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Doubly equivalent designs

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Abstract

Mendelsohn and Liang, in *J. Combin. Des.* 11 (2003), introduced a new class of difference sets and designs: $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-difference sets and $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-designs, and mainly discussed designs with a relationship called λ-equivalence. In this paper, we introduce the concepts of a doubly equivalent design as well as a super class. We describe the structure of super classes and discuss properties of doubly equivalent designs. We generalize results of simply equivalent designs in the paper of Mendelsohn and Liang to doubly equivalent designs. The main result is Theorem 3.7, which claims that a particular class of doubly equivalent designs can produce singly equivalent designs.

1 Introduction

Designs or difference sets with two values of λ's have been studied extensively, under the names partial difference sets, divisible difference sets, relative difference sets, or near difference sets, by authors such as Arasu, Davis, Jungnickel, and Pott [1], Elliott and Butson [2], Koukouvinos and Whiteman [3], Koukouvinos and Whiteman [4], Ray-Chaudhuri and Xiang [6] etc. In 2003, Mendelsohn and Liang [5] introduced the concepts of a design and a difference set with an arbitrary positive integer number of λ's. A $(v, b, r, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-design is an arrangement of v distinct objects $a_1, a_2, \ldots, a_v$ into b blocks $B_1, B_2, \ldots, B_b$ such that each block contains exactly k distinct objects, each object occurs in exactly r different blocks, and every pair of distinct objects a_i, a_j occurs together in exactly $\lambda_1, \lambda_2, \ldots,$ or λ_m blocks, while for each λ_r $(r = 1, 2, \ldots, m)$, there exists at least one pair a_i, a_j $(i \neq j)$ appearing together in exactly λ_r blocks. A $(v, b, r, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-design is called a $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-design if $v = b$ (and thus $k = r$). Let $S = \{a_1, a_2, a_3, \ldots, a_k\}$ be a set of k distinct residues mod v. We say that S is a $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-difference set if

(1) For each $d \not\equiv 0 \pmod v$ there are exactly $\lambda_1, \lambda_2, \ldots$, or λ_m ordered pairs (a_i, a_j), where $a_i, a_j \in S$, such that $a_i - a_j \equiv d \pmod v$; and

(2) For each λ_r $(r = 1, 2, \ldots, m)$, there exists at least one $d \not\equiv 0 \pmod v$ such that there are exactly λ_r ordered pairs (a_i, a_j), where $a_i, a_j \in S$, satisfying $a_i - a_j \equiv d \pmod v$.

Theorem 1.1. (Mendelsohn and Liang [5]) *Let $S = \{a_1, a_2, a_3, \ldots, a_k\}$ be a $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-difference set and $[i]$ be the set: $\{a_1+i, a_2+i, a_3+i, \ldots, a_k+i\}$ mod v, $i = 0, \ldots, v-1$. If $0 \le j < i \le v-1$ and $1 \le r \le m$, then the following three statements are equivalent:*

(1) $i - j$ appears λ_r times as a difference in S;

(2) $[i]$ and $[j]$ have λ_r objects in common;

(3) i and j appear together in λ_r blocks.

Corollary 1.2. (Mendelsohn and Liang [5]) *For a $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-difference set, $\lambda_r \le k$ for $r = 1, 2, \ldots, m$.*

By Theorem 1.1, a $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-difference set S can generate a $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-design D in the following way: the objects of the design are the integers $0, 1, \ldots, v-1$ mod v and the blocks of the design are $[0], [1], \ldots, [v-1]$, where the block $[i]$ consists of the k points $i, i+a_2, i+a_3, \ldots, i+a_k$ mod v $(i = 0, \ldots, v-1)$. S is called a *base set* of D, while i is called a *point* of D and $[i]$ is also called a *line* of D, where $i = 0, \ldots, v-1$. A $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-difference set S is called a $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m]; t_1, t_2, \ldots, t_m)$*-difference set* if for each r $(r = 1, 2, \ldots, m)$ there are exactly t_r nonzero residues appearing exactly λ_r times in the multiset of differences of S. The $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-design generated by a $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m]; t_1, t_2, \ldots, t_m)$-difference set is called a $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m]; t_1, t_2, \ldots, t_m)$*-design*. In addition, a $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m]; t)$*-difference set* is defined to be a $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m]; t_1, t_2, \ldots, t_m)$-difference set with $t_m = t$.
A $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m]; t)$*-design* can be defined similarly.

Theorem 1.3. (Mendelsohn and Liang [5]) *A $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m]; t_1, t_2, \ldots, t_m)$-design has*

$$t_1 + t_2 + \cdots + t_m = v - 1, \tag{1}$$

and

$$\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_m t_m = k(k-1). \tag{2}$$

In a $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-design D, if two lines are the same or intersect in λ_r $(1 \le r \le m)$ points, then $[i]$ and $[j]$ are called λ_r*-intersecting*. D is called λ_r*-equivalent* if the relation λ_r-intersection is an equivalence relation. In this case, if two lines $[i]$ and $[j]$ in D are λ_r-intersecting, we denote this by $[i] \overset{\lambda_r}{\sim} [j]$. If $\lambda_r = 0$, we also call a 0-equivalent design *parallel equivalent*.

A $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-difference set S is called λ_r*-equivalent* if the design generated by S is λ_r-equivalent.

A λ_r-equivalent difference set or design is also called a *singly equivalent difference set* or *design* respectively.

Theorem 1.4. (Mendelsohn and Liang [5]) *A* $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$*-design* D *generated by a* $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$*-difference set* S *is* λ_r*-equivalent* $(1 \leq r \leq m)$ *if and only if* $(t+1) \mid v$ *and all the residues appearing* λ_r *times as differences of* S *are:*

$$\alpha, 2\alpha, 3\alpha, \ldots, t\alpha,$$

where t *is the number of residues appearing* λ_r *times as differences of* S *and* $\alpha = \frac{v}{t+1}$.

In a 0-equivalent $(v, k, [\ldots, 0]; t)$-design, the set of all lines which are parallel to the line $[i]$ is called the *parallel class* $\langle i \rangle$, 0*-equivalence class* $\langle i \rangle$ or *line class* $\langle i \rangle$ $(i = 0, 1, \ldots, \frac{v}{t+1} - 1)$. We say that a line class $\langle i \rangle$ *misses* a point x if no line in $\langle i \rangle$ contains the point x.

Theorem 1.5. (Mendelsohn and Liang [5]) *In a* 0*-equivalent* $(v, k, [\ldots, 0]; t)$*-design, the point* 0 *is missing from the line class* $\langle l \rangle$ *if and only if the point* a *is missing from the line class* $\langle l + a \rangle \pmod{\frac{v}{t+1}}$*, where* $(0 \leq a < v)$.

In a 0-equivalent $(v, k, [\ldots, 0]; t)$-design, let $\boldsymbol{u}$ be the number of line classes from which the point 0 is missing, where u is a nonnegative integer. We have:

Lemma 1.6. (Mendelsohn and Liang [5]) *For a* 0*-equivalent* $(v, k, [\ldots, 0]; t)$*-design,* $v = (k + u)(t + 1)$.

Theorem 1.7. (Mendelsohn and Liang [5]) *For a* 0*-equivalent* $(v, k, [\lambda, 0]; t)$*-design with* $u > 1$*, assume the point* 0 *is missing from the parallel classes* $\langle i_1 \rangle$, $\langle i_2 \rangle$, $\ldots$, $\langle i_u \rangle$*, where* $0 \leq i_1 < i_2 < \ldots < i_u < \alpha$*; then* $\{i_1, i_2, \ldots, i_u\}$ *is an* (α, u, μ)*-difference set, where* $u = k - \lambda(t+1) + \mu$ *and* $\mu(\alpha - 1) = u(u - 1)$.

2 Concept of doubly equivalent design

A $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_m])$-design with $m > 2$ can be both λ_1-equivalent and λ_2-equivalent.

Example 2.1. $\{0, 1, 2, 4, 12, 17\} \bmod 21$ *is a* $(21, 6, [2, 1, 0]; 12, 6, 2)$*-difference set. Each of*

$$3, 6, 9, 12, 15, 18$$

appears in the multiset of its differences just once and both 7 *and* 14 *are missing while all other nonzero residues appear twice.*

By Theorem 1.4, the design generated by this difference set is both 1*-equivalent and* 0*-equivalent.*

Definition 2.2. *A* $(v, k, [\lambda_1, \lambda_2, \ldots]; s, t, \ldots)$*-design which is both* λ_1*-equivalent and* λ_2*-equivalent is called a* doubly equivalent design*. A* doubly equivalent difference set *is defined similarly.*

Clearly, a doubly equivalent design (or difference set) is also a singly equivalent design (or difference set respectively). In this paper, from now on, we always denote $\frac{v}{s+1}$ by $\boldsymbol{\alpha}$ and $\frac{v}{t+1}$ by $\boldsymbol{\beta}$ respectively.

Theorem 2.3. *If there exists a $(v, k, [\lambda_1, \lambda_2, \ldots]; s, t, \ldots)$-design which is both λ_1- and λ_2-equivalent, then $(s+1, t+1) = 1$ and $(s+1)(t+1) \mid v$.*

Proof. By Theorem 1.4, we have $(s+1) \mid v$ and $(t+1) \mid v$. Let $\alpha = \frac{v}{s+1}$ and $\beta = \frac{v}{t+1}$. Since $\lambda_1 \neq \lambda_2$, a nonzero residue cannot appear both λ_1 and λ_2 times as a difference. Hence, we must have $v = [\alpha, \beta]$, where $[\alpha, \beta]$ is the least common multiple of α and β. Thus $(s+1, t+1) = (\frac{v}{\alpha}, \frac{v}{\beta}) = 1$. Therefore, $(s+1)(t+1) \mid v$. □

3 Super Classes

Definition 3.1. *In a λ-equivalent $(v, k, [\ldots, \lambda]; t)$-design, the set $\mathcal{C}$ of all lines which are λ-intersecting to the line $[i]$ is called the* λ-equivalence class $\langle i \rangle_\lambda$ $(i = 0, 1, \ldots, \frac{v}{t+1} - 1)$. *If $\lambda = 0$, then we just simply denote $\langle i \rangle_\lambda$ by $\langle i \rangle$.*

Definition 3.2. *Let D be a $(v, k, [\lambda_1, \lambda_2, \ldots]; s, t, \ldots)$-design which is both λ_1- and λ_2-equivalent. A* super class $\mathcal{S}$ *containing a line $[i]$ in D is the λ_1-equivalence class $\langle i \rangle_{\lambda_1}$ union all the λ_2-equivalence classes containing a line in $\langle i \rangle_{\lambda_1}$. We denote the super class $\mathcal{S}$ containing the line $[i]$ by $\hat{i}$.*

Theorem 3.3. *Let D be a $(v, k, [\lambda_1, \lambda_2, \ldots]; s, t, \ldots)$-design which is both λ_1- and λ_2-equivalent and $\mathcal{S}$ be a super class in D. Then the following five statements are equivalent to each other ($\mathbf{Z}$ is the set of all integers):*

(1) $[i_1], [i_2] \in \mathcal{S}$;

(2) $\exists\ [i_3]$ *such that* $[i_1] \overset{\lambda_1}{\sim} [i_3]$ *and* $[i_2] \overset{\lambda_2}{\sim} [i_3]$;

(3) $\exists\ [i_4]$ *such that* $[i_1] \overset{\lambda_2}{\sim} [i_4]$ *and* $[i_2] \overset{\lambda_1}{\sim} [i_4]$;

(4) $i_1 - i_2 = a\alpha + b\beta$, *where* $a, b \in \mathbf{Z}$;

(5) $d \mid (i_1 - i_2)$, *where* $d = (\alpha, \beta)$.

Proof. Let $\alpha = \frac{v}{s+1}$ and $\beta = \frac{v}{t+1}$.

(1) ⇒ (2). Since $[i_1], [i_2] \in \mathcal{S}$, there exist $a, b \in \mathbf{Z}$ such that $[i_1 + a\beta] \overset{\lambda_1}{\sim} [i_2 + b\beta]$. Thus $[i_1] \overset{\lambda_1}{\sim} [i_2 + (b-a)\beta]$. Take $i_3 = i_2 + (b-a)\beta$. Then $[i_1] \overset{\lambda_1}{\sim} [i_3]$ and $[i_2] \overset{\lambda_2}{\sim} [i_3]$.

(2) ⇒ (1). If $\exists\ [i_3]$ such that $[i_1] \overset{\lambda_1}{\sim} [i_3]$ and $[i_2] \overset{\lambda_2}{\sim} [i_3]$, then, by Definition 3.2, we have $[i_1], [i_2] \in \widehat{i_1}$.

Therefore, (1) and (2) are equivalent. Similarly, (1) and (3) are equivalent.

(2) $\Rightarrow$ (4). If $[i_1] \overset{\lambda_1}{\sim} [i_3]$ and $[i_2] \overset{\lambda_2}{\sim} [i_3]$, then $i_1 - i_3 = a\alpha$ and $i_2 - i_3 = b'\beta$, where $a, b' \in \mathbf{Z}$. So, $i_1 - i_2 = a\alpha - b'\beta = a\alpha + b\beta$, where $b = -b'$.

(4) $\Rightarrow$ (2). If $i_1 - i_2 = a\alpha + b\beta$, then set $i_3 = i_1 - a\alpha = i_2 + b\beta$. Hence, $[i_1] \overset{\lambda_1}{\sim} [i_3]$ and $[i_2] \overset{\lambda_2}{\sim} [i_3]$.

Thus, (2) and (4) are equivalent.

(4) $\Rightarrow$ (5). Since $i_1 - i_2 = a\alpha + b\beta$ and $d \mid \alpha, \beta$, so, $d \mid (i_1 - i_2)$.

(5) $\Rightarrow$ (4). Because $d = (\alpha, \beta)$, we have $d = a_1\alpha + b_1\beta$, where $a_1, b_1 \in \mathbf{Z}$. Let $i_1 - i_2 = n_1 d$, where $n_1 \in \mathbf{Z}$. Then, $i_1 - i_2 = (n_1 a_1)\alpha + (n_1 b_1)\beta = a\alpha + b\beta$, where $a = n_1 a_1, b = n_1 b_1 \in \mathbf{Z}$.

Accordingly, (4) and (5) are equivalent.

Therefore, all five statements are equivalent to each other. □

Corollary 3.4. *Let D be a $(v, k, [\lambda_1, \lambda_2, \ldots]; s, t, \ldots)$-design which is both λ_1- and λ_2-equivalent. Then all super classes in D constitute a partition of all lines of D. Let $\mathcal{S}$ be a super class in D. Then $\mid \mathcal{S} \mid = (s+1)(t+1)$.*

Proof. By the fact that (1) and (4) of Theorem 3.3 are equivalent, the relation that two lines are in the same super class is an equivalence relation. So, all super classes in D constitute a partition of all lines of D. Because a super class contains $s+1$ λ_2-equivalence classes, we have that $\mid \mathcal{S} \mid = (s+1)(t+1)$. □

Let D be a $(v, k, [\lambda_1, \lambda_2, \ldots]; s, t, \ldots)$-design which is both λ_1- and λ_2-equivalent. Since the lines in the same λ_1-equivalence class are in the same super class, we see that all super classes in D constitute a partition of all λ_1-equivalence classes in D. Similarly, we also see that all super classes in D constitute a partition of all λ_2-equivalence classes in D.

Theorem 3.5. *Let D be a $(v, k, [\lambda_1, \lambda_2, \ldots]; s, t, \ldots)$-design which is both λ_1- and λ_2-equivalent and $d = (\alpha, \beta)$, where $\alpha = \frac{v}{s+1}$ and $\beta = \frac{v}{t+1}$. Then $v = (s+1)(t+1)d$.*

Proof. Since (1) and (5) of Theorem 3.3 are equivalent, two λ_2-equivalent classes $\langle i \rangle_{\lambda_2}$ and $\langle j \rangle_{\lambda_2}$ are in the same super class if and only if $d \mid (i - j)$. So, two λ_2-equivalent classes in

$$\langle 0 \rangle_{\lambda_2}, \langle 1 \rangle_{\lambda_2}, \ldots, \langle \beta - 1 \rangle_{\lambda_2}$$

are in the same super class if and only if their "distance" is a multiple of d. Hence, we have exactly d super classes:

$$\hat{0}, \hat{1}, \ldots, \widehat{d-1}.$$

Counting the total lines of D, by Corollary 3.4, we have $v = (s+1)(t+1)d$. □

Although $v = (s+1)(t+1)d$ can be proved directly by number theory, the above proof is from the point of view of the super classes. Meanwhile, the equivalence relation generated by super classes is the join of two equivalence relations: λ_1- and λ_2-equivalence. So, we can also obtain the above results from ring theory.

Example 3.6. $\{0, 1, 2, 4, 6, 7, 11, 17\}$ mod 24 *is a* $(24, 8, [3, 2, 1, 0]; 14, 6, 2, 1)$*-difference set. Each of* $8, 16$ *appears in the multiset of its differences just once and* 12 *is missing while all other nonzero residues appear either twice or three times.*

By Theorem 1.4, the design generated by this difference set is both 1*- and* 0*-equivalent.*

Since $\alpha = \frac{v}{s+1} = 8$ *and* $\beta = \frac{v}{t+1} = 12$*, we have* $d = (\alpha, \beta) = 4$*. By Theorem 3.3, the super class* $\hat{0}$ *contains* 6 *lines:* [0], [4], [8], [12], [16] *and* [20]*. By Theorem 3.5,*

$$v = (s + 1)(t + 1)d = 3 \cdot 2 \cdot 4 = 24.$$

Theorem 3.7. *For a* λ_2*- and* 0*-equivalent* $(v, k, [\lambda_1, \lambda_2, 0]; r, s, t)$*-design* D*, let* $\alpha = \frac{v}{s+1}$ *and* $\beta = \frac{v}{t+1}$*. If* $d = \frac{v}{(s+1)(t+1)} > 1$*, then* $u > 1$*. In addition, assume the point* 0 *is missing from the parallel classes* $\langle i_1 \rangle$*,* $\langle i_2 \rangle$*, …,* $\langle i_u \rangle$*, where* $0 \le i_1 < i_2 < \ldots < i_u < \beta$*, then* $\{i_1, i_2, \ldots, i_u\}$ *is a* μ_2*-equivalent* $(\beta, u, [\mu_1, \mu_2]; s)$*-difference set and* $u = k - \lambda_1(t + 1) + \mu_1$*,* $u = k - \lambda_1 t - \lambda_2 + \mu_2$*. Thus we have* $\lambda_1 - \lambda_2 = \mu_1 - \mu_2$ *(when* $\lambda_1 < \lambda_2$*, we may write* $\lambda_2 - \lambda_1 = \mu_2 - \mu_1$*).*

Proof. Since $v = (s + 1)(t + 1)d$, by Theorem 3.5, $d = (\alpha, \beta)$. D has d super classes:

$$\hat{0}, \hat{1}, \ldots, \widehat{d-1}.$$

Since $d > 1$, we have that $\langle 0 \rangle$ and $\langle d \rangle$ are in the same super class while $\langle 0 \rangle$ and $\langle 1 \rangle$ are in different super classes. So, every line in $\langle d \rangle$ meets one line in $\langle 0 \rangle$ in λ_2 points while it meets every other line in $\langle 0 \rangle$ in λ_1 points. Every line in $\langle 1 \rangle$ meets every line in $\langle 0 \rangle$ in λ_1 points. If $\lambda_1 > \lambda_2$, then there are some points not on any lines of $\langle 0 \rangle$ and $\langle 1 \rangle$; if $\lambda_1 < \lambda_2$, then there are some points not on any lines of $\langle 0 \rangle$ and $\langle d \rangle$. In either case, by Theorem 1.5, 0 is missing from more than one parallel class, that is, $u > 1$.

Assume 0 is missing from the line classes $\langle i_1 \rangle$, $\langle i_2 \rangle$, …, $\langle i_u \rangle$, where $0 \le i_1 < i_2 < \ldots < i_u < \beta$. By Theorem 1.5 again, we have (in the following discussion, for each line class $\langle a \rangle$, the value a should be taken mod β):

$$\begin{array}{lllll}
\langle i_1 \rangle, & \langle i_2 \rangle, & \ldots, & \langle i_u \rangle & \text{miss} \\
0, & \beta, & 2\beta, & \ldots, & t\beta; \\
 & & & & \\
\langle i_1 + 1 \rangle, & \langle i_2 + 1 \rangle, & \ldots, & \langle i_u + 1 \rangle & \text{miss} \\
1, & 1 + \beta, & 1 + 2\beta, & \ldots, & 1 + t\beta; \\
 & & & & \\
\langle i_1 + 2 \rangle, & \langle i_2 + 2 \rangle, & \ldots, & \langle i_u + 2 \rangle & \text{miss} \\
2, & 2 + \beta, & 2 + 2\beta, & \ldots, & 2 + t\beta; \\
 & & & & \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
 & & & & \\
\langle i_1 + \beta - 1 \rangle, & \langle i_2 + \beta - 1 \rangle, & \ldots, & \langle i_u + \beta - 1 \rangle & \text{miss} \\
\beta - 1, & \beta - 1 + \beta, & \beta - 1 + 2\beta, & \ldots, & \beta - 1 + t\beta.
\end{array}$$

So, two line classes $\langle l\rangle$ and $\langle m\rangle$ $(l < m)$ have a common missing point if and only if $m - l \equiv \pm(i_q - i_p) \pmod{\beta}$ for some p and q, where $1 \le p < q \le u$.

Let $T = \{i_1, i_2, \ldots, i_u\}$. Any line class contains $k(t+1)$ points. Given two line classes $\langle l\rangle$ and $\langle m\rangle$ in the same super class, we have that every line in $\langle l\rangle$ meets one line in $\langle m\rangle$ in λ_2 points while it meets every other line in $\langle m\rangle$ in λ_1 points. Thus between them they cover $k(t+1)+(k-\lambda_1 t-\lambda_2)(t+1)$ points. Let n be the number of common missed points of $\langle l\rangle$ and $\langle m\rangle$, then $n = v-k(t+1)-(k-\lambda_1 t-\lambda_2)(t+1)$, which is independent of the choice of $\langle l\rangle$ and $\langle m\rangle$ as far as they are in the same super class. Meanwhile, two line classes $\langle l\rangle$ and $\langle m\rangle$ are in the same super class if and only if $d \mid (m-l)$. Notice that $(s+1)d = \beta$. Then, as in the proof, given in Mendelsohn and Liang [5], of Theorem 1.7, we have that each of d, $2d$, …, sd must appear the same number of times, say μ_2 times, as a difference of T. Similarly, each nonzero residue not in $\{d, 2d, \ldots, sd\}$ must appear the same number of times, say μ_1 times, as a difference of T. Therefore, $T = \{i_1, i_2, \ldots, i_u\}$ is a $(\beta, u, [\mu_1, \mu_2]; s)$-difference set. By Theorem 1.4, it is a μ_2-equivalent difference set.

If $\langle i_1\rangle, \langle i_2\rangle, \ldots, \langle i_u\rangle$ miss a point, say x, other than $0, \beta, 2\beta, \ldots, t\beta$, then they also miss t other points: $x+\beta, x+2\beta, \ldots, x+t\beta$. We may assume $0 < x < \beta$. Since $\langle i_1\rangle, \langle i_2\rangle, \ldots, \langle i_u\rangle$ miss $x, x+\beta, \ldots, x+t\beta$, we have that $\langle i_1+(\beta-x)\rangle, \langle i_2+(\beta-x)\rangle, \ldots, \langle i_u+(\beta-x)\rangle$ also miss $0, \beta, 2\beta, \ldots, t\beta$. Thus $\langle i_1+(\beta-x)\rangle, \langle i_2+(\beta-x)\rangle, \ldots, \langle i_u+(\beta-x)\rangle$ are the same as $\langle i_1\rangle, \langle i_2\rangle, \ldots, \langle i_u\rangle$. In other words, if we use B_i to denote the i-th block of the design generated by T $(i = 0, 1, \ldots, \beta-1)$, then B_0 and $B_{\beta-x}$ have u objects in common. By Theorem 1.1, $\beta - x$ appears u times as a difference in T. If $d \nmid (\beta-x)$, then $\mu_1 = u$. Thus $\mid B_0 \cap B_1 \mid= u$. So, $\langle i_u+1\rangle = \langle 0\rangle = \langle i_1\rangle$. However, $\langle i_1\rangle = \langle 0\rangle$ implies that $\langle 0\rangle$ misses 0, which is impossible. Therefore, $d \mid (\beta-x)$. So, $\mu_2 = u$ and $d \mid x$.

If $\lambda_1 > \lambda_2$, then $\mu_1 > \mu_2 = u$, which is a contradiction by Corollary 1.2. Hence $\lambda_1 < \lambda_2$. Since $\mu_2 = u$, we have $B_0 = B_d$. Thus, $\langle i_1\rangle, \langle i_2\rangle, \ldots, \langle i_u\rangle$ also miss $d, d+\beta, d+2\beta, \ldots, d+t\beta$. Similarly, they also miss $id, id+\beta, id+2\beta, \ldots, id+t\beta$, for $i = 2, 3, \ldots s$. Accordingly, since $d \mid x$, all the common missed points of $\langle i_1\rangle, \langle i_2\rangle, \ldots, \langle i_u\rangle$ are:

$$\begin{array}{lllll}
0, & \beta, & 2\beta, & \ldots, & t\beta, \\
d, & d+\beta, & d+2\beta, & \ldots, & d+t\beta, \\
2d, & 2d+\beta, & 2d+2\beta, & \ldots, & 2d+t\beta, \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
sd, & sd+\beta, & sd+2\beta, & \ldots, & sd+t\beta.
\end{array}$$

Hence, counting the total number of points of D, whether they are covered by $\langle 0\rangle$ and $\langle d\rangle$ or not, by Lemma 1.6 we have

$$(k+u)(t+1) = k(t+1) + (k-\lambda_1 t-\lambda_2)(t+1) + \frac{\mu_2}{s+1}(s+1)(t+1).$$

Thus

$$u = k - \lambda_1 t - \lambda_2 + \mu_2.$$

Similarly, counting the total number of points of D, whether they are covered by $\langle 0\rangle$ and $\langle 1\rangle$ or not, we obtain

$$u = k - \lambda_1(t+1) + \mu_1.$$

(Notice that in this case we have $\mu_2 = u$ and $k - \lambda_1 t - \lambda_2 = 0$.)

If $\langle i_1\rangle, \langle i_2\rangle, \ldots, \langle i_u\rangle$ do not have any common missed points other than $0, \beta, 2\beta, \ldots, t\beta$, then counting the total number of points of D, whether they are covered by $\langle 0\rangle$ and $\langle d\rangle$ or not, we get

$$(k+u)(t+1) = k(t+1) + (k - \lambda_1 t - \lambda_2)(t+1) + \mu_2(t+1).$$

So,

$$u = k - \lambda_1 t - \lambda_2 + \mu_2.$$

Similarly, counting the total number of points of D, whether they are covered by $\langle 0\rangle$ and $\langle 1\rangle$ or not, we obtain

$$u = k - \lambda_1(t+1) + \mu_1.$$

□

4 λ- and 0-equivalent designs with $u = 0$

Theorem 4.1. *Let D be a λ_2- and 0-equivalent $(v, k, [\lambda_1, \lambda_2, 0]; r, s, t)$-design with $u = 0$, then $v = k(t+1)$, $k = t\lambda_1 + \lambda_2$, $k = s+1$, $r = st$.*

Proof. Since $u = 0$, by Lemma 1.6, $v = (k+u)(t+1) = k(t+1)$. Because $u = 0$ again, by Theorem 3.7, we have that $d = 1$. Accordingly, a line $[l]$ not in the parallel class $\langle 0\rangle$ meets one line in $\langle 0\rangle$ in λ_2 points while meeting every other line in $\langle 0\rangle$ in λ_1 points. Since $u = 0$, every point on the line $[l]$ should be on some line in $\langle 0\rangle$ by Theorem 1.5. Thus, $k = t\lambda_1 + \lambda_2$. Since $v = k(t+1) = (s+1)(t+1)d$ and $d = 1$, we have $k = (s+1)d = s+1$. Since $v = (s+1)(t+1)$ and $r + s + t = v - 1$, we have $r = st$. □

We can also derive $k = t\lambda_1 + \lambda_2$ from the equalities $k(k-1) = r\lambda_1 + s\lambda_2$, $k = s+1$ and $r = st$.

Under the assumption of Theorem 4.1, since $d = 1$, D just has one superclass, which contains all the lines of D.

By Theorem 4.1, given λ_1, λ_2 and t, we can determine the values of the other parameters v, k, r and s. Then we run a computer program to see whether there exist any λ_2- and 0-equivalent difference sets $(v, k, [\lambda_1, \lambda_2, 0]; r, s, t)$ with $u = 0$.

Example 4.2. $\{0,1,2,4,14,15,19,21\}$ mod 24 *is a* $(24,8,[3,2,0];14,7,2)$*-difference set. Each of*

$$3,6,9,12,15,18,21$$

appears in the multiset of its differences exactly twice and both 8 *and* 16 *are missing, while all other nonzero residues appear three times.*

By Theorem 1.4, the design generated by this difference set is both 2*-equivalent and* 0*-equivalent.*

5 Designs generated by λ- and 0-equivalent designs with $u = 1$

Theorem 5.1. (Mendelsohn and Liang [5]) *Let* $S = \{0, a_2, a_3, \ldots, a_k\}$ *be a base set of a* 0*-equivalent* $(v,k,[\lambda,0];t)$*-design* D *with* $u = 1$*, let* $t+1$ *points which are not on any line in the parallel class* $\langle 0\rangle$ *be:* $b, b+\alpha, b+2\alpha, \ldots, b+t\alpha$*. Then the set* $T = \{0, a_2, a_3, \ldots, a_k, b, b+\alpha, b+2\alpha, \ldots, b+t\alpha\}$*, obtained by adding those* $t+1$ *points to* S*, generates a* $(t+1)$*-equivalent* $(v, k+t+1, [\lambda+2, t+1]; t)$*-design* E *when* $\lambda+2 \neq t+1$*; otherwise it generates a* $(v, k+t+1, \lambda+2)$*-design* E*.*

If $S = \{0, a_2, a_3, \ldots, a_k\}$ is a base set of a λ- and 0-equivalent $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_{m-2}, \lambda, 0]; t)$-design with $u = 1$, then, since

$$v = (k+1)(t+1) = k(t+1) + (t+1),$$

there are exactly $t+1$ points which are not on any lines in the parallel class $\langle 0\rangle$. So, we can extend Theorem 5.1 to the case of doubly equivalent designs.

Theorem 5.2. *Let* $S = \{0, a_2, a_3, \ldots, a_k\}$ *be a base set of a* λ*- and* 0*-equivalent* $(v, k, [\lambda_1, \lambda_2, \ldots, \lambda_{m-2}, \lambda, 0]; \ldots, s, t)$*-design* D *with* $u = 1$*, let* $t+1$ *points which are not on any lines in the parallel class* $\langle 0\rangle$ *be:*

$$b, b+\alpha, b+2\alpha, \ldots, b+t\alpha.$$

Then the set $T = \{0, a_2, a_3, \ldots, a_k, b, b+\alpha, b+2\alpha, \ldots, b+t\alpha\}$*, obtained by adding those* $t+1$ *points to* S*, generates another design:*

(1) if $t+1 = \lambda_i+2$ *for some* i $(1 \leq i \leq m-2)$*, then it generates a* $(\lambda+2)$*-equivalent* $(v, k+t+1, [\lambda_1+2, \lambda_2+2, \ldots, \lambda_{m-2}+2, \lambda+2]; s)$*-design;*

(2) if $t+1 = \lambda+2$*, then it generates a* $(v, k+t+1, [\lambda_1+2, \lambda_2+2, \ldots, \lambda_{m-2}+2, \lambda+2]; s+t)$*-design;*

(3) if $t+1 \neq \lambda_1+2, \lambda_2+2, \ldots, \lambda_{m-2}+2, \lambda+2$*, then it generates a* $(\lambda+2)$*- and* $(t+1)$*-equivalent* $(v, k+t+1, [\lambda_1+2, \lambda_2+2, \ldots, \lambda_{m-2}+2, \lambda+2, t+1]; \ldots, s, t)$*-design.*

Proof. The proof is similar to the proof, given in Mendelsohn and Liang [5], of Theorem 5.1. □

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