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Altitude of small complete and complete bipartite graphs

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Abstract

An *edge-ordering* of a graph $G = (V, E)$ is a one-to-one function f from E to the set of positive integers. A path of length k in G is called a (k, f)-*ascent* if f increases along the edge sequence of the path. The *altitude* $\alpha(G)$ of G is the greatest integer k such that for all edge-orderings f, G has a (k, f)-ascent.

We obtain upper bounds for the altitude of complete and complete bipartite graphs, and exact values for some small graphs.

1 Introduction

A one-to-one function f from E to the set of positive integers is called an *edge-ordering* of the graph $G = (V, E)$. For $e \in E$, we call $f(e)$ the *label* of e, and use e and $f(e)$ interchangeably. Denote the set of all edge-orderings of G by $\mathcal{F}$. For $f \in \mathcal{F}$, a path of G for which f increases along the edge sequence, is called an f-*ascent* of G, and a (k, f)-*ascent* if it has length k. The *height* $h(f)$ of f is the maximum length of an f-ascent. The parameter of principal interest in this work is $\alpha(G)$, the *altitude* of G, defined by

$$\alpha(G) = \min_{f \in \mathcal{F}} h(f).$$

Observe that $\alpha(G)$ is the greatest integer k such that G has a (k, f)-ascent for each edge-ordering $f \in \mathcal{F}$.

Clearly, $\alpha(G) \geq 2$ for any graph G with a vertex of degree at least two. It is also evident that if H is a subgraph of G, then $\alpha(H) \leq \alpha(G)$. The altitude of some classes of graphs is easy to determine, for example, (trivially) $\alpha(K_2) = 1$, $\alpha(K_3) = 2$ (since K_3 has no path of length three), $\alpha(C_{2n}) = 2$ and $\alpha(C_{2n+1}) = 3$ for all $n \geq 2$. Let E_1, E_2, E_3 be (the edge sets of) any 1-factorization of K_4 and f an edge-ordering such that the labels of the edges in E_i are $2i - 1$ and $2i$, $i = 1, 2, 3$. It is easy to see that $h(f) = 2$ and so $\alpha(K_4) = 2$.

The problem of determining $\alpha(K_n)$ was first posed by Chvatál and Komlós [5], and Calderbank, Chung and Sturtevant [4] obtained the asymptotic bound

$$\alpha(K_n) \leq (\tfrac{1}{2} + o(1))n.$$

The general bounds

$$\tfrac{1}{2}(\sqrt{4n-3} - 1) < \alpha(K_n) < \tfrac{3n}{4}$$

were obtained by Graham and Kleitman [6], but the proof of the upper bound is incorrect.

Calderbank et al. state that the upper bound of $\frac{3n}{4}$ has been improved to $\frac{7n}{12}$ by Alspach, Heinrich and Graham. However, this improved bound does not hold for $n = 5, ..., 8$ (see Section 4), and its proof does not appear in the literature.

In this paper we prove that for $m \leq n$,

$$\alpha(K_{m,n}) \leq \min\{2m, \lceil \tfrac{3}{2} \lceil \tfrac{n}{2} \rceil \rceil \},$$

and for n even,

$$\alpha(K_{n-1}) \leq \alpha(K_n) \leq \begin{cases} \lceil \frac{11n}{16} \rceil & \text{if } n \equiv 10 \pmod{16} \\ \lfloor \frac{11n-1}{16} \rfloor & \text{otherwise.} \end{cases}$$

These bounds enable us to determine $\alpha(K_{m,n})$ and $\alpha(K_n)$ for certain small values of m and n. For work on the altitude of other classes of graphs the reader is referred to [1, 8, 9].

2 Determination of upper bounds for altitude

The principal results of Sections 3 and 4 will be established using the methods of this section. These techniques have also been exploited in [4, 6, 8, 9].

Let $\mathbf{P} = (E_1, ..., E_t)$ be an ordered partition of the edge set E of G and let f be any edge-ordering of G satisfying

$$e_i \in E_i \text{ and } e_j \in E_j, \text{ where } i < j, \text{ implies } f(e_i) < f(e_j).$$

Such an edge-ordering is called $\mathbf{P}$-*consistent.*

For $i = 1, ..., t$ we use the abbreviations $f_i = f \restriction E_i$ and $G_i = G[E_i]$ (the subgraph of G induced by E_i). Observe that f_i is an edge-ordering of G_i. In the edge-sequence X of an f-ascent of G, for each $i < j$, edges in E_i precede edges in E_j. Hence $X = X_1, ..., X_t$, where X_i (possibly empty) is an f_i-ascent of G_i.

Proposition 1 *For any graph G, $\alpha(G) \leq \sum_{i=1}^{t} \alpha(G_i)$.*

Proof. Let f be $\mathbf{P}$-consistent and satisfy $h(f_i) = \alpha(G_i)$ for each $i = 1, ..., t$. Suppose that λ is an f-ascent of G with maximum length $h(f)$. Then

$$\alpha(G) \leq h(f) = \sum_{i=1}^{t} |X_i| \leq \sum_{i=1}^{t} h(f_i) = \sum_{i=1}^{t} \alpha(G_i). \qquad \blacksquare$$

In many cases judicious choices of the ordered partition $\mathbf{P}$ and the $\mathbf{P}$-consistent edge-ordering f enable us to improve the upper bound of Proposition 1. More specifically, these choices may allow us to find consecutive sets $E_j, ..., E_k$ so that the maximum length of an ascent in $f \upharpoonright (E_j \cup ... \cup E_k)$ is equal to $\Sigma_{i=j}^{k} \alpha(G_i) - c$ for some $c > 0$. In such a case it is easily seen that the bound may be improved to $\Sigma_{i=1}^{t} \alpha(G_i) - c$. Situations of this type involving just two consecutive sets E_i, E_{i+1} of the partition include:

(*i*) G_i and G_{i+1} are vertex disjoint. In this case no edge of E_{i+1} may follow an edge of E_i in an f-ascent λ. Hence λ (considered as an edge set) satisfies $\lambda \cap E_i = \phi$ or $\lambda \cap E_{i+1} = \phi$, and the upper bound may be decreased by $\min\{\alpha_i, \alpha_{i+1}\}$.

(*ii*) Property (*i*) does not hold, but there is no vertex which is both the terminal vertex of an (α_i, f_i)-ascent in G_i and the initial vertex of an (α_{i+1}, f_{i+1})-ascent in G_{i+1}.

(*iii*) Properties (*i*) and (*ii*) do not hold. However, paths which negate Property (*ii*) have more than one common vertex.

The above method easily establishes the following result. Part (*iii*) was proved in [9].

Proposition 2 (*i*) *If G has components $G_1, ..., G_t$, then $\alpha(G) = \max_{i=1}^{t}\{\alpha(G_i)\}$.*

(*ii*) *If $\mathbf{P} = (E_1, ..., E_t)$ is a partition of E such that G_i is 1-regular for each i, then $\alpha(G_i) = 1$ and hence $\alpha(G) \leq t$.*

(*iii*) *If G has maximum degree Δ, then $\alpha(G) \leq \Delta + 1$.*

Proof. Statements (*i*) and (*ii*) are obvious and (*iii*) follows from Vizing's theorem (cf. [2]) that states that E can be partitioned into at most $\Delta + 1$ matchings. ■

Our final observation of this section relates altitude and the independence number β.

Proposition 3 *For any graph G of order n, $\alpha(G) \leq 2(n - \beta)$.*

Proof. If I is an independent set of G, then any vertex in $V - I$ is incident with at most two edges of any path λ, and λ contains no edges of $G[I]$. ■

3 Altitude of complete bipartite graphs

It is easy to see that for $m \leq n$, $E(K_{m,n})$ can be partitioned into n sets $E_1, ..., E_n$ such that $|E_i| = m$ and $K_{m,n}[E_i]$ is 1-regular for each i. Thus by Proposition 2(*ii*), $\alpha(K_{m,n}) \leq n$. Also, by Proposition 3, $\alpha(K_{m,n}) \leq 2m$. We therefore have

Proposition 4 *If $m \leq n$, then $\alpha(K_{m,n}) \leq \min\{2m, n\}$.*

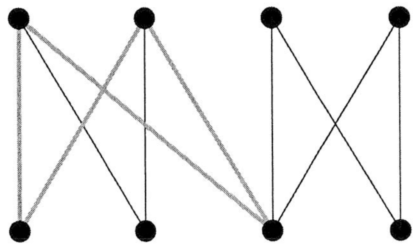

Figure 1: Two edges in E_1 followed by two edges in E_2 form a C_4 in $K_{4,4}$.

Another simple application of the partition method of Section 2 decreases the bound of Proposition 4 in many cases.

Theorem 5 *(i)* $\alpha(K_{2n,2n}) \leq \lceil 3n/2 \rceil$.
(ii) If $m \leq n$, *then* $\alpha(K_{m,n}) \leq \min\{2m, \lceil 3 \lceil \frac{n}{2} \rceil /2 \rceil\}$.

Proof. *(i)* Let $(F_1, ..., F_n)$ be any ordered 1-factorization of $K_{n,n}$ and let $\mathbf{P} = (E_1, ..., E_n)$ be the ordered 2-factorization of $K_{2n,2n}$ obtained by replacing each edge of F_i with a 4-cycle. Note that for each i, $G_i \cong nC_4$ and so by Proposition 2(i), $\alpha(G_i) = \alpha(C_4) = 2$. Let f be a $\mathbf{P}$-consistent edge-ordering of $K_{2n,2n}$ with $h(f) = k$, and such that $h(f_i) = 2$ for each $i = 1, ..., n$. If λ is a (k, f)-ascent of $K_{2n,2n}$, then λ contains edges from at most one 4-cycle in each 2-factor and so λ contains at most two edges in each E_i. Suppose that for some i, λ contains two edges in E_i and two edges in E_{i+1}. Then, as illustrated for $K_{4,4}$ in Figure 1, where only the edges of G_1 and λ (grey thicker edges) are shown, λ contains a 4-cycle, a contradiction. Thus whenever λ contains two edges of E_i, it contains at most one edge of E_{i+1} and it follows that $k \leq \lceil \frac{3n}{2} \rceil$.
(ii) The result follows immediately from *(i)* and Proposition 4. ■

Corollary 6 *(i)* $\alpha(K_{2,3}) = \alpha(K_{3,3}) = \alpha(K_{2,4}) = \alpha(K_{3,4}) = \alpha(K_{4,4}) = 3$.
(ii) $\alpha(K_{2,n}) = 4$ *for* $n \geq 5$.

Proof. *(i)* As shown in in [1], $\alpha(K_{2,3}) \geq 3$ and the result follows from Theorem 5 and the observation that if H is a subgraph of G, then $\alpha(H) \leq \alpha(G)$.
(ii) The value $\alpha(K_{2,5}) = 4$ was obtained in [8] and Proposition 4 asserts that $\alpha(K_{2,n}) \leq 4$ for all n. ■

Our final result of this section establishes the altitude of some other small complete bipartite graphs.

Theorem 7 *(i) For* $3 \leq m \leq 4$, $5 \leq n \leq 6$, $\alpha(K_{m,n}) = 4$.
(ii) $\alpha(K_{5,5}) = 4$.

Proof. (*i*) Since $\alpha(K_{2,5}) = 4$, it suffices to show that $\alpha(K_{4,6}) \leq 4$. Let H be the 6-vertex edge-ordered tree in Figure 2. It is easy to see that $\alpha(H) = 2$, but we will use the given ordering of $E(H)$ of height three. Consider the ordered partition $\mathbf{P} = (E_1, E_2, E_3)$ of $E(K_{4,6})$, where $H_i = K_{4,6}[E_i] \cong C_4 \cup H$, $i = 1, 3$, $H_2 \cong C_4 \cup P_3$, and $K_{4,6}[E_i \cup E_j]$, $i \neq j$, are the graphs $G_{i,j}$ shown in Figure 2. (The edges in E_i, $i < j$, are thinner than those in E_j and the edges in E_2 are grey.) Let f be an ordering of $K_{4,6}$ such that

- f is $\mathbf{P}$-consistent,
- the restriction of f to E_2 and to each C_4 component of H_i, $i = 1, 3$, has height two,
- in the component $H_1[\{a, b, c, d, e\}] \cong H$ of H_1 we have $a < b < c$ and $e < d < c$ (the given edge-ordering of H satisfies this ordering),
- in the component $H_3[\{p, q, r, s, t\}] \cong H$ of H_3 we have $r < q < p$ and $r < s < t$ (for example, subtract each given label of H from 25),
- $k < l$ in E_2.

In this and subsequent proofs, $(uvw...)$ will denote an f-ascent whose edges (in sequence) have the labels $u, v, w, \ldots$. Suppose $\lambda = (uvxyz)$ is a $(5, f)$-ascent in $K_{4,6}$. If $u, v, x \in E_1$, then $(uvx) = (edc)$, so $y \in \{p, q, r\} \subseteq E_3$ and hence $z \in E_3$. But then $\lambda = (edcrs)$, which is not a path. Similarly, λ does not contain the subpath (rst). Consequently λ contains edges in each E_i, hence $u \in E_1$, $x \in E_2$ and $z \in E_3$.

Suppose $v \in E_1$. The ordering imposed on H_1 and the fact that $x \in E_2$ show that the initial vertex of (uvx) is one of A, B, D or R, and the terminal vertex is one of C, D or Y.

If the initial vertex is A or B, then the terminal vertex is C or D, and $y \notin E_2$, for otherwise λ contains a 4-cycle. Hence $y \in E_3$. However, as can be seen from $G_{2,3}$, then $y \in \{p, t\}$, and with the given ordering of H_3, $(uvxy)$ cannot be extended to a $(5, f)$-ascent.

If the initial vertex of (uvx) is D, then $(uvx) = (abk)$ and so $y \in \{l, s, t\}$. But if $y = l$, then the only possibility for z is $z = q$, a contradiction since $(bklq)$ is a 4-cycle, if $y = t$, then $(abkt)$ is a 4-cycle, and $r < s$ implies that $(abks)$ cannot be extended.

If the initial vertex of (uvx) is R, then $x = l$, and since $k < l$, $y \in E_3$. Thus $y \in \{t, s\}$, but $(uvls)$ is a 4-cycle and $(uvlt)$ cannot be extended to a $(5, f)$-ascent.

Hence there is no $(5, f)$-ascent with $v \in E_1$. We conclude that $v \in E_2$ and deduce that $y \in E_3$.

Consider the possible $(4, f)$-ascents $(vxyz)$ in the graph $G_{2,3}$. To avoid 4-cycles, $(vxyz)$ does not have initial vertex A or B. If the initial vertex is C, then $y = t$ and since $s < t$, (vxt) cannot be extended to a $(4, f)$-ascent. Similarly, since $p > q > r$, the initial vertex is not D. The only other possible initial vertex is Q, in which case $(vx) = (kl)$, which extends uniquely to the $(4, f)$-ascent $(klqp)$. The only edge in E_1 adjacent to k is b. However, $(bklqp)$ contains a 4-cycle, and with this contradiction the proof is complete.

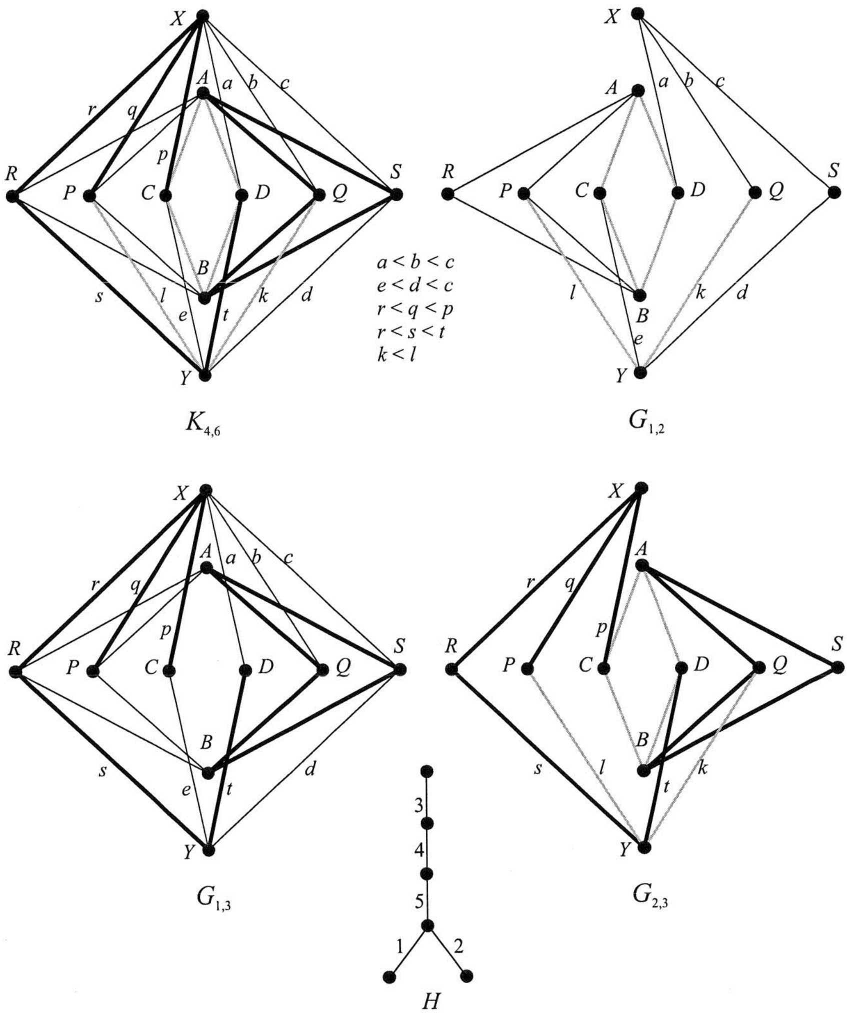

Figure 2: An edge-ordering of $K_{4,6}$ of height four

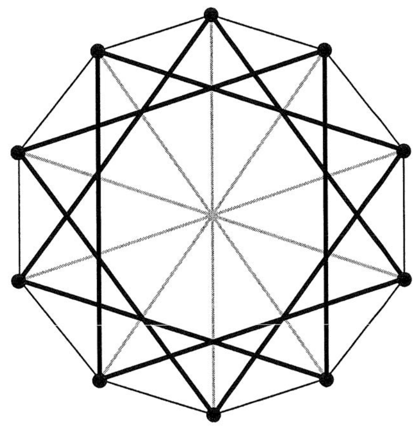

Figure 3: An edge-ordering of $K_{5,5}$ of height four

(ii) Let $\mathbf{P} = (E_1, E_2, E_3)$ be the ordered partition of $E(K_{5,5})$ such that for $i = 1, 3$, $K_{5,5}[E_i] \cong C_{10}$ and $K_{5,5}[E_2] \cong 5K_2$. Suppose f is a $\mathbf{P}$-consistent edge-ordering such that for $i = 1, 3$, $h(f_i) = 2$. Any $(5, f)$-ascent contains one edge in E_2 and two edges in each of E_1 and E_3. However, any such choice of edges contains a 4-cycle (see Figure 3, which uses the same convention as Figure 2) and hence does not form a path of length 5. Therefore $\alpha(K_{5,5}) \leq h(f) \leq 4$. Since $\alpha(K_{5,5}) \geq \alpha(K_{2,5}) = 4$ (Corollary 6(ii)), we have $\alpha(K_{5,5}) = 4$ as required. ■

4 Altitude of complete graphs

Since there exist K_4-resolvable block designs of K_n for all $n \equiv 4 \pmod{12}$ (see [7]), there exists, for each $k \geq 1$, a factorization $G_1 \oplus \cdots \oplus G_{4k+1}$ of K_{12k+4} such that $G_i \cong (3k+1)K_4$ for each $i = 1, ..., 4k+1$. Thus by Propositions 1 and 2 and the fact that $\alpha(K_4) = 2$, we have $\alpha(K_{12k+4}) \leq (4k+1)\alpha(K_4) = 8k+2$, that is, $\alpha(K_n) \leq \frac{2}{3}(n-1)$ for $n \equiv 4 \pmod{12}$. Hence in general, since $\alpha(H) \leq \alpha(G)$ if H is a subgraph of G,

$$\alpha(K_n) \leq \tfrac{2}{3}\left(12\left\lceil \tfrac{n-4}{12} \right\rceil + 3\right). \tag{1}$$

We now use the bound for $\alpha(K_{n,n})$ in Theorem 5(ii) to establish another upper bound for $\alpha(K_n)$. Although this bound is better than the bound in (1) for only finitely many values of n ($n = 270$ being the largest integer for which the bound in Corollary 9 is smaller than that in (1)), it is required to establish exact values of $\alpha(K_n)$ for some small n.

Theorem 8 *For any $n \geq 2$, $\alpha(K_{2n}) \leq n + \left\lceil \frac{\alpha(K_{n,n})-2}{2} \right\rceil$.*

Proof. Consider the ordered partition $\mathbf{P} = (E_1, E_2, E_3)$ of $E(K_{2n})$ with $K_{2n}[E_1] \cong K_{n,n}$ and $K_{2n}[E_2] \cong K_{2n}[E_3] \cong K_n$. Let f be a $\mathbf{P}$-consistent edge-ordering of K_{2n}. A (k, f)-ascent λ cannot contain edges from both E_2 and E_3 since $K_{2n}[E_2]$ and $K_{2n}[E_3]$ are vertex-disjoint and the edges (in E_1) between these two sets have smaller labels than any edges in $E_2 \cup E_3$. Without loss of generality say λ contains no edges in E_3 and x edges in E_1. These x edges use $\lceil \frac{x+1}{2} \rceil$ vertices in $K_{2n}[E_2]$ and thus λ contains at most $n - \lceil \frac{x+1}{2} \rceil$ edges (the number of vertices remaining) in E_2. Therefore $k \leq x + n - \lceil \frac{x+1}{2} \rceil = n + \lceil \frac{x-2}{2} \rceil$. The maximum of this expression over the permitted values of x occurs when $x = \alpha(K_{n,n})$. ■

Theorem 5(ii) immediately gives

Corollary 9 *For any $n \geq 2$, $\alpha(K_{2n-1}) \leq \alpha(K_{2n}) \leq n + \left\lceil \frac{\lceil 3\lceil n/2 \rceil /2 \rceil - 2}{2} \right\rceil$. Thus, for m even,*

$$\alpha(K_{m-1}) \leq \alpha(K_m) \leq \begin{cases} \lceil \frac{11m}{16} \rceil & \text{if } m \equiv 10 \pmod{16} \\ \lfloor \frac{11m-1}{16} \rfloor & \text{otherwise.} \end{cases}$$

Using the method of the proof of Theorem 8 and the value $\alpha(K_{5,5}) = 4$ obtained in Theorem 7(ii), we also have the following result, an improvement on the bound above.

Corollary 10 $\alpha(K_9) \leq \alpha(K_{10}) \leq 6$.

We next provide proof that $\alpha(K_5) = 3$ and $\alpha(K_6) = 4$.

Proposition 11 $\alpha(K_5) = 3$.

Proof. Let (E_1, E_2) be any 2-factorization of K_5 and f an edge-ordering such that the edges of the 5-cycle induced by E_1 (E_2, respectively) are labelled, in sequence, 1,5,2,4,3 (6,10,7,9,8). Any $(4, f)$-ascent contains edges from both E_1 and E_2 (because $h(f_i) = 3$ for $i = 1, 2$). Let λ be a (k, f)-ascent in K_5. If v_1, v_2, v_3, v_4 is the vertex sequence of λ in $C_5[E_1]$, then both edges in E_2 incident with v_4 are incident with v_1 or v_2, so $k = 3$. If v_1, v_2, v_3 is the vertex sequence of λ in $C_5[E_1]$, then the only edge in E_2 incident with v_3 but not v_1, is followed by an edge of E_2 incident with v_2. Hence again $k = 3$. By symmetry, no path of length two or three in $C_5[E_2]$ can be extended to a $(4, f)$-ascent, and so $\alpha(K_5) \leq 3$. Since C_5 is a subgraph of K_5, it follows that $\alpha(K_5) = 3$. ■

Theorem 12 $\alpha(K_6) = 4$.

Proof. The upper bound of Corollary 9 gives $\alpha(K_6) \leq 4$ and it remains to show that there is no edge-ordering of K_6 of height three. Suppose to the contrary that f is an edge-ordering of K_6 with $h(f) = 3$ and (abc) is a $(3, f)$-ascent as in Figure 4(a). Without loss of generality, assume $y_2 < y_1$. We will repeatedly use the following lemma. The trivial proof of each part follows its statement; proofs of some parts require preceding parts.

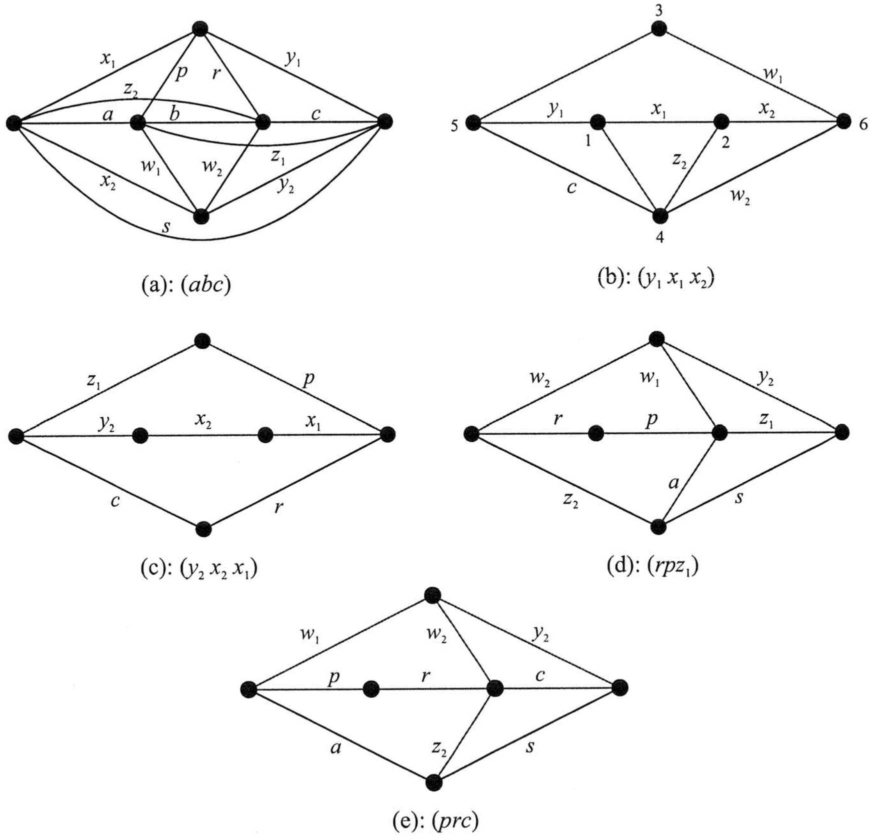

Figure 4: 3-ascents in K_6 in the proof of Theorem 12

Lemma 12.1 *Considering* (abc), *we have*

(i)	$y_1 < c$,	$y_2 < c$	(*avoid* $(abcy_1)$, $(abcy_2)$)
(ii)	$x_1 > a$,	$x_2 > a$	(*avoid* (x_1abc), (x_2abc))
(iii)	$x_1 > y_1$,	$x_2 > y_2$	(*avoid* (ax_1y_1c), (ax_2y_2c))
(iv)	$z_2 < x_1$		(*avoid* $(y_2y_1x_1z_2)$)
(v)	$w_1 > y_2$,	$w_2 > y_2$	(*avoid* $(w_1y_2y_1x_1)$, $(w_2y_2y_1x_1)$)
(vi)	$w_1 > a$		(*avoid* $(y_2w_1ax_1)$)
(vii)	$w_2 > z_2$		(*avoid* $(y_2w_2z_2x_1)$).

We also prove the following two lemmas.

Lemma 12.2 $x_2 < x_1$.

Proof. Suppose the contrary, i.e. $x_1 < x_2$. We apply Lemma 12.1 to $(y_1x_1x_2)$. It is helpful to redraw f as in Figure 4(b). If $w_2 < w_1$, Lemma 12.1(v) gives $z_2 > w_2$ which contradicts Lemma 12.1(vii). Therefore $w_1 < w_2$. Lemma 12.1(iii) gives $c > w_2$ and we have (aw_1w_2c), a contradiction. □

Lemma 12.3 (i) $r < x_1$ (ii) $z_1 > p$ (iii) $c > r$ (iv) $p < r$.

Proof. To establish (i), (ii) and (iii), we apply Lemma 12.1 to $(y_2x_2x_1)$. The existence of this $(3, f)$-ascent is asserted by Lemmas 12.1(iii) and 12.2. We redraw $(y_2x_2x_1)$ as in Figure 4(c). By Lemma 12.1(i), $r < x_1$. By Lemma 12.1(iii), $z_1 > p$ and $c > r$.

(iv) Suppose that $r < p$. Then by (ii) we have the path (rpz_1) which is drawn in Figure 4(d). By Lemma 12.1(vii), $w_2 > z_2$ and it follows from Lemma 12.2 that $s < y_2$. Now, by Lemma 12.1(vi), $w_1 > a$. By Lemma 12.1(v) applied to (rpz_1), $a > s$ and by Lemma 12.3 applied to (rpz_1), $w_1 < w_2$. Hence (saw_1w_2), a contradiction.□

We complete the proof of Theorem 12. By Lemma 12.3 we have (prc), redrawn in Figure 4(e). Since $w_1 > a$, it follows from Lemma 12.2 that $s < y_2$. We apply the preceding lemmas to (prc): By Lemma 12.1(v), $z_2 > s$, by Lemma 12.3(i), $w_2 < w_1$, and the original Lemma 12.1(vii) states that $w_2 > z_2$. Therefore $(sz_2w_2w_1)$, the final contradiction which shows that no edge-ordering with height three exists, hence $\alpha(K_6) = 4$. ■

Using a computer program we found that $\alpha(K_7) = 5$. (In fact, we found an edge-ordering of K_7 with exactly one 5-ascent.) Hence, using the upper bound in Corollary 9, we have

Corollary 13 $\alpha(K_7) = \alpha(K_8) = 5$.

We also found an edge-ordering of K_{11} without 8-ascents, hence $\alpha(K_{11}) \leq 7$.

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References

[1] A. Bialostocki and Y. Roditty, A monotone path in an edge-ordered graph, *Internat. J. Math. & Math. Sci.* **10**(1987), 315–320.

[2] B. Bollobás, *Extremal Graph Theory.* Academic Press, 1978.

[3] A. P. Burger, E. J. Cockayne and C. M. Mynhardt, Altitude of complete and complete bipartite graphs, *University of South Africa Research Report* **341/2003(1)**, 2003.

[4] A. R. Calderbank, F. R. K. Chung and D. G. Sturtevant, Increasing sequences with non-zero block sums and increasing paths in edge-ordered graphs, *Discrete Math.* **50**(1984), 15–28.

[5] V. Chvatál and J. Komlós, Some combinatorial theorems on monotonicity, *Canad. Math. Bull.* **14**(1971), 151–157.

[6] R. L. Graham and D. J. Kleitman, Increasing paths in edge ordered graphs, *Periodica Mathematica Hungarica* **3**(1973), 141–148.

[7] H. Hanani, D. K. Ray-Chaudhuri and R. M. Wilson, On resolvable designs, *Discrete Math.* **3**(1972), 343–357.

[8] Y. Roditty, B. Shoham and R. Yuster, Monotone paths in edge-ordered sparse graphs, *Discrete Math.* **226**(2001), 411–417.

[9] R. Yuster, Large monotone paths in graphs with bounded degree, *Graphs Combin.* **17**(2001), 579–587.

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