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Squaring the terms of an ℓ^{th} order linear recurrence

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1 Introduction and the main result

An ℓ^{th} *order linear recurrence* is a sequence in which each is a linear combination of the ℓ previous terms. The symbolic representation of an ℓ^{th} order linear recurrence defined by

$$a_n = \sum_{j=1}^{\ell} p_j a_{n-j} = p_1 a_{n-1} + p_2 a_{n-2} + \cdots + p_\ell a_{n-\ell}, \tag{1}$$

is $(a_n(c_0, \ldots, c_{\ell-1}; p_1, \ldots, p_\ell))_{n\geq 0}$, or briefly $(a_n)_{n\geq 0}$, where the p_i are constant coefficients, with given $a_j = c_j$ for all $j = 0, 1, \ldots, \ell - 1$, and $n \geq \ell$; in such a context, $(a_n)_{n\geq 0}$ is called an *ℓ-sequence*.

In the case $\ell = 2$, this sequence is called Horadam's sequence and was introduced, in 1965, by Horadam [4, 5], and it generalizes many sequences (see [1, 6]). Examples of such sequences are the Fibonacci numbers $(F_n)_{n\geq 0}$, the Lucas numbers $(L_n)_{n\geq 0}$, and the Pell numbers $(P_n)_{n\geq 0}$, when one has the following initial conditions: $p_1 = p_2 = c_1 = 1$, $c_0 = 0$; $p_1 = p_2 = c_1 = 1$, $c_0 = 2$; and $p_1 = 2$, $p_2 = c_1 = 1$, $c_0 = 0$; respectively. In 1962, Riordan [8] found the generating function for powers of Fibonacci numbers. He proved that the generating function $\mathcal{F}_k(x) = \sum_{n\geq 0} F_n^k x^n$ satisfies the recurrence relation

$$(1 - a_k x + (-1)^k x^2)\mathcal{F}_k(x) = 1 + kx \sum_{j=1}^{[k/2]} (-1)^j \frac{a_{kj}}{j} \mathcal{F}_{k-2j}((-1)^i x)$$

for $k \geq 1$, where $a_1 = 1$, $a_2 = 3$, $a_s = a_{s-1} + a_{s-2}$ for $s \geq 3$, and $(1 - x - x^2)^{-j} = \sum_{k\geq 0} a_{kj} x^{k-2j}$. Horadam [5] gave a recurrence relation for $\mathcal{H}_k(x)$ (see also [3]). Haukkanen [2] studied linear combinations of Horadam's sequences and the generating function of the ordinary product of two of Horadam's sequences. Recently,

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Mansour [7] found a formula for the generating functions of powers of Horadam's sequence. In this paper we are interested in studying the generating function for squaring the terms of the ℓ-sequence, that is,

$$\mathcal{A}_\ell(x) = \mathcal{A}_\ell(x; c_0, \dots, c_{\ell-1}; p_1, \dots, p_\ell) = \sum_{n\geq 0} a_n^2(c_0, \dots, c_{\ell-1}; p_1, \dots, p_\ell)x^n.$$

The main result of this paper can be formulated as follows.
Let $\Delta_\ell = (\Delta_\ell(i,j))_{0\leq i,j\leq \ell-1}$ be the $\ell \times \ell$ matrix

$$\Delta_\ell(i,j) = \begin{cases} 1 - \sum_{s=1}^{\ell} p_j^2 x^j, & i = j = 0 \\ -2xv_j, & i = 0 \text{ and } 1 \leq j \leq \ell - 1 \\ -p_i x^i, & 1 \leq i \leq \ell - 1 \text{ and } j = 0 \\ \delta_{i,j} - p_{i-j}x^{i-j} - p_{i+j}x^i, & 1 \leq i \leq \ell - 1 \text{ and } 1 \leq j \leq \ell - i \\ \delta_{i,j}, & 1 \leq i \leq \ell - 1 \text{ and } \ell + 1 - i \leq j \leq \ell - 1 \end{cases}$$

where v_j is given by

$$v_j = p_1 p_{j+1} + p_2 p_{j+2} x + \dots + p_{\ell-j} p_\ell x^{\ell-j-1},$$

for all $j = 1, 2, \dots, \ell - 1$, we define $p_i = 0$ for $i \leq 0$, and $\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.

Let $\Gamma_\ell = (\Gamma_\ell(i,j))_{0\leq i,j\leq \ell-1}$ be the $\ell \times \ell$ matrix

$$\Gamma_\ell(i,j) = \begin{cases} x\sum_{s=0}^{\ell-1}(c_s^2 - w_{s-1}^2)x^j, & i = j = 0 \\ x^{i+1}\sum_{s=0}^{\ell-1-i} c_s(c_{s+i} - w_{s+i-1})x^s, & j = 0 \text{ and } 1 \leq i \leq \ell - 1 \\ \Delta_\ell(i,j), & 0 \leq i \leq \ell - 1 \text{ and } 1 \leq j \leq \ell - 1 \end{cases}$$

where w_j is given by

$$w_j = p_1 c_j + p_2 c_{j-1} + \dots + p_{j+1} c_0 = \sum_{s=1}^{j+1} p_s c_{j+1-s},$$

for $j = 0, 1, \dots, \ell - 2$ with $w_{-1} = 0$.

Theorem 1.1 *The generating function $\mathcal{A}_\ell(x)$ is given by*

$$\frac{\det(\Gamma_\ell)}{x\det(\Delta_\ell)}.$$

The paper is organized as follows. In Section 2 we give the proof of Theorem 1.1 and in Section 3 we give some applications for Theorem 1.1.

2 Proofs

Let $(a_n)_{n\geq 0}$ be a sequence satisfying Relation (1) and ℓ be any positive integer. We define a family $\{f_d(n)\}_{d=0}^{\ell-1}$ of sequences by

$$f_d(n) = a_{n-1}a_{n-1-s},$$

and a family $\{F_d(x)\}_{d=0}^{\ell-1}$ of generating functions by

$$F_d(x) = \sum_{n\geq 1} a_{n-1}a_{n-1-d}x^n. \tag{2}$$

Now we state two relations (Lemma 2.1 and Lemma 2.2) between the generating functions $F_d(x)$ and $F_0(x) = x\mathcal{A}_\ell(x)$ that play the crucial roles in the proof of Theorem 1.1.

Lemma 2.1 *We have*

$$F_0(x) = F_0(x)\sum_{j=1}^{\ell} p_j^2 x^j + 2x\sum_{j=1}^{\ell-1} v_j F_j(x) + x\sum_{j=0}^{\ell-1}(c_j^2 - w_{j-1}^2)x^j.$$

Proof. Since the sequence $(a_n)_{n\geq 0}$ satisfying Relation (1) we get that

$$a_n^2 = \sum_{j=1}^{\ell} p_j^2 a_{n-j}^2 + 2\sum_{1\leq i<j\leq \ell} p_i p_j a_{n-i}a_{n-j},$$

for all $n \geq \ell$. Multiplying by x^n and summing over $n \geq \ell$ together with the following facts:

1. $\sum_{n\geq \ell} a_n^2 x^n = \frac{1}{x}\sum_{n\geq \ell} f_0(n+1)x^{n+1} = \frac{1}{x}\left(F_0(x) - \sum_{j=1}^{\ell} a_{j-1}^2 x^j\right),$

2. $\sum_{n\geq \ell} a_{n-j}^2 x^n = \sum_{n\geq \ell} f_0(n-j+1)x^n = x^{j-1}\left(F_0(x) - \sum_{t=1}^{\ell-j} a_{t-1}^2 x^t\right),$

3. $\sum_{n\geq \ell} a_{n-i}a_{n-j}x^n = \frac{1}{x}\sum_{n\geq \ell+1} f_{j-i}(n-i)x^n = x^{i-1}\left(F_{j-i}(x) - \sum_{d=j-i+1}^{\ell-i} a_{d-1}a_{d-j+i-1}x^d\right),$

we have that

$$\begin{aligned} F_0(x) &= F_0(x)\sum_{j=1}^{\ell} p_j^2 x^j + 2\sum_{1\leq i<j\leq \ell} p_i p_j x^i F_{j-i}(x) \\ &\quad + \sum_{j=1}^{\ell} a_{j-1}^2 x^j - \sum_{j=1}^{\ell}\sum_{i=1}^{\ell-j} p_j^2 a_{i-1}^2 x^{j+i} - 2\sum_{1\leq i<j\leq \ell}\sum_{d=j-i+1}^{\ell-i} p_i p_j a_{d-1}a_{d-(j-i)-1}x^{i+d} \\ &= F_0(x)\sum_{j=1}^{\ell} p_j^2 x^j + 2x\sum_{j=1}^{\ell-1} v_j F_j(x) + x\sum_{j=0}^{\ell-1}(a_j^2 - w_{j-1}^2)x^j. \end{aligned}$$

Hence, using the fact that $a_j = c_j$ for $j = 0, 1, \ldots, \ell-1$ we obtain the desired result. □

Lemma 2.2 *For any* $i = 1, 2, \dots, \ell - 1$,

$$F_i(x) = p_i x^i F_0(x) + \sum_{j=1}^{\ell-i} (p_{i-j} x^{i-j} + p_{i+j} x^i) F_j(x) + x^{i+1} \sum_{j=0}^{\ell-1-i} c_j (c_{i+j} - w_{i+j-1}) x^j .$$

Proof. By direct calculations we have for $n \geq \ell + 1$,

$$f_i(n) = a_{n-1} a_{n-1-i} = \sum_{j=1}^{\ell} p_j a_{n-1-j} a_{n-1-i};$$

equivalently, $f_i(n) =$

$p_1 f_{i-1}(n-1) + p_2 f_{i-2}(n-2) + \cdots + p_i f_0(n-i) + p_{i+1} f_1(n-i) + \cdots + p_\ell f_{\ell-i}(n-i)$. As in Lemma 2.1, multiplying by x^n and summing over $n \geq \ell + 1$ we get

$$F_i(x) - \sum_{j=i+1}^{\ell} a_{j-1} a_{j-1-i} x^j = \sum_{j=1}^{i} p_j x^j \left(F_{i-j}(x) - \sum_{d=i-j+1}^{\ell-j} a_{d-1} a_{d-(i-j)-1} x^d \right)$$
$$+ \sum_{j=i+1}^{\ell} p_j x^i \left(F_{j-i}(x) - \sum_{d=j-i+1}^{\ell-i} a_{d-1} a_{d-(j-i)-1} x^d \right)$$

The rest is easy to check from the definitions. □

Proof. (Theorem 1.1) Using the above lemmas together with the definitions we have

$$\Delta_k \cdot [F_0(x), F_1(x), F_2(x), \dots, F_{\ell-1}(x)]^{\mathrm{T}} = \mathbf{w},$$

where the vector $\mathbf{w}$ is given by

$$\begin{bmatrix} x \sum_{j=0}^{\ell-1} (c_j^2 - w_{j-1}) x^j \\ x^2 \sum_{j=0}^{\ell-2} c_j (c_{j+1} - w_j) x^j \\ x^3 \sum_{j=0}^{\ell-3} c_j (c_{j+2} - w_{j+1}) x^j \\ \vdots \\ x^{\ell-1} \sum_{j=0}^{0} c_j (c_{j+\ell-1} - w_{j+\ell-2}) x^j \end{bmatrix}.$$

Hence, the solution of the above equation gives the generating function $F_0(x) = \frac{\det(\Gamma_\ell)}{\det(\Delta_\ell)}$; equivalently, $\mathcal{A}_\ell(x) = \frac{\det(\Gamma_\ell)}{x \det(\Delta_\ell)}$, as claimed in Theorem 1.1. □

3 Applications

In this section we present some applications of Theorem 1.1.

Fibonacci numbers. Let $F_{k,n}$ be the n^{th} k-Fibonacci number which is given by

$$F_{k,n} = \sum_{j=1}^{k} F_{k,n-j},$$

for $n \geq k$, with $F_{k,0} = 0$ and $F_{k,j} = 1$ for $j = 1, 2, \ldots, k-1$; in such a context, $F_{2,n}$, $F_{3,n}$, and $F_{4,n}$ are usually called the n^{th} Fibonacci numbers, tribonacci numbers, and tetranacci numbers; respectively. Using Theorem 1.1 with $c_0 = 0$ and

$$c_1 = c_2 = \cdots = c_{k-1} = p_1 = p_2 = \cdots = p_k = 1$$

gives the generating function $\sum_{n\geq 0} F_{k,n}^2 x^n$ (see Table 1).

k	The generating function $\sum_{n>0} F_{k,n}^2 x^n$
2	$\frac{x(1-x)}{(1+x)(1-3x+x^2)}$
3	$\frac{x(1-x-x^2-x^3)}{(1+x+x^2-x^3)(1-3x-x^2-x^3)}$
4	$\frac{x(1-x-5x^2-2x^3-x^4-2x^5+3x^7+x^8)}{1-2x-4x^2-5x^3-8x^4+4x^5+6x^6+x^8-x^{10}}$
5	$\frac{x(1-x-5x^2-12x^3-8x^4-10x^5-7x^6-17x^7-8x^8+13x^9+10x^{10}+3x^{11}+9x^{12}+4x^{13})}{1-2x-4x^2-7x^3-11x^4-16x^5+4x^6+7x^7+4x^8+4x^9+7x^{10}-x^{12}-x^{13}-x^{15}}$

Table 1: The generating function for the square of the k^{th}-Fibonacci numbers

From Table 1, for $k = 3$ we obtain

$$\sum_{n\geq 0} nF_{3,n}^2 x^n = \frac{x(1-2x+2x^2+12x^3+8x^5+2x^6+4x^7+3x^8+2x^9)}{(x^3-x^2-x-1)^2(x^3+x^2+3x-1)^2}.$$

Pell numbers. Let $P_{k,n}$ be the n^{th} k-Pell number which is given by

$$P_{k,n} = 2P_{k,n-1} + \sum_{j=2}^{k} P_{k,n-j},$$

for $n \geq k$, with $P_{k,j} = 1$ for $j = 0, 1, \ldots, k-1$; in such a context, $P_{2,n}$ is usually called the n^{th} Pell number. Using Theorem 1.1 with $c_j = 1$ for $j = 0, 1, \ldots, k-1$ and $p_j = 1$ for $j = 1, 2, \ldots, k$ gives the generating function $\sum_{n\geq 0} P_{k,n}^2 x^n$ (see Table 2).

From Table 2, for $k = 2$ we have

$$\sum_{n\geq 0} nP_{2,n}^2 x^n = \frac{x(1-2x+10x^2-2x^3+x^4)}{(x+1)^2(x^2-6x+1)^2}.$$

k	The generating function $\sum_{n>0} P_{k,n}^2 x^n$
2	$\frac{1-4x-x^2}{(1+x)(1-6x+x^2)}$
3	$\frac{1-4x-11x^2-13x^3-5x^4-4x^5}{(1-6x-3x^2-x^3)(1-x+2x^2-x^3)}$
4	$\frac{1-4x-12x^2-25x^3-29x^4-3x^5-9x^6-12x^7+13x^8+9x^9}{(1-5x-8x^2-13x^3-20x^4+2x^5+14x^6+x^7+x^8-x^{10})}$
5	$\frac{(1+x)(9x^{13}+4x^{12}+2x^{11}+13x^{10}+6x^9-26x^8-6x^7-2x^6-5x^5-14x^4-9x^3-3x^2-2x+1)}{1-2x-4x^2-7x^3-11x^4-16x^5+4x^6+7x^7+4x^8+4x^9+7x^{10}-x^{12}-x^{13}-x^{15}}$

Table 2: The generating function for the square of the k^{th}-Pell numbers

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