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On some finite linear spaces with few lines

Vito Napolitano*

Università degli studi della Basilicata
Dipartimento di Matematica
Campus Macchia Romana
Contrada Macchia Romana – 85100 POTENZA
ITALY
vnapolitano@unibas.it

Abstract

A linear space with few lines is a finite linear space in which the number b of lines is a bit larger than the number v of points (K. Metsch, Finite linear spaces with few lines, Lec. Notes in Math. 1490, Springer-Verlag, 1991). In this paper some new classes of finite linear spaces with few lines are investigated and classified.

1 Introduction

A *linear space* is a pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a set of *points* and $\mathcal{L}$ is a family of subsets of $\mathcal{P}$, called *lines*, such that: *any two points are on a unique line, each line has at least two points and there are at least two lines.*

If $(\mathcal{P}, \mathcal{L})$ is a finite linear space, that is $|\mathcal{P}| < \infty$, the *degree* of a point p is the number $[p]$ of lines through p and the *length* of a line L is its cardinality. The numbers $v = |\mathcal{P}|$, $b = |\mathcal{L}|$, $[p]$ for all $p \in \mathcal{P}$, and $|\ell|$ for all $\ell \in \mathcal{L}$, are called the *parameters* of $(\mathcal{P}, \mathcal{L})$. Moreover let m and k denote the minimum point degree and the maximal line length, respectively.

A finite linear space is *irreducible* if any line has length at least three.

A *projective plane* is an irreducible linear space in which any two lines intersect in a point.

A *near-pencil* on v points is a linear space on v points with a line of length $v-1$.

Clearly the near-pencils and the projective planes are the only linear spaces in which any two lines intersect in a point.

An (h, k)-*cross* $(h, k \geq 3)$ is the linear space on $v = h + k - 1$ points, with two lines of length h and k respectively, intersecting in a point p, and any line not on p has length 2.

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An *affine plane* is a linear space such that for any point-line pair (p, ℓ), with $p \notin \ell$, the number of lines passing through p and missing ℓ is exactly one.

Let $(\mathcal{P}, \mathcal{L})$ be a linear space, and let X be a set of points of $\mathcal{P}$ such that outside of X there are at least three non-collinear points. Deleting X from $(\mathcal{P}, \mathcal{L})$, one obtains a new linear space called the *complement* of X in $(\mathcal{P}, \mathcal{L})$.

For example, the complement of a line in a projective plane is an affine plane.

A *punctured* (respectively *doubly-punctured*) linear space $(\mathcal{P}, \mathcal{L})$ is the complement of a point (respectively two points) in $(\mathcal{P}, \mathcal{L})$.

In 1948 de Bruijn and Erdős [5] proved (the *Fundamental Theorem*) that each finite linear space has at least as many lines as points, with equality only if it is a projective plane or a near pencil. This theorem was the starting point for the study of the following problem.

Characterize or classify finite linear spaces with $b = v + s$ for a fixed value of s.

Since the early 1970s a number of results have been obtained on this question (see for example [1, 3, 8, 9, 10, 15, 18, 19, 23]).

In this paper we give a complete classification of finite linear spaces with $b-v \leq k$ (Theorem 2.5), (k being the maximum line length), and as a corollary we also obtain the classification of finite linear spaces with $b - v \leq 1 + \sqrt{v}$ (Theorem 2.6).

2 Preliminary results and notions

In this section we recall some definitions and results on finite linear spaces we need in the following.

Throughout the paper we assume that $k \geq m+1$. If $k \leq m$, then we can use the classification of Lemma 2.1 below.

The *order* of a finite linear space $(\mathcal{P}, \mathcal{L})$ is the integer n such that $n+1 = \max_{p \in \mathcal{P}}[p]$.

If $(\mathcal{P}, \mathcal{L})$ is a finite projective plane, then there is an integer n such that $[p] = |\ell| = n + 1$ for all $p \in \mathcal{P}$ and for all $\ell \in \mathcal{L}$, and so the order of a finite projective plane is the length of one of its lines less one.

As an affine plane is the complement of a line in a projective plane, we have $[p] = |\ell| + 1 = n + 1$ for all $p \in \mathcal{P}$ and for all $\ell \in \mathcal{L}$, so the order of an affine plane is the size of any of its lines.

Two lines ℓ and ℓ' of a linear space $(\mathcal{P}, \mathcal{L})$ are *parallel* if $\ell = \ell'$ or $\ell \cap \ell' = \emptyset$.

In an affine plane $(\mathcal{P}, \mathcal{L})$, the set of lines parallel to a given line ℓ partitions the set of points of $(\mathcal{P}, \mathcal{L})$, and this set of lines is called a *parallel class* of ℓ. A parallel class consists of n lines.

An *affine plane with a point at infinity* is a linear space obtained from an affine plane by adding a "new point", called a *point at infinity*, to the point set of the affine plane and to all the lines of the parallel class of a line ℓ.

An *affine plane with a linear space at infinity* (or *an inflated affine plane*) $(\mathcal{P}, \mathcal{L})$ consists of an affine plane together with a linear space imposed on some of its infinite points.

If the linear space at infinity is a near-pencil or a projective plane, then $(\mathcal{P}, \mathcal{L})$ is called a *projectively* inflated affine plane.

If the linear space at infinity consists of all points at infinity of the affine plane, then $(\mathcal{P}, \mathcal{L})$ is called a *complete* inflated affine plane.

Let $(\mathcal{P}, \mathcal{L})$ be a linear space; the linear space obtained from $(\mathcal{P}, \mathcal{L})$ by adding a new point and connecting it to every old point by a line of length 2 is called a *simple extension of type* 1 [10].

The linear space obtained from a linear space $(\mathcal{P}, \mathcal{L})$ by adding a new point to some line L of $(\mathcal{P}, \mathcal{L})$ and connecting it to every point outside of L by a line of length 2 is called a *simple extension of type* 2 [10].

In both cases the new point is called the *special point* of the extension.

A *subspace* of a linear space $(\mathcal{P}, \mathcal{L})$ is a linear space $(\mathcal{P}', \mathcal{L}')$ with $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{L}' = \{\ell \cap \mathcal{P}' \mid \ell \in \mathcal{L}\}$.

Suppose that $(\mathcal{P}, \mathcal{L})$ has $s \geq 1$ subspaces $(\mathcal{P}_j, \mathcal{L}_j)$, $j = 1, \ldots, s$ and that there exists a point q with $\mathcal{P}_i \cap \mathcal{P}_j = \{q\}$ for $i \neq j$. If $s = 1$ then we suppose furthermore that $q \in \mathcal{P}_1$ and $(\mathcal{P}_1, \mathcal{L}_1) \neq (\mathcal{P}, \mathcal{L})$. Then we can smooth $(\mathcal{P}, \mathcal{L})$ in the way we replace $(\mathcal{P}_j, \mathcal{L}_j)$ by a line (i.e. we remove the lines of $(\mathcal{P}_j, \mathcal{L}_j)$ and adjoin the set $\mathcal{P}_j$ as a new line), $j = 1, \ldots, s$, and obtain a linear space $(\mathcal{P}', \mathcal{L}')$. Suppose that $(\mathcal{P}', \mathcal{L}')$ has $n^2 + n + 1$ lines and that $(\mathcal{P}', \mathcal{L}')$ can be extended to a projective plane π of order n. Then $(\mathcal{P}, \mathcal{L})$ is called an *s-fold inflated projective plane of order* n [10]. In $(\mathcal{P}', \mathcal{L}')$, the point q lies on the lines $\mathcal{P}_j$ and on $n + 1 - s$ further lines $L_0, \ldots, L_{n-s}$. If $n + 1 - d_j$ is the length of L_j in $(\mathcal{P}, \mathcal{L})$, then $d_0 + \cdots + d_{n-s}$ is called the *deficiency* of $(\mathcal{P}, \mathcal{L})$.

The spaces $(\mathcal{P}_j, \mathcal{L}_j)$ are called the *main subspaces* of $(\mathcal{P}, \mathcal{L})$.

Hence an s-fold projective plane with deficiency d can be obtained from a projective plane of order n as follows: we fix a point q and s lines $H_1, \ldots, H_s$ through q. For $j \in \{1, \ldots, s\}$, we remove the line H_j and some of the points of H_j (but not q) and impose a linear space $(\mathcal{P}_j, \mathcal{L}_j)$ on the remaining points of H_j. Furthermore, we remove d points which do not lie on any line H_j in such a way that we do not produce lines of size less than 2. Notice that 1-fold inflated projective planes with deficiency 0 are the inflated affine planes.

2.1 Restricted linear spaces

A finite linear space is called *restricted* if it fulfills the inequality $b - v \leq \sqrt{v}$. In the 1970's De Witte [22] started the study of restricted linear spaces, and after him J. Totten [18] was able to classify all restricted linear spaces proving the following famous theorem.

Theorem 2.1 (Totten, (1976), [18]) *Suppose that* $(\mathcal{P}, \mathcal{L})$ *is a linear space satisfying* $b - v \leq \sqrt{v}$. *Then the following hold.*

1) $(\mathcal{P}, \mathcal{L})$ *is a near pencil.*

2) $(\mathcal{P}, \mathcal{L})$ *is a projective plane of order* n *with at most* n *points deleted but not more than* $n - 1$ *from the same line.*

3) $(\mathcal{P},\mathcal{L})$ is an affine plane, or an affine plane with one infinite point, or a punctured affine plane with one infinite point.

4) $(\mathcal{P},\mathcal{L})$ is a complete projectively inflated affine plane.

5) $(\mathcal{P},\mathcal{L})$ is the $(3,4)$–cross.

2.2 Weakly restricted linear spaces

A finite linear space with $b-v \leq \sqrt{b}$ is called *weakly–restricted* [10]. The classification of such linear spaces is due to Metsch [10], who proved the following classification theorem.

Theorem 2.2 (Metsch, (1991) [10], Thm. 8.6, pp. 79–84) *Suppose that $(\mathcal{P},\mathcal{L})$ is a linear space satisfying $b-v \leq \sqrt{b}$, with $n^2-n+2 \leq v \leq n^2+n+1$. Then $(\mathcal{P},\mathcal{L})$ is one of the following linear spaces.*

1) A restricted linear space.

2) An affine plane with a punctured projective plane on n or $n+1$ points at infinity.

3) A complete projectively inflated punctured affine plane.

4) An affine plane of order 4 or 5 with the $(3,3)$–cross at infinity.

5) The linear space obtained from the projective plane of order 3 by deleting two lines, their point of intersection, and two more points from each of these lines.

6) The $(3,5)$-cross.

7) The linear space on $v=7$ points, $b=10$ lines, with a line of length 4, three of length 3 and the remaining of length 2.

8) The linear space on $v=8$ points, $b=11$ lines, with a line of length 4, six of length 3, the remaining of length 2, and with a point not on any line of length 3.

2.3 Further generalizations

At the end of the 1980's, Metsch [11], in order to solve the Dowling–Wilson conjecture,[1] proved the following interesting classification result.

Theorem 2.3 (Metsch, (1991) [10], Thm. 17.1, pp. 181–187) *Suppose that $(\mathcal{P},\mathcal{L})$ is a linear space satisfying $b-v \leq [q]-2$ for some point q. Then one of the following cases occurs.*

1) $(\mathcal{P},\mathcal{L})$ can be obtained from a projective plane of order $n=[q]-1$ by removing $b-v$ $(\leq n-1)$ points and q is any point of the plane.

[1] Let $(\mathcal{P},\mathcal{L})$ be a finite linear space with v points and b lines. For every point-line pair (p,ℓ), with $p \in \ell$, let $\pi(p,\ell)$ denote the number of lines through p missing ℓ. Then the *Dowling-Wilson conjecture* [7] states that $b \geq v+\pi(p,\ell)$ for each of its non-incident point–line pairs.

2) $(\mathcal{P}, \mathcal{L})$ is a near-pencil and q is any point of the near pencil.

3) $(\mathcal{P}, \mathcal{L})$ is a simple extension of type 2 of a generalized projective plane and q is the special point.

4) $(\mathcal{P}, \mathcal{L})$ is an affine plane of order n with one point at infinity and q is not the point at infinity.

5) $(\mathcal{P}, \mathcal{L})$ is an s-fold inflated projective plane of order n for some integers s and n with $1 \leq s < n$. If $(\mathcal{P}_1, \mathcal{L}_1), \ldots, (\mathcal{P}_s, \mathcal{L}_s)$ are its main subspaces then q is a point of every subspace $(\mathcal{P}_j, \mathcal{L}_j)$. Furthermore, if b_j is the number of lines of $(\mathcal{P}_j, \mathcal{L}_j)$, v_j its number of points, and r_j the degree of q in $(\mathcal{P}_j, \mathcal{L}_j)$, and if d is the deficiency of $(\mathcal{P}, \mathcal{L})$, then $b_j \leq v_j + r_j - 2$ for all j, and

$$2 + d + (s-1)(n+1) + \sum_{j=1}^{s}(b_j - v_j - r_j) \leq 0.$$

Conversely every linear space satisfying 1), 2), 3), 4) and 5) is a linear space satisfying $b - v \leq [q] - 2$ for some point q.

After this result, other cases have been considered, (see for example [15, 13]). Recently Durante [6] has investigated the case $s = k$, k being the maximum line length, and Theorem 2.4 below contains his results.

2.4 Finite linear spaces with $b - v \leq k$

In this section we give a list of finite linear spaces with v points and $b - v \leq k$. Before giving the list, we recall some definitions.

A *Bridges space* is a linear space with $b - v = 1$, and in view of Bridges theorem [3], it is either the $(3,3)$-cross or a punctured projective plane.

The *Nwpanka-plane* is the linear space obtained from the complete graph K_6 by adding to each of its five parallel classes (each of them formed by three lines of length 2) a new point and putting the five new points so obtained on a line.

A near-pencil is also called a *degenerate projective plane.*

Clearly, in what follows $PG(2,q)$ denotes the desarguesian projective plane of order q. Now we are ready to give the list of finite linear spaces with $b - v \leq k$.

E1. A near-pencil.
E2. A linear space with $\mathcal{P} = \ell \cup \ell'$, with $|\ell| = k$ and $|\ell'| = 2$.
E3. The $(4,4)$-cross.
E4. The $(3,k)$-cross.
E5. An extension of type 2 of the Fano plane.
E6. The Nwpanka-plane.
E7. The linear space obtained from the affine plane of order 3 by deleting a line ℓ but one point $q \in \ell$ and by adding a triangle at infinity.
E8. The linear space is obtained from $PG(2,3)$ by deleting two lines ℓ, ℓ' but two of their points $q \in \ell \setminus \ell \cap \ell'$ and $q' \in \ell' \setminus \ell \cap \ell'$.

E9. The linear space obtained from $PG(2,3)$ by deleting all its points except four collinear points of a line L and except for three points outside of L forming a triangle.
E10. The linear space obtained from $PG(2,4)$ by deleting two lines ℓ, ℓ' but two of their points $q \in \ell \setminus \ell \cap \ell'$ and $q' \in \ell' \setminus \ell \cap \ell'$.
E11. The linear space obtained from $PG(2,4)$ by deleting all its points except five collinear points of a line L and except for three points outside of L forming a triangle.
E12. An affine plane of order k.
E13. An affine plane of order $k-1$ with a point at infinity.
E14. A punctured affine plane of order $k-1$ with a triangle at infinity.
E15. An affine plane of order $k-1$ with a (possibly degenerate) projective plane at infinity.
E16. A linear space obtained from a projective plane of order $k-1$ by deleting $k-1$ non-collinear points.
E17. A punctured affine plane of order $k-1$ with a point at infinity.
E18. An affine plane of order $k-1$ with a Bridges space at infinity.
E19. A punctured affine plane of order $k-1$ with a (possibly degenerate) projective plane at infinity.
E20. A linear space obtained obtained from a projective plane of order $k-1$ by deleting k points, with at most $k-2$ of them collinear.
E21. A linear space obtained from a projective plane of order $k-1 \geq 3$ by deleting a line L but one of its points, and two more points outside of L.
E22. A projective plane of order $k-1$ with at most $k-2$ points deleted.

Theorem 2.4 (Durante, 2002 [6]) *Let $(\mathcal{P},\mathcal{L})$ be a finite linear space satisfying $b-v \leq k$. Then either $(\mathcal{P},\mathcal{L})$ is one of the linear spaces $E1,\ldots,E22$, or $b-v=k$, the maximum point degree is $k+1$, there are points of different degree and $b \geq k^2-k+3$, (and if $b=k^2-k+3$ then $k \leq 10$).*

In this paper we prove the following two results.

Theorem 2.5 *Let $(\mathcal{P},\mathcal{L})$ be a finite linear space, and let k be the maximum line length. If $b-v \leq k$, then $(\mathcal{P},\mathcal{L})$ is one of the linear spaces described in $E1,\ldots,E22$.*

Theorem 2.6 *Let $(\mathcal{P},\mathcal{L})$ be a finite linear space satisfying $b-v=1+\sqrt{v}$. Let k and m denote the maximum line length and the minimum point degree, respectively. Then either $b-v \leq m$ or $b-v \leq k$.*

Thus, in view of Lemma 2.1, Theorem 2.5, and Totten's Theorem [18], one has that, apart from a few cases with a small number of points, a finite linear space with $b-v \leq 1+\sqrt{v}$ is either obtained from a projective plane inflating one of its lines and deleting (possibly) some points, or it is obtained from a projective plane by removing a suitable set of points (possibly also some lines).

2.5 Two useful lemmas

In this section we recall a theorem on the classification of finite linear spaces with $b - v \leq m$ [12], and also a result on finite linear spaces with $b - v = 5$, which will be two useful lemmas in the following.

Lemma 2.1 *Let $(\mathcal{P}, \mathcal{L})$ be a finite linear space on v points, such that $b - v \leq m$. Then one of the following holds.*

1) *$(\mathcal{P}, \mathcal{L})$ is the near pencil on v points, the $(3,3)$-cross, the $(3,4)$-cross or the linear space on $v = 5$ points, with a line of length 3 and all the other of length 2.*

2) *$(\mathcal{P}, \mathcal{L})$ is an affine plane of order $m - 1$ or a punctured affine plane of order $m - 1$.*

3) *$(\mathcal{P}, \mathcal{L})$ is an affine plane of order m with a point at infinity.*

4) *$(\mathcal{P}, \mathcal{L})$ is a punctured affine plane of order m with a point at infinity.*

5) *$(\mathcal{P}, \mathcal{L})$ is obtained from a finite projective plane of order $m - 1$ by deleting at most m points.*

6) *$(\mathcal{P}, \mathcal{L})$ is an affine plane of order $m-1$ with either a punctured projective plane, or the $(3,3)$-cross at infinity.*

7) *$(\mathcal{P}, \mathcal{L})$ is a projectively inflated punctured affine plane of order $m - 1$, with at least four points at infinity.*

8) *A projectively inflated affine plane of order $m - 1$.*

9) *$(\mathcal{P}, \mathcal{L})$ is a punctured affine plane of order $m - 1$ with a triangle at infinity.*

10) *$(\mathcal{P}, \mathcal{L})$ is the linear space on $v = 7$ points, $b = 10$ lines, $m = 3$, with a single line of length $k = m + 1 = 4$, three lines of length 3 and the remaining lines of length 2.*

11) *$(\mathcal{P}, \mathcal{L})$ is the linear space on $v = 8$ points, $b = 11$ lines, $m = 3$, with a single line L of length $k = m + 1 = 4$, six lines of length 3 and the remaining lines of length 2, and on each point of L there is a line of length 3.*

12) *$(\mathcal{P}, \mathcal{L})$ is the linear space on $v = 8$ points, $b = 11$ lines, $m = 3$, with a single line L of length $k = m + 1 = 4$, six lines of length 3 and the remaining lines of length 2, and with a point L on which there is no line of length 3.*

Finally, from the results contained in [14] one obtains the following statement.

Lemma 2.2 *If $(\mathcal{P}, \mathcal{L})$ is a finite linear space with v points, $b = v + 5$ lines, $v \geq 10$, with at least two points of degree m, $k \geq m + 1$ and $m \geq 4$, then $(\mathcal{P}, \mathcal{L})$ is one of the following two linear spaces.*

(i) *The Nwpanka-plane.*

(ii) *The linear space obtained from the projective plane of order* 4 *deleting two lines* ℓ *and* ℓ' *and their points, but a point* p *on* ℓ *and a point* p' *on* $\ell \setminus \ell'$.

2.6 Some embedding results

A finite linear space is *embeddable* in a finite projective plane if it is possible to add points and lines to obtain a finite projective plane. For example a finite affine plane is embeddable in a finite projective plane; actually adding to each parallel class a new point and imposing these new points to form a new line one obtains a finite projective plane.

The embedding problem for finite linear spaces in finite projective planes is one of the most interesting problems in finite geometries, and in the literature we can find a number of papers devoted to this question (the interested reader is referred to [2, 10] in which he can find many of the most important results on this question and also a complete bibliography on this topic). In this section we collect some embedding results.

Theorem 2.7 (Vanstone [20]) *Every finite linear space with constant point degree* $n+1$, $b = n^2+n+1$ *lines and* $v \geq n^2$ *can be embedded into a projective plane of order* n.

Theorem 2.8 (Metsch [10], Lemma 3.6) *Every finite linear space with constant point degree* $n+1$, $b \leq n^2+n$ *lines and* $v \geq n^2-n+1$ *points can be embedded into a projective plane of order* n.

Theorem 2.9 (Metsch [10], Corollary 5.7) *If* $(\mathcal{P}, \mathcal{L})$ *is a finite linear space with* $v \geq n^2-n+2$ *points and* $b \leq n^2+n$ *lines, then one of the following cases occurs:*

(i) $(\mathcal{P}, \mathcal{L})$ *is a near-pencil.*

(ii) $(\mathcal{P}, \mathcal{L})$ *can be embedded into a projective plane of order* n.

(iii) $(\mathcal{P}, \mathcal{L})$ *is the* $(3,6)$*-cross or a simple extension of type* 2 *of the Fano plane.*

3 The result

In this section we are going to prove Theorems 2.5 and 2.6.

3.1 Finite Linear Spaces with $b - v \leq k$

In this section we are going to characterize finite linear spaces with $b - v \leq k$. We proceed as follows.

- *First we study the case in which there are two lines containing all the points of* $(\mathcal{P}, \mathcal{L})$ *showing that the linear space is one of the linear spaces described in* $E1, \ldots, E4$.

After, we study the case $b - v \geq m + 1$, *and we prove that*

- *either* $(\mathcal{P}, \mathcal{L})$ *is embeddable in a finite projective plane of order* m, *or* $k = m+2$ *and* $(\mathcal{P}, \mathcal{L})$ *is the linear space* $E11$, *or* $k = m+1$ *and* $(\mathcal{P}, \mathcal{L})$ *is one of the linear spaces described in* $E5, E6, E7, E9, E10$.

Hence, together with Lemma 2.1 we get the complete classification of finite linear spaces with $b - v \leq k$.

Proposition 3.1 *Let* $(\mathcal{P}, \mathcal{L})$ *be a finite linear space with* $b - v \leq k$. *If there are two lines* ℓ *and* ℓ' *such that* $\mathcal{P} = \ell \cup \ell'$ *then* $(\mathcal{P}, \mathcal{L})$ *is one of the following linear spaces.*

1) The linear space on $v = k + 2$ *points, with a single line of length* k, *all the remaining lines of length* 2 *and with a line of length* 2 *parallel to the line of length* k.

2) The near-pencil on v *points, a* $(3, k)$*-cross or the* $(4, 4)$*-cross.*

Proof. Put $|\ell| = k$ and $|\ell'| = h$. So by definition $h \leq k$. We distinguish two cases.
CASE 1. $\ell \cap \ell' = \emptyset$.

In such a case we have that $v = h + k$ and $b = hk + 2$. So from $b \leq v + k$ it follows that

$$hk + 2 \leq h + 2k,$$

that is,

$$h(k - 1) \leq 2(k - 1),$$

from which it follows that $h = 2$, and so we have the linear space described in 1).
CASE 2. $\ell \cap \ell' \neq \emptyset$.

In this case we have $v = h + k - 1$ and $b = (h - 1)(k - 1) + 2$. From $b - v \leq k$ it follows that

$$(h - 1)(k - 1) + 2 \leq h + 2k - 1,$$

from which

$$(h - 1)(k - 1) \leq h - 1 + 2(k - 1),$$

and so

$$(k - 1)(h - 1 - 2) \leq h - 1.$$

Since $h \leq k$, it follows that

$$(h - 1)(h - 3) \leq h - 1,$$

and so $h \leq 4$.

If $h = 2$,then $v = k + 1$ and $(\mathcal{P}, \mathcal{L})$ is a near-pencil.

If $h = 3$, then $v = k + 2$ and $(\mathcal{P}, \mathcal{L})$ is a $(3, k)$-cross.

If $h = 4$, then $v = k + 3$. Thus $b = 3k - 1 \leq 2k + 3$, and so it follows that $k = h = 4$ and $(\mathcal{P}, \mathcal{L})$ is the $(4, 4)$-cross. Hence the assertion is completely proved. □

So from now on we may assume that for any given pair of lines there is at least a point outside of them, and so in particular $m \geq 3$.

In view of Lemma 2.1 we may assume $b - v \geq m + 1$, and so $k \geq m + 1$.

Since $k \geq m+1$, it follows that if x is a point of degree m and ℓ is a line of length k, then $x \in \ell$.

3.2 There is a single point p of degree m

Let p be the unique point of degree m. In such a case all the lines of length k pass through p, and counting the lines meeting a line of length k we have that $b \geq km$.

Proposition 3.2 *There are at least two lines of length k.*

Proof. Assume to the contrary that there is a single line ℓ of length k.

First we prove that $k \geq m + 2$.

If $k = m + 1$, then $b - v \leq m + 1$. And so by our assumption $b - v = m + 1$. Hence $b \geq m^2 + m$, and so $v \geq m^2 - 1$. Since $v \leq m + 1 + (m - 1)^2 = m^2 - m + 2$, it follows that $m \leq 3$. So $m = 3$, $v = 8$ and $b = 12$. From the Doyen list of finite linear spaces on at most nine points, we have that such a finite linear space cannot exist (see for example the Appendix in [2]).

Now we are going to prove the assertion. We distinguish two cases:

(i) *There is a line of length $k - 1$ on p.*

(ii) *There is no line of length $k - 1$ on p.*

Case (i). Let h be a line of length $k - 1$ on p. Then each point of h has degree at least k, and since there is a point outside of $\ell \cup h$, it follows that h has at least a parallel line. Thus, counting the lines meeting h or parallel to h, one obtains $b \geq m + (k - 2)(k - 1) + 1$. Since $b \leq v + k \leq mk + k - 2m + 2$, we obtain

$$1 + k(k - 3) - k \leq m(k - 3),$$

and so

$$(k - m)(k - 3) \leq k - 1.$$

From $k \geq m + 2$ it follows that $2k - 5 \leq k$ and so $m = 3$ and $k = 5$. Moreover h has exactly one parallel line, and ℓ has no parallel lines, one point of degree m, one point of degree $k = m + 2$ and all the other points of degree $k - 1 = m + 1$. Since points outside of ℓ have degree $m + 2$ and h has one parallel line, it follows that the third line on p has length 2. Hence $v = 5 + 3 + 1 = 9$. But $b \geq km = 15 > v + k = 14$, a contradiction.

Case (ii). Now each line on p, different from ℓ, has length at most $k - 2$.

Counting v via the lines on p gives $v \leq k + (m - 1)(k - 3)$.

If each point of ℓ different from p, has degree at least $m + 2$, then $b \geq m + (k - 1)(m + 1)$, and so $m + (k - 1)(m + 1) \leq 2k + (m - 1)(k - 3)$. It follows that $3m \leq 4$, a contradiction.

Thus ℓ has a point y of degree $m+1$, and so $k-2 \leq m+1$. Hence $v \leq k+(m-1)m$, from which it follows that $b \leq 2k + m(m - 1)$.

Since $b \geq km$, we have

$$k(m-2) \leq m(m-1). \tag{1}$$

From $k \geq m+2$, it follows that $m \leq 4$.

If $m = 3$, then $v \leq 3k-6$, and so $b \leq 4k-6$.

Since $b \geq km = 3k$, it follows that $k \geq 6 = m+3$. Thus by Equation (1) we have $k = m+3 = 6$, $b = km = 18$ and $v = b-k = 12$. Hence every point of ℓ different from p has degree $m+1$, and every point outside of ℓ has degree $m+3$. On a point of degree $m+1$ there are ℓ and three lines of length $m = 3$. Let x be a point not on ℓ, then $12 = v = |px| + (m+2)(m-1) = |px| + 10$.

Thus the lines on p different from ℓ have length 2, a contradiction.

Therefore $m = 4$. Thus $k = m+2 = 6$, and $b \geq 24$. Counting v via the lines on p gives $v \leq k + 3(k-3) = 6 + 3 \cdot 3 = 15$, contradicting $v \geq b - 6 = 18$.

Hence there are at least two lines of length k. □

Proposition 3.3 *$k = m+1$ and either $(\mathcal{P}, \mathcal{L})$ is the unique linear space on $v = 10$ $b = 14$ lines and with $m = 3$, or $(\mathcal{P}, \mathcal{L})$ is embeddable in a finite projective plane of order m.*

Proof. Since there are two lines of length k, each point different from p has degree at least k. Counting the lines meeting a line of maximal length gives $b \geq m + (k-1)^2$, and counting v via the lines on p, we have $v \leq m(k-1)+1$.

Since $b \leq m(k-1)+1+k$, we have

$$m + (k-1)^2 \leq m(k-1) + 1 + k,$$

$$(k-1)(k-1-m) \leq k+1-m \leq k-2$$

and so $k = m+1$. Hence $b - v \leq m+1$. Thus by our assumptions we have $b - v = m+1$. Then

$$m^2 + m \leq b \leq m^2 + m + 2.$$

If $b = m^2+m+2$, then $v = m^2+1$ and from a theorem of Stinson [16] it follows that $(\mathcal{P}, \mathcal{L})$ is the unique linear space with $m = 3, v = 10$ and $b = 14$.

If $b = m^2+m+1$, then $v = m^2$. The case with a point of degree at least $m+2$ cannot occur (by [10] Thm. 7.4). So $m+1$ is the maximum point degree, and from Theorem 2.7 we have that $(\mathcal{P}, \mathcal{L})$ is obtained from a projective plane of order m by deleting at most m points and no line.

If $b = m^2+m$, then $v = m^2-1$ and by Theorem 2.9 $(\mathcal{P}, \mathcal{L})$ is embeddable in a finite projective plane of order m. □

3.3 There are at least two points of degree m

In such a case there is a single line ℓ of length k.

Proposition 3.4 *Either $(\mathcal{P},\mathcal{L})$ is the Nwpanka-plane, or on a point of degree m there is at least one line of length m.*

Proof. Let p be a point of degree m. Assume on the contrary that on p there is no line of length m. Then $v \leq k+(m-1)(m-2) = k+m^2-3m+2$. Thus, $b \leq 2k+m^2-3m+2$. From $b \geq 1+k(m-1)$ it follows that

$$k(m-3) \leq m(m-3)+1.$$

Hence, either $m=3$, or $k \leq m + \dfrac{1}{m-3}$, $k=m+1$ and $m=4$.

If $m=3$, then $v \leq k+2$, a contradiction since $(\mathcal{P},\mathcal{L})$ is not the union of two lines.

If $m=4$, then $v=11$, $b=16$, every point of ℓ has degree $m=4$, and ℓ has no parallel line. Thus, the lines have length 3 and 5, points have degree 4 and 5. From Lemma 2.2 it follows that $(\mathcal{P},\mathcal{L})$ is the Nwpanka-plane. □

Thus, from now on we may assume that $(\mathcal{P},\mathcal{L})$ is not the Nwpanka-plane.

Proposition 3.5 $b \geq 2+k(m-1)$.

Proof. Let p be a point of degree m of ℓ, and L be a line of length m on p. Since $(\mathcal{P},\mathcal{L})$ is not the union of ℓ and L, we have that there is a line t parallel to L, and so either ℓ has a point of degree $m+1$ or a parallel line. □

Proposition 3.6 *$(\mathcal{P},\mathcal{L})$ is one of the following linear spaces.*

(i) The linear space obtained from the projective plane of order 4 by deleting all its points except five collinear points of a line ℓ and except for three points outside of ℓ forming a triangle[2].

(ii) A simple extension of type 2 of the Fano–plane.

(iii) The linear space obtained from the projective plane of order 4 by deleting two lines t and t' but one point on t and one point on $t' \setminus t$.

(iv) The linear space obtained from the projective plane of order 3 by deleting all its points except four collinear points of a line L and except for three points outside of L forming a triangle.

Proof. If $k \geq m+2$, then the line L of the previous proposition has two parallel lines, and so $b \geq 3+k(m-1)$. It follows that

$$3+k(m-1) \leq 2k+(m-1)^2, \tag{2}$$

from which, it follows that, either $m=3$, or

$$k \leq m+1+\frac{1}{m-3}.$$

[2]Notice that in this case we have $k=m+2$

Thus, either $k = m + 2$ and $m = 4$, or $m = 3$.

From Equation (2) it follows that the points have degree at most $m + 3$. If there is a point of degree $m + 3$ outside of ℓ, then $b \geq 4 + k(m - 1)$. Thus Equation (2) gives $k \leq m + 1$, a contradiction. So points outside ℓ have at most degree $m + 2$. Hence $k = m + 2$, and ℓ has no parallel line.

If $m = 3$, then $v \leq k + 4$. Since $(\mathcal{P}, \mathcal{L})$ is not the union of two lines, we have $v \geq k + 3$, and so $8 = k + 3 \leq v \leq k + 4 = 9$.

If $v = k+4 = 9$ then on a point of degree m there are at least two lines of length m. So, the line L of the previous propositions, has at least four parallel lines, and since ℓ has no parallel line, it follows that $k + v \geq b \geq 5 + k(m-1) = 2k + 5 = 15 > k + v$. Thus, $v = k + 3 = 8$. It is easy to see that $(\mathcal{P}, \mathcal{L})$ is the linear space described in (i).

If $m = 4$, then $v = k + (m - 1)^2$ and $b = 2k + (m - 1)^2 = 3 + k(m - 1)$. Thus on a point of degree m there are at least two lines L and H of length m. Let x be a point of $H \setminus \{\ell\}$. The two parallel lines on x to L give two points of degree $m+1$ on ℓ, and since $b = 3 + k(m - 1)$ it follows that $m = |H| = 2$, a contradiction. Hence $k = m + 1$, and so $b - v = m + 1$.

It follows that $v \leq m + 1 + (m - 1)^2 = m^2 - m + 2$, and so $b \leq m^2 + 3$.

CASE 1. $b = m^2 + 3$.

In such a case $v = m^2 - m + 2$, and $b = m^2 + 3 \leq m^2 + m$, and by Theorem 2.9 we have that either $(\mathcal{P}, \mathcal{L})$ is embeddable in a projective plane of order m, or it is a simple extension of type 2 of the Fano plane, that is the linear space described in (ii). Consider the case in which $(\mathcal{P}, \mathcal{L})$ is embeddable in a projective plane of order m. Then maximum point degree is $m + 1$ and so ℓ has no parallel line, and it has exactly three points of degree $m + 1$. Since $v = m^2 - m + 2$ on a point of degree m there are $m - 1$ lines of length m. Let H and K be two lines of length m meeting ℓ in a point of degree m; then since ℓ has three points of degree m, K has length at most 4, otherwise the parallel lines to H and meeting K would meet ℓ in at least four points of degree $m + 1$. So $m \leq 4$. Since there are at least two points of degree m on ℓ, it follows that $m = 4$. So $(\mathcal{P}, \mathcal{L})$ is the linear space described in (iii).

CASE 2. $b = m^2 + 2$.

We have $v = m^2 - m + 1$. If ℓ has a parallel line, then there is a point of degree at least $m + 2$, and so ℓ has exactly one parallel line t and exactly one point of degree $m + 1$. Let L be a line of length m meeting ℓ. Since ℓ has exactly one point of degree $m + 1$, a contradiction follows. So ℓ has no parallel lines. Hence each point outside of ℓ has degree $m + 1$. If on a point of degree m there are two lines of length m, then since there are at most two points of degree $m + 1$ on ℓ we have $m = 3$, a contradiction, since $v = m^2 - m + 1$. So on a point of degree m there is a single line of length m, and so again $m = 3$, $b - v = 4$, $v = 7$. Hence, $(\mathcal{P}, \mathcal{L})$ is the linear space described in (iv).

CASE 3. $b = m^2 + 1$.

Now $v = m^2 - m$. In this case ℓ has either a parallel line or a single point of degree $m+1$. Since there are lines of length m, it follows that ℓ has no parallel line. Thus, ℓ has a single point of degree $m+1$, and all the points outside of it have degree $m+1$. It follows that on a point of degree m there is a single line of length m. Thus $m^2 - m \leq v \leq m+1+m-1+(m-2)^2$, from which it follows that $m \leq 4$. If $m = 3$, then $v \leq 6$, and it is easy to see that this case is not possible.

If $m = 4$, then $v = 12$, $b = 17$, and by Lemma 2.2 this case cannot occur. □

3.4 The case $b - v \leq 1 + \sqrt{v}$

In this section we study finite linear spaces with $b - v \leq 1 + \sqrt{v}$. The classification of such finite linear spaces is, clearly, a slight generalization of Totten's classification theorem (see [10], page 78), and, as other results of this type, it may be useful in the study of finite linear spaces with a "small" number of points.

In view of Totten's theorem we have to consider only the case $b - v = 1 + \sqrt{v}$.

We are going to show that such a linear space fulfills either $b - v \leq m$ or $b - v \leq k$, and so Lemma 2.1 and the results of the previous section give the required classification. So let $(\mathcal{P}, \mathcal{L})$ be a finite linear space with $b - v = 1 + \sqrt{v}$.

Proposition 3.7 *There is no finite linear space with $b - v = 1 + \sqrt{v}$ such that there are two lines ℓ and ℓ' such that $\mathcal{P} = \ell \cup \ell'$.*

Proof. If $\ell \cap \ell' \neq \emptyset$, then $m = 2$, $v = h+k-1(\leq 2k-1)$, and $b = 2+(h-1)(k-1)$. It follows that

$$2 + (h-1)(k-1) = h + k + \sqrt{v},$$

and so

$$(k-2)(h-2) = 1 + \sqrt{v}.$$

Thus

$$k \neq 2, \ \ h \neq 2.$$

If $h = 3$, then $k - 3 = \sqrt{k+2}$, a contradiction.

If $h = 4$, than $2k - 5 = \sqrt{k+3}$, again a contradiction.

If $h \geq 5$, then $3k - 7 \leq \sqrt{v} \leq \sqrt{2k-1} \leq k$, and so $k = 3$. Using an argument as above we get a contradiction.

If $\ell \cap \ell' = \emptyset$, then $v = h + k$ and $b = hk + 2$. From $b = v + \sqrt{v} + 1$ it follows that

$$(h-1)^2(k-1)^2 = h + k.$$

If $h = 2$, then $k^2 - 3k + 1 = 0$, a contradiction since k is an integer.

If $h \geq 3$, then $2k^2 - 5k + 2 \leq 0$, from which it follows that $k \leq 2$, a contradiction since $3 \leq h \leq k$. □

As a consequence we have that for any two given lines there is a point outside of them. In particular $m \geq 3$.

3.5 The case $k \leq m$

In such a case counting the number of points v via the lines on a point of minimum degree gives $v \leq m(m-1)+1$. Hence $v < m^2$. Thus $\sqrt{v} < m$, and so $b - v \leq m$.

3.6 The case $k \geq m+1$

If there are two points of degree m, then there is a single line of length k, and so

$$v \leq k + (m-1)^2 \leq k + (k-2)^2 = k^2 - k + 4.$$

If $k \geq 5$ then $\sqrt{v} < k$, and so $b - v \leq k$.
If $k = 4$, then $m = 3$ and $v \leq k + 4 < (k-1)^2$.

If there is a single point of degree m, then

$$v \leq 1 + m(k-1) \leq 1 + (k-1)^2.$$

If $v \leq (k-1)^2$, then $b - v \leq k$.

Finally, from $v = 1 + (k-1)^2 < k^2$ it follows that $b - v \leq k$.

Thus, in view of the results of the previous section and of Lemma 2.1 we have the classification of finite linear spaces with $b - v \leq 1 + \sqrt{v}$.

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