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A formula for the generating functions of powers of Horadam's sequence

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1 Introduction and the main result

The second-order linear recurrence sequence $(w_n(a,b;p,q))_{n\geq 0}$, or briefly $(w_n)_{n\geq 0}$, is defined by

$$w_{n+2} = pw_{n+1} + qw_n, \tag{1}$$

with $w_0 = a, w_1 = b$ and $n \geq 0$. This sequence was introduced in 1965 by Horadam [3, 4], and it generalizes many sequences (see [1, 5]). Examples of such sequences are Fibonacci number sequences $(F_n)_{n\geq 0}$, Lucas number sequences $(L_n)_{n\geq 0}$, and Pell number sequences $(P_n)_{n\geq 0}$, when one has $p = q = b = 1$, $a = 0$; $p = q = b = 1$, $a = 2$; and $p = 2$, $q = b = 1$, $a = 0$; respectively. In this paper we are interested in studying the generating function for powers of Horadam's sequence, that is, $\mathcal{H}_k(x; a, b, p, q) = \mathcal{H}_k(x) = \sum_{n\geq 0} w_n^k x^n$.

In 1962, Riordan [7] found the generating function for powers of Fibonacci numbers. He proved that the generating function $\mathcal{F}_k(x) = \sum_{n\geq 0} F_n^k x^n$ satisfies the recurrence relation

$$(1 - a_k x + (-1)^k x^2)\mathcal{F}_k(x) = 1 + kx \sum_{j=1}^{[k/2]} (-1)^j \frac{a_{kj}}{j} \mathcal{F}_{k-2j}((-1)^i x)$$

for $k \geq 1$, where $a_1 = 1$, $a_2 = 3$, $a_s = a_{s-1} + a_{s-2}$ for $s \geq 3$, and $(1 - x - x^2)^{-j} = \sum_{k\geq 0} a_{kj} x^{k-2j}$. Horadam [4] gave a recurrence relation for $\mathcal{H}_k(x)$ (see also [6]). Recently, Haukkanen [2] studied linear combinations of Horadam's sequences and the generating function of the ordinary product of two of Horadam's sequences. The main result of this paper can be formulated as follows.

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Let $\Delta_k = (\Delta_k(i,j))_{1\le i,j\le k} = \Delta_k(p,q)$ be the $k\times k$ matrix

$$\begin{pmatrix}
1-p^kx-q^kx^2 & -xp^{k-1}q^1\binom{k}{1} & -xp^{k-2}q^2\binom{k}{2} & \cdots & -xp^2q^{k-2}\binom{k}{k-2} & -xpq^{k-1}\binom{k}{k-1}\\
-p^{k-1}x & 1-xp^{k-2}q^1\binom{k-1}{1} & -xp^{k-3}q^2\binom{k-1}{2} & \cdots & -xpq^{k-2}\binom{k-1}{k-2} & -xq^{k-1}\binom{k-1}{k-1}\\
-p^{k-2}x & -xp^{k-3}q^1\binom{k-2}{1} & 1-xp^{k-4}q^2\binom{k-2}{2} & \cdots & -xq^{k-2}\binom{k-2}{k-2} & 0\\
-p^{k-3}x & -xp^{k-4}q^1\binom{k-3}{1} & -xp^{k-5}q^2\binom{k-3}{2} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & & \vdots & \vdots\\
-p^2x & -xpq^1\binom{2}{1} & -xq^2\binom{2}{2} & \cdots & 1 & 0\\
-p^1x & -xq^1\binom{1}{1} & 0 & \cdots & 0 & 1
\end{pmatrix},$$

and let $\delta_k = \delta_k(p,q,a,b)$ be the $k\times k$ matrix

$$\begin{pmatrix}
a^k+g_kx & -xp^{k-1}q^1\binom{k}{1} & -xp^{k-2}q^2\binom{k}{2} & \cdots & -xp^2q^{k-2}\binom{k}{k-2} & -xpq^{k-1}\binom{k}{k-1}\\
g_{k-1}x & 1-xp^{k-2}q^1\binom{k-1}{1} & -xp^{k-3}q^2\binom{k-1}{2} & \cdots & -xpq^{k-2}\binom{k-1}{k-2} & -xq^{k-1}\binom{k-1}{k-1}\\
g_{k-2}x & -xp^{k-3}q^1\binom{k-2}{1} & 1-xp^{k-4}q^2\binom{k-2}{2} & \cdots & -xq^{k-2}\binom{k-2}{k-2} & 0\\
g_{k-3}x & -xp^{k-4}q^1\binom{k-3}{1} & -xp^{k-5}q^2\binom{k-3}{2} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & & \vdots & \vdots\\
g_2x & -xpq^1\binom{2}{1} & -xq^2\binom{2}{2} & \cdots & 1 & 0\\
g_1x & -xq^1\binom{1}{1} & 0 & \cdots & 0 & 1
\end{pmatrix},$$

where $g_j = (b^j - a^jp^j)a^{k-j}$ for all $j = 1,2,\ldots,k$.

Theorem 1.1 *The generating function $\mathcal{H}_k(x)$ is given by* $\dfrac{\det(\delta_k)}{\det(\Delta_k)}$.

The paper is organized as follows. In Section 2 we give the proof of Theorem 1.1 and in Section 3 we give some applications for Theorem 1.1.

2 Proofs

Let $(w_n)_{n\ge0}$ be a sequence satisfying relation (1) and k be any positive integer. We define a family $\{A_{k,d}\}_{d=1}^{k}$ of generating functions by

$$A_{k,d}(x) = \sum_{n\ge0} w_n^{k-d}w_{n+1}^{d}x^{n+1}. \tag{2}$$

Now we introduce two relations (Lemma 2.1 and Lemma 2.2) between the generating functions $A_{k,d}(x)$ and $\mathcal{H}_k(x)$ that play the crucial roles in the proof of Theorem 1.1.

Lemma 2.1 *For any $k\ge1$,*

$$(1-p^kx-q^kx^2)\mathcal{H}_k(x) - x\sum_{j=1}^{k-1}\binom{k}{j}p^{k-j}q^jA_{k,k-j}(x) = a^k + x(b^k - a^kp^k).$$

Proof. Using the binomial theorem we get

$$w_{n+2}^k = (pw_{n+1} + qw_n)^k = p^k w_{n+1}^k + \sum_{j=1}^{k-1} \binom{k}{j} p^{k-j} q^j w_{n+1}^{k-j} w_n^j + q^k w_n^k.$$

Multiplying by x^{n+2} and summing over all $n \geq 0$, using Definition (2), we have

$$\mathcal{H}_k(x) - b^k x - a^k = p^k x(\mathcal{H}_k(x) - a^k) + x \sum_{j=1}^{k-1} \binom{k}{j} p^{k-j} q^j A_{k,k-j}(x) + q^k x^2 \mathcal{H}_k(x),$$

as required. □

Lemma 2.2 *For any $k - 1 \geq d \geq 1$,*

$$A_{k,d}(x) - a^{k-d} b^d x = p^d x(\mathcal{H}_k(x) - a^k) + x \sum_{j=1}^{d} \binom{d}{j} p^{d-j} q^j A_{k,k-j}(x).$$

Proof. Using the binomial theorem we have

$$w_n^{k-d} w_{n+1}^d = w_n^{k-d}(pw_n + qw_{n-1})^d = w_n^{k-d} \sum_{j=0}^{d} \binom{d}{j} p^{d-j} q^j w_n^{d-j} w_{n-1}^j.$$

Multiplying by x^{n+1} and summing over all $n \geq 1$ we get

$$A_{k,d}(x) - a^{k-d} b^d x = p^d x(\mathcal{H}_k(x) - a^k) + x \sum_{j=1}^{d} \binom{d}{j} p^{d-j} q^j A_{k,k-j}(x),$$

as required. □

Proof. (Theorem 1.1) By using the above lemmas together with definitions we obtain

$$\Delta_k \cdot [\mathcal{H}_k(x), A_{k,k-1}(x), A_{k,k-2}(x), \ldots, A_{k,1}(x)]^{\mathrm{T}} = v_k,$$

where v_k is given by

$$\left[a^k + x(b^k - a^k p^k), (a^1 b^{k-1} - p^{k-1} a^k)x, (a^2 b^{k-2} - p^{k-2} a^k)x, \ldots, (a^{k-1} b^1 - p^1 x a^k)x\right]^{\mathrm{T}}.$$

Hence the solution of the above equation gives the generating function $\mathcal{H}_k(x) = (\det(\delta_k))/(\det(\Delta_k))$, as claimed in Theorem 1.1. □

k	The generating function $\mathcal{H}_k(x;0,1,1,1)$
1	$\frac{x}{1-x-x^2}$
2	$\frac{x(1-x)}{(1+x)(1-3x+x^2)}$
3	$\frac{x(1-2x-x^2)}{(1+x-x^2)(1-4x-x^2)}$
4	$\frac{x(1+x)(1-5x+x^2)}{(1-x)(1+3x+x^2)(1-7x+x^2)}$
5	$\frac{x(1-7x-16x^2+7x^3+x^4)}{(1-x-x^2)(1+4x-x^2)(1-11x-x^2)}$
6	$\frac{x(1-x)(1-11x-64x^2-11x^3+x^4)}{(1+x)(1-3x+x^2)(1+7x+x^2)(1-18x+x^2)}$

Table 1. The generating function for the powers of Fibonacci numbers

k	The generating function $\mathcal{H}_k(x;2,1,1,1)$
1	$\frac{2-x}{1-x-x^2}$
2	$\frac{4-7x-x^2}{(1+x)(1-3x+x^2)}$
3	$\frac{8-13x-24x^2+x^3}{(1+x-x^2)(1-4x-x^2)}$
4	$\frac{16-79x-164x^2+76x^3+x^4}{(1-x)(1+3x+x^2)(1-7x+x^2)}$
5	$\frac{32-255x-1045x^2+960x^3+235x^4-x^5}{(1-x-x^2)(1+4x-x^2)(1-11x-x^2)}$
6	$\frac{64-831x-5940x^2+11155x^3+5485x^4-716x^5-x^6}{(1+x)(1-3x+x^2)(1+7x+x^2)(1-18x+x^2)}$

Table 2. The generating function for the powers of Lucas numbers

3 Applications

In this section we present some applications for Theorem 1.1.

Fibonacci numbers. If $a = 0$ and $p = q = b = 1$, then Theorem 1.1 for $k = 1,2,3,4,5,6$ yields Table 1.

Lucas numbers. If $a = 2$ and $p = q = b = 1$, then Theorem 1.1 for $k = 1,2,3,4,5,6$ yields Table 2.

Pell numbers. If $a = 0$, $b = q = 1$ and $p = 2$, then Theorem 1.1 for $k = 1,2,3,4,5,6$ yields Table 3.

Chebyshev polynomials of the second kind. If $a = 1$, $b = p = 2t$ and $q = -1$, then Theorem 1.1 for $k = 1,2,3,4,5,6$ yields Table 4.

More generally, if applying Theorem 1.1 for $k = 1,2,3,4$, then we get the following corollary.

Corollary 3.1 *Let $k = 1,2,3,4$. Then the generating function $\mathcal{H}_k(x)$ is given by $(\mathcal{A}_k(x))/(\mathcal{B}_k(x))$ where*

k	The generating function $\mathcal{H}_k(x;0,1,2,1)$
1	$\frac{x}{1-2x-x^2}$
2	$\frac{x(1-x)}{(1+x)(1-6x+x^2)}$
3	$\frac{x(1-4x-x^2)}{(1+2x-x^2)(1-14x-x^2)}$
4	$\frac{x(1+x)(1-14x+x^2)}{(1-x)(1+6x+x^2)(1-34x-x^2)}$
5	$\frac{x(1-38x-130x^2+38x^3+x^4)}{(1-2x-x^2)(1-82x-x^2)(1+14x-x^2)}$
6	$\frac{x(1-x)(1-104x-1210x^2-104x^3+x^4)}{(1+x)(1+34x+x^2)(1-6x+x^2)(1-198x+x^2)}$

Table 3. The generating function for the powers of Pell numbers

k	The generating function $\mathcal{H}_k(x;1,2t,2t,-1)$
1	$\frac{1}{1-2tx+x^2}$
2	$\frac{1+x}{(1-x)((1+x)^2-4xt^2)}$
3	$\frac{1+4tx+x^2}{(1-2tx+x^2)(1+2t(3-4t^2)x+x^2)}$
4	$\frac{(1+x)((1-x)^2+12t^2x)}{(1-x)((1+x)^2-4t^2x)(16t^2(1-t^2)x+(1-x)^2))}$
5	$\frac{1-6tx+2x^2+32t^3x+96t^4x^2+32t^3x^3-32t^2x^2-6x^3t+x^4}{(1+2t(3-4t^2)x+x^2)(1-2tx+x^2)(1-8t^3(4t^2-5)x-10tx+x^2)}$
6	$\frac{(1+x)(x^4+80t^4x^3-24x^3t^2-2x^2-480t^4x^2+640t^6x^2+88t^2x^2+80t^4x-24t^2x+1)}{(1-x)((1+x)^2-4t^2x)((1-x)^2+16t^2(1-t^2)x)((1+x)^2-4t^2(4t^2-3)^2x)}$

Table 4. The generating function for the powers of Chebyshev polynomials of the second kind

$$\begin{aligned}
\mathcal{A}_1(x) &= a + x(b-ap),\\
\mathcal{A}_2(x) &= (a^2+xb^2)(xq-1)a^2 + a^2p^2x(xq+1) - 2x^2pqab,\\
\mathcal{A}_3(x) &= (a^3+b^3x-a^3p^3x)(1-q^3x^2) - 2xpq(a^3+b^3x) - x^2a^3p^4q + 3ab^2x^2p^2q\\
&\quad +3ab^2x^3pq^3 - 3a^2bx^3p^2q^3 + 3a^2bx^2pq^2 - 3p^2x^2a^3q^2,\\
\mathcal{A}_4(x) &= a^4 + (b^4 - a^4(p^4+3p^2q+q^2))x - q(5qa^4p^4 + b^4q + a^4q^3 + a^4p^6 + 7q^2a^4p^2\\
&\quad -6qb^2a^2p^2 - 4b^3ap^3 - 4q^2ba^3p + 3b^4p^2)x^2 + q^3(-8qba^3p^3 - 3b^4p^2 + a^4q^3\\
&\quad +5qa^4p^4 - 6b^2a^2p^4 - b^4q + a^4p^6 - 4q^2ba^3p + 8b^3ap^3 + 4q^2a^4p^2 + 4qb^3ap)x^3\\
&\quad +q^6(ap-b)^4x^4
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}_1(x) &= 1 - px - x^2q,\\
\mathcal{B}_2(x) &= (1+xq)(p^2x - (xq-1)^2),\\
\mathcal{B}_3(x) &= (1+pqx-q^3x^2)(1-3pqx-p^3x-q^3x^2),\\
\mathcal{B}_4(x) &= (1-q^2x)((1+q^2x)^2+p^2qx)((1-q^2x)^2-p^2x(p^2+4q)).
\end{aligned}$$

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