2}+A^{2k+1}+\cdots+A^{k+1})B_0;$$
$$B_1^T(A^{2k+1}+A^{2k}+\cdots+A^{k+1})=B_1^T(A^{2k+2}+A^{2k+1}+\cdots+A^{k+1});$$
$$A_L^{2k+2}+A_L^{2k+1}+\cdots+A_L^{k+1}=A_L^{2k+3}+A_L^{2k+2}+\cdots+A_L^{k+2}.$$

Hence $A_M^{2k+2}=A_M^{2k+3}$. This shows that $k(M(D))\le 2k+2=\max\{2p,2k+2\}$.

Combining Case 1 and Case 2, we have

$$k(M(D))\le \max\{2p(D),2k(D)+2\}.$$

Similarly, we can prove assertions 2 and 3.

The third assertion in Theorem 3.1 was proved by Yan and Zhang [15].

Remark 1 Let $D_1=(V(D_1),A(D_1))$, $V(D_1)=\{1,2,3\}$, and $A(D_1)=\{(1,2),(2,1),(2,3),(3,2)\}$. Then $p(D_1)=2$, $k(D_1)=1$, $k(M(D_1))=k(M'(D_1))=4$, $k(T(D_1))=3$. This example shows that the upper bounds in Theorem 3.1 are obtained.

Theorem 3.2 Let D be a primitive digraph; then

$$k(M(D))\le k(D)+1, k(M'(D))\le k(D)+2, k(T(D))\le k(D)+1.$$

Proof Let $k(D)=k$, J_n be the $n\times n$ matrix with all entries equal one. Then $A^k=J_n$. By Proposition 1.2, $A_L^{k+1}=J_m$. If k is odd, then

$$A_M^{k+1}=\begin{pmatrix} A^k+A^{k-1}+\cdots+A^{\frac{k+1}{2}} & (A^k+A^{k-1}+\cdots+A^{\frac{k+1}{2}})B_0 \\ B_1^T(A^k+A^{k-1}+\cdots+A^{\frac{k+1}{2}}) & A_L^{k+1}+A_L^k+\cdots+A_L^{\frac{k+1}{2}} \end{pmatrix}.$$

Noting that $J_nB_0=J_{nm}$, $B_1^TJ_n=J_{mn}$, where J_{nm} and J_{mn} denote the $n\times m$ matrix with all entries equal one and the $m\times n$ matrix with all entries equal one, respectively. Hence $A_M^{k+1}=J_{n+m}$. Similarly, if k is even then $A_M^{k+1}=J_{n+m}$. This shows that $k(M(D))\le k(D)+1$. Similarly, we can prove that $k(M'(D))\le k(D)+2$, $k(T(D))\le k(D)+1$. Our theorem is thus proved.

The third assertion in Theorem 3.2 was obtained by Yan and Zhang [16] and You and Liu et al. [17].

Remark 2 Let $D_2=(V(D_2),A(D_2)),V(D_2)=\{1,2\},andA(D_2)=\{(1,1),(1,2),(2,1)\}$. It is obvious that D_2 is a primitive digraph and $k(D_2)=2,k(M(D_2))=k(T(D_2))=3,k(M'(D_2))=4$. This example shows that the upper bounds in Theorem 3.2 are obtained.

Theorem 3.3 Let D be a digraph. Then

1. If D is a directed cycle with n vertices, then $k(M(D))=k(M'(D))=k(T(D))+1=2n$.
2. If there exist no directed cycles in D, then $k(M(D))=k(M'(D))=k(T(D)=2k(D)-1=2l(D)+1$, where $l(D)$ denotes the length of the longest directed path in D.

Proof It is easy to prove that the first assertion holds.

Assume that there exist no directed cycles in D. By Lemma 2.5, then $p(M(D)) = p(M'(D)) = p(T(D)) = 1$, $l(M(D)) = l(M'(D)) = l(T(D)) = 2l(D)$. By Proposition 1.2, then $k(M(D)) = k(M'(D)) = k(T(D)) = 2l(D)+1$, and $k(D) = l(D)+1$. Hence $k(M(D)) = k(M'(D)) = k(T(D)) = 2k(D)-1 = 2l(D)+1$. Thus the second assertion holds. Our proof thus follows.

Partial results on $T(D)$ in Theorem 3.3 were obtained by Yan and Zhang [16] and You and Liu et al. [17].

Theorem 3.4 Let D be a strongly connected digraph. Then
$k(M(D)) \geq Diam(D)+1, k(M'(D)) \geq Diam(D)+2, k(T(D)) \geq Diam(D)+1$.

Proof Let $Diam(D) = d$. If D is a directed cycle with n vertices, then by Theorem 3.3 we have $k(M(D)) = k(M'(D)) = k(T(D))+1 = 2n \geq d+1$. If D is not a cycle, then by Proposition 1.3 we have $Diam(L(D)) = d+1$. Let $x_i x_{i+1} \cdots x_{i+d+1}$ be the directed path of length $d+1$ in $L(D)$. Then the (i,i+d+1)-entry of matrix $I + A_L + A_L^2 + \cdots + A_L^d$ equals zero. Hence $I + A_L + A_L^2 + \cdots + A_L^d \neq J_m$. By Lemma 2.2, then $A_M^d \neq J_{n+m}$. By Lemma 2.6, $M(D)$ is a primitive digraph. Hence $k(M(D)) \geq d+1$. Similarly, we can prove that $k(M'(D)) \geq d+2$ and $k(T(D)) \geq d+1$. Our proof thus follows.

The third result in Theorem 3.4 was obtained by Yan and Zhang [16].

Remark 3 Let $D_3 = (V(D_3), A(D_3))$,$V(D_3) = \{1,2\}$, $A(D_3) = \{(1,1),(1,2),(2,1),(2,2)\}$. It is easy to prove that $k(M(D_3)) = k(T(D_3)) = 2$, $k(M'(D_3)) = 3$, and $Diam(D_3) = 1$. Hence $k(M(D_3)) = k(T(D_3)) = Diam(D_3)+1$ and $k(M'(D_3)) = Diam(D_3)+2$. This example shows that the lower bounds in Theorem 3.4 are obtained.

Theorem 3.5 Let D be a digraph. Then
$k(T(D)) \leq k(M(D)) \leq k(T(D))+1, \quad k(T(D)) \leq k(M'(D)) \leq k(T(D))+1$.

Proof First, we prove that $k(T(D)) \leq k(M(D)) \leq k(T(D))+1$. Let $k(M(D)) = k$. By Corollary 2.7, then $A_M^k = A_M^{k+1}$. We distinguish the following two cases:

Case 1 Let k be odd. Since $A_M^k = A_M^{k+1}$, by Lemma 2.2, we have

$$A_M^k = \begin{pmatrix} A^{k-1} + A^{k-2} + \cdots + A^{\frac{k+1}{2}} & (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}})B_0 \\ B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}}) & A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}} & (A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}})B_0 \\ B_1^T(A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}}) & A_L^{k+1} + A_L^k + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix} = A_M^{k+1}.$$

Hence

(1) $\quad A^{k-1} + A^{k-2} + \cdots + A^{\frac{k+1}{2}} = A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}}$;

(2) $\quad (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}})B_0 = (A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}})B_0$;

(3) $\quad B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}}) = B_1^T(A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}})$;

(4) $\quad A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}} = A_L^{k+1} + A_L^k + \cdots + A_L^{\frac{k+1}{2}}$.

By (1), we have

$$A(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k+1}{2}}) + A^{\frac{k+1}{2}} = A(A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}}) + A^{\frac{k+1}{2}}.$$

Thus

(1′) $A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}} = A^{k+1} + A^k + \cdots + A^{\frac{k+1}{2}}.$

By Lemma 2.4, then

$$A_T^k = \begin{pmatrix} A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}} & (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}})B_0 \\ B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}}) & A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix};$$

$$A_T^{k+1} = \begin{pmatrix} A^{k+1} + A^k + \cdots + A^{\frac{k+1}{2}} & (A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}})B_0 \\ B_1^T(A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}}) & A_L^{k+1} + A_L^k + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix}.$$

By (1′), (2), (3), and (4), then $A_T^k = A_T^{k+1}$. Thus $k(T(D)) \le k = k(M(D))$.

Case 2 Let k be even. Similarly, we can prove that $k(T(D)) \le k = k(M(D))$.

Hence we proved that $k(T(D)) \le k(M(D))$.

Now we prove that $k(M(D)) \le k(T(D)) + 1$. Let $k(T(D)) = k'$. By Corollary 2.7, then $A_T^{k'} = A_T^{k'+1}$. We distinguish the following two cases:

Case a Let k' be odd. Since $A_T^{k'} = A_T^{k'+1}$, by Lemma 2.4 we have

$$A_T^{k'} = \begin{pmatrix} A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}} & (A^{k'-1} + A^{k'-2} + \cdots + A^{\frac{k'-1}{2}})B_0 \\ B_1^T(A^{k'-1} + A^{k'-2} + \cdots + A^{\frac{k'-1}{2}}) & A_L^{k'} + A_L^{k'-1} + \cdots + A_L^{\frac{k'+1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} A^{k'+1} + A^{k'} + \cdots + A^{\frac{k'+1}{2}} & (A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}})B_0 \\ B_1^T(A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}}) & A_L^{k'+1} + A_L^{k'} + \cdots + A_L^{\frac{k'+1}{2}} \end{pmatrix} = A_T^{k'+1}.$$

Hence

(5) $A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}} = A^{k'+1} + A^{k'} + \cdots + A^{\frac{k'+1}{2}};$

(6) $(A^{k'-1} + A^{k'-2} + \cdots + A^{\frac{k'-1}{2}})B_0 = (A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}})B_0;$

(7) $B_1^T(A^{k'-1} + A^{k'-2} + \cdots + A^{\frac{k'-1}{2}}) = B_1^T(A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}});$

(8) $A_L^{k'} + A_L^{k'-1} + \cdots + A_L^{\frac{k'+1}{2}} = A_L^{k'+1} + A_L^{k'} + \cdots + A_L^{\frac{k'+1}{2}}.$

By (6), then

$$(A^{k'-1} + A^{k'-2} + \cdots + A^{\frac{k'-1}{2}})B_0B_1^T = (A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}})B_0B_1^T.$$

Hence

(5′) $A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}} = A^{k'+1} + A^{k'} + \cdots + A^{\frac{k'+3}{2}}.$

By (5), then

(6′) $(A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}})B_0 = (A^{k'+1} + A^{k'} + \cdots + A^{\frac{k'+1}{2}})B_0;$

(7′) $B_1^T(A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}}) = B_1^T(A^{k'+1} + A^{k'} + \cdots + A^{\frac{k'+1}{2}}).$

By (7), then

$$B_1^T(A^{k'-1} + A^{k'-2} + \cdots + A^{\frac{k'-1}{2}})B_0 = B_1^T(A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}})B_0.$$

Hence

(9) $A_L^{k'} + A_L^{k'-1} + \cdots + A_L^{\frac{k'+1}{2}} = A_L^{k'+1} + A_L^{k'} + \cdots + A_L^{\frac{k'+3}{2}}.$

By (8), then

(10) $$A_L^{k'+1} + A_L^{k'} + \cdots + A_L^{\frac{k'+3}{2}} = A_L^{k'+2} + A_L^{k'+1} + \cdots + A_L^{\frac{k'+3}{2}}.$$

We plus both sides of the equation (9) by $A_L^{k'+1}$, then we have

(11) $$A_L^{k'+1} + A_L^{k'} + \cdots + A_L^{\frac{k'+1}{2}} = A_L^{k'+1} + A_L^{k'} + \cdots + A_L^{\frac{k'+3}{2}}.$$

By (10) and (11), thus

(8′) $$A_L^{k'+1} + A_L^{k'} + \cdots + A_L^{\frac{k'+1}{2}} = A_L^{k'+2} + A_L^{k'+1} + \cdots + A_L^{\frac{k'+3}{2}}.$$

By Lemma 2.2, then

$$A_M^{k'+1} = \begin{pmatrix} A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}} & (A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}})B_0 \\ B_1^T(A^{k'} + A^{k'-1} + \cdots + A^{\frac{k'+1}{2}}) & A_L^{k'+1} + A_L^{k'} + \cdots + A_L^{\frac{k'+1}{2}} \end{pmatrix};$$

$$A_M^{k'+2} = \begin{pmatrix} A^{k'+1} + A^{k'} + \cdots + A^{\frac{k'+3}{2}} & (A^{k'+1} + A^{k'} + \cdots + A^{\frac{k'+1}{2}})B_0 \\ B_1^T(A^{k'+1} + A^{k'} + \cdots + A^{\frac{k'+1}{2}}) & A_L^{k'+2} + A_L^{k'+1} + \cdots + A_L^{\frac{k'+3}{2}} \end{pmatrix}.$$

By $(5'), (6'), (7')$ and $(8')$, then $A_M^{k'+1} = A_M^{k'+2}$. Hence $k(M(D)) \le k'+1 = k(T(D))+1$.

Case b Let k' be even. Similarly, we can prove that $k(M(D)) \le k(T(D)) + 1$.

Hence we have proved that $k(T(D)) \le k(M(D)) \le k(T(D)) + 1$.

Similarly we can prove that $k(T(D)) \le k(M'(D)) \le k(T(D)) + 1$. Our proof thus follows.

Remark 4 Let $D_4 = (V(D_4), A(D_4)), V(D_4) = \{1, 2, 3, 4\}$, $A(D_4) = \{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$. Then $k(M(D_4)) = k(M'(D_4)) = k(T(D_4)) = 6$. Note that the digraph D_1 in Remark 1 shows that there are digraphs D such that $k(M(D)) = k(T(D)) + 1$ or $k(M'(D)) = k(T(D)) + 1$. Thus the bounds in Theorem 3.5 are obtained.

Corollary 3.6 Let D be a digraph. Then $k(M(D))-1 \le k(M'(D)) \le k(M(D))+1$.

Proof By Theorem 3.5, it follows that $k(T(D)) \le k(M(D)) \le k(T(D)) + 1$, and $k(T(D)) \le k(M'(D)) \le k(T(D))+1$. Hence $k(M(D))-1 \le k(M'(D)) \le k(M(D))+1$.

Remark 5 Let $D_5 = (V(D_5), A(D_5)), V(D_5) = \{1, 2, 3\}, and A(D_5) = \{(1, 2), (2, 2), (2, 3)\}$. Then $k(M(D_5)) = 3, k(M'(D_5)) = 2$. Hence $k(M'(D_5)) = k(M(D_5)) - 1$. This shows that there are digraphs D such that $k(M'(D)) = k(M(D)) - 1$. The digraph D_4 in Remark 4 shows that there are digraphs D such that $k(M'(D)) = k(M(D))$. The digraph D_2 in Remark 2 shows that there are digraphs D such that $k(M'(D)) = k(M(D)) + 1$.

Theorem 3.7 Let D be a digraph. If there are not both sources and sinks in D, then $k(M(D)) \le k(M'(D)) \le k(M(D)) + 1$.

Proof By Corollary 3.6, we only need to prove that $k(M(D)) \le k(M'(D))$. Let $k(M'(D)) = k$; then $A_{M'}^k = A_{M'}^{k+1}$. Hence we only need to prove that $A_M^k = A_M^{k+1}$. We distinguish the following two cases:

Case 1 We suppose that there are no sources in D.

Subcase 1.1 Let k be odd. Since $A^k_{M'} = A^{k+1}_{M'}$, by Lemma 2.3 we have

$$A^k_{M'} = \begin{pmatrix} A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}} & (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}})B_0 \\ B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}}) & A_L^{k-1} + A_L^{k-2} + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} A^{k+1} + A^k + \cdots + A^{\frac{k+1}{2}} & (A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}})B_0 \\ B_1^T(A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}}) & A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix} = A^{k+1}_{M'}.$$

Hence

(1) $A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}} = A^{k+1} + A^k + \cdots + A^{\frac{k+1}{2}}$;

(2) $(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}})B_0 = (A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}})B_0$;

(3) $B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}}) = B_1^T(A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}})$;

(4) $A_L^{k-1} + A_L^{k-2} + \cdots + A_L^{\frac{k+1}{2}} = A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}}$.

We multiply both sides of the equality (4) by A_L; then

$$A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+3}{2}} = A_L^{k+1} + A_L^k + \cdots + A_L^{\frac{k+3}{2}}.$$

Hence

(1′) $A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}} = A_L^{k+1} + A_L^k + \cdots + A_L^{\frac{k+1}{2}}$.

By (4), then

$$(A_L^{k-1} + A_L^{k-2} + \cdots + A_L^{\frac{k+1}{2}})B_1^T = (A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}})B_1^T.$$

by Lemma 2.1, then

(5) $B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k+1}{2}}) = B_1^T(A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}})$.

Let $C_1 = A^{k-1} + A^{k-2} + \cdots + A^{\frac{k+1}{2}}$, and $C_2 = A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}}$. By (5), then $B_1^TC_1 = B_1^TC_2$. Since there are not sources in D, By Lemma 2.8, then $C_1 = C_2$. Hence

(2′) $A^{k-1} + A^{k-2} + \cdots + A^{\frac{k+1}{2}} = A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}}$.

By (2′), (2), (3), and (1′), then

$$A^k_M = \begin{pmatrix} A^{k-1} + A^{k-2} + \cdots + A^{\frac{k+1}{2}} & (A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}})B_0 \\ B_1^T(A^{k-1} + A^{k-2} + \cdots + A^{\frac{k-1}{2}}) & A_L^k + A_L^{k-1} + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}} & (A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}})B_0 \\ B_1^T(A^k + A^{k-1} + \cdots + A^{\frac{k+1}{2}}) & A_L^{k+1} + A_L^k + \cdots + A_L^{\frac{k+1}{2}} \end{pmatrix} = A^{k+1}_M.$$

Hence $k(M(D)) \le k = k(M'(D))$.

Subcase 1.2 Let k be even. Similarly, we can prove $k(M(D)) \le k = k(M'(D))$.

Case 2 We suppose that there are no sinks in D.

Let D^* denote the converse digraph of D. Then there are no sources in D^*. By Case 1, we have $k(M(D^*)) \le k(M'(D^*))$. It is easy to prove that $M(D^*) = M^*(D)$,

$M'(D^*) = M'^*(D)$, $k(M(D)) = k(M^*(D))$ and $k(M'(D)) = k(M'^*(D))$. Hence $k(M(D)) \leq k(M'(D))$.

By Cases 1 and 2, Theorem 3.7 holds.

By Theorem 3.7, the following corollary is obvious.

Corollary 3.8 Let D be a strongly connected digraph; then

$$k(M(D)) \leq k(M'(D)) \leq k(M(D)) + 1.$$

4 Some problems

In this section, we will pose some problems on the classification of digraphs by their indices of convergence.

Let D be a digraph. Theorem 3.5 shows that $k(T(D)) \leq k(M(D)) \leq k(T(D))+1$ and $k(T(D)) \leq k(M'(D)) \leq k(T(D)) + 1$. Corollary 3.6 shows that $k(M(D)) - 1 \leq k(M'(D)) \leq k(M(D)) + 1$ and Theorem 3.7 shows that if there are not both sources and sinks in D then $k(M(D)) \leq k(M'(D)) \leq k(M(D)) + 1$. Hence the following problems of characterization of digraphs are worth considering.

Problem 1 Determine the digraphs D such that $k(M(D)) = k(T(D))$ or $k(M(D))$=$k(T(D)) + 1$, respectively.

Problem 2 Determine the digraphs D such that $k(M'(D)) = k(T(D))$ or $k(M'(D))$=$k(T(D) + 1$, respectively.

Problem 3 Determine the digraphs D such that $k(M'(D)) = k(M(D)) - 1$, or $k(M'(D))$=$k(M(D))$, or $k(M'(D))$=$k(M(D)) + 1$, respectively.

Problem 4 Suppose that there are not both sources and sinks in D. Determine the digraphs D such that $k(M'(D)) = k(M(D))$ or $k(M'(D))$=$k(M(D))$+1, respectively.

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