AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **30** (2004), Pages 197–206

Crossing numbers of sequences of graphs I: general tiles

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Abstract

A *tile* T is a connected graph together with two specified sequences of vertices, the left and right walls. The crossing number $tcr(T)$ of a tile T is the minimum number of crossings among all drawings of T in the unit square with the left wall in order down the left hand side and the right wall in order down the right hand side. The tile T^n is obtained by gluing n copies of T in a linear fashion, while the graph $\circ(T^n)$ is obtained by gluing n copies of T in a circular fashion. Our main theorem is: $\lim_{n\to\infty} tcr(T^n)/n = \lim_{n\to\infty} cr(\circ(T^n))/n$. Thus, for any tile T, there are constants $acr(T)$ and c_T such that

$$n \cdot acr(T) - c_T \le cr(\circ(T^n)) \le n \cdot acr(T) + c_T.$$

1 Introduction

Many well-known examples of infinite sequence of graphs, including the Generalized Petersen Graphs $P(n,k)$ [2, 8] and the Cartesian product $C_m \times C_n$ of two cycles [1, 4], are constructed by gluing many copies of a small piece (the "tile") in a circular fashion. It is the purpose of this work to provide a theory of tiles and to show how asymptotic estimates for the crossing numbers of the graphs so obtained can be given.

In Section 2 we shall give the precise definitions of a tile T and the "large" graphs obtained by gluing together many copies of the tile. The key points are: (a) that the crossing numbers of the long linear tiles T^n constructed are subadditive, so that the average number of crossings per copy of T has a limiting value ("the average crossing number $acr(T)$ of T") (Section 3); and (b) the crossing number of the circular version of T^n, which is a graph $\circ(T^n)$, is closely related to the crossing number of T^n, and therefore $\lim_{n\to\infty} cr(\circ(T^n))/n = acr(T)$ (Section 4).

The theory was motivated by examples and in turn sheds light on examples. In Section 5 we mention some examples.

2 Tiles and Sequences of Tiles

A *tile* is a 3-tuple $T = (G, L, R)$, where

- G is a connected graph;
- $L = L[1], L[2], \dots, L[|L|]$ is a finite sequence of vertices of $V(G)$, called the *left-wall*;
- $R = R[1], R[2], \dots, R[|R|]$ is a finite sequence of vertices of $V(G)$, called the *right-wall*; and
- all the vertices in L and R are distinct.

A *tile drawing* of a tile $T = (G, L, R)$ is a drawing of T in the unit square $S = \{(x, y) : -1 \le x \le 1 \text{ and } -1 \le y \le 1\}$ such that: the intersection with the boundary $\{(x, y) \in S : \text{either } x \in \{-1, 1\} \text{ or } y \in \{-1, 1\}\}$ is precisely $L \cup R$; the vertices in L occur in the line $x = -1$, with the y-coordinates of $L[1], L[2], ..., L[\,|L|\,]$ decreasing; and the vertices in R occur in the line $x = 1$, with the y-coordinates of $R[1], R[2], ..., R[\,|R|\,]$ also decreasing. The *tile crossing number* $tcr(T)$ of a tile T is the smallest number of crossings in any tile drawing of T.

A tile $T_1 = (G_1, L_1, R_1)$ is *compatible* with a tile $T_2 = (G_2, L_2, R_2)$ if the function $f : R_1 \to L_2$ defined by $f(R_1[i]) = L_2[i]$ is a bijection and

$$\forall\, v_1, v_2 \in R_1, v_1v_2 \in E(G_1) \Leftrightarrow f(v_1)f(v_2) \in E(G_2).$$

A tile is *self-compatible* if it is compatible to itself.

We remark that the preservation of edges in the definition of compatibility is not especially relevant in this work. It is more important in the study of planar tiles [6], where a variation applicable to planar tiles is considered. A model is provided there to easily compute the crossing number of certain infinite families of graphs that are crossing-critical for the same crossing number.

Let G be a graph. Let v_1, $v_2 \in V(G)$ and $v \notin V(G)$. We define $G\{v_1, v_2\}$ to be the contraction G^+/v_1v_2, where G^+ is G if $v_1v_2 \in E(G)$ and $G + v_1v_2$ otherwise.

Thus, $G\{v_1, v_2\}$ identifies the vertices v_1 and v_2 of G into a new vertex v. Our graphs are all simple, so if both v_1 and v_2 are adjacent to a vertex w, we keep only one copy of the edge joining w to the identification vertex.

Let G and H be (disjoint) graphs and let X and Y be sequences of distinct vertices of G and H, respectively, having the same length t. Then $(G \cup H)\{X, Y\}$ is the graph $(\ldots(((G \cup H)\ \{X[1], Y[1]\})\ \{X[2], Y[2]\})\ \ldots)\ \{X[t], Y[t]\}$.

Let $S = (G, L, Q)$ and $T = (H, M, R)$ be compatible tiles. Then the tile ST is $((G \cup H)\ \{Q, M\}, L, R)$. (That is, we simply identify the right wall of S with the left wall of T.) If $T = (G, L, R)$ is a self-compatible tile, then:

1. $\circ(T)$ is the graph $G\{L, R\}$; and
2. T^n is defined inductively by $T^1 = T$ and $T^n = T^{n-1}T$.

Observe that $\circ(T)$ is a graph, not a tile.

We note the following basic facts about the relationships between the crossing numbers of these constructs.

Lemma 1. *If T_1 is compatible with T_2, then $tcr(T_1T_2)) \leq tcr(T_1) + tcr(T_2)$ and if T is self-compatible, then $cr(\circ(T)) \leq tcr(T)$.*

3 Average Crossing Number

In this section, we prove that $\lim_{n\to\infty} tcr(T^n)/n$ exists, and show that simultaneously permuting the left and right walls in the same way does not change the limit.

We need an elementary standard result, sometimes referred to as Fekete's Lemma, which can be found in [5].

Lemma 2. *Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers which is subadditive, i.e., $a_{n+m} \leq a_n + a_m$. Then $\lim_{n\to\infty} a_n/n$ exists and is equal to $\inf_{n\geq 1}\{a_n/n\}$.* □

By Lemma 1, $tcr(T^{m+n}) \leq tcr(T^m) + tcr(T^n)$, so an immediate consequence of Lemma 2 is the following.

Theorem 3. *Let T be a self-compatible tile. Then $\lim_{n\to\infty} tcr(T^n)/n$ exists and is equal to $\inf_{n\geq 1}\{tcr(T^n)/n\}$.* □

The *average crossing number* $acr(T)$ of a tile T is $\lim_{n\to\infty} tcr(T^n)/n$.

Let $T = (G, L, R)$ be a self-compatible tile. Let $t = |L| = |R|$. Let π be a permutation of $\{1, 2, \ldots, t\}$. Let L_π be the sequence $L[\pi(1)], L[\pi(2)], \ldots,\ L[\pi(t)]$

and similarly define R_π. Let $T_\pi = (G, L_\pi, R_\pi)$. Note that if $T^n = (G_1, L, R)$ and $T^n_\pi = (G_2, L_\pi, R_\pi)$, then G_1 and G_2 are isomorphic. We have the following simple result.

Theorem 4. *For any permutations π, π', $acr(T_\pi) = acr(T_{\pi'})$.*

***Proof*:** For each n, let $\mathcal{D}_n$ be a tile drawing of T^n_π having $tcr(T^n_\pi)$ crossings. We can get a tile drawing $\mathcal{D}'_{n+2}$ of $T^{n+2}_{\pi'}$ by adding a tile at each end of T^n_π such that $cr(\mathcal{D}'_{n+2}) \le cr(\mathcal{D}_n) + C(\pi, \pi')$, where $C(\pi, \pi')$ is the sum of the crossing numbers of these two added tiles (so $C(\pi, \pi')$ is independent of n). Therefore,

$$\begin{aligned} acr(T_{\pi'}) &\le \lim_{n\to\infty} \frac{tcr(\mathcal{D}'_{n+2})}{n+2} = \lim_{n\to\infty} \frac{tcr(\mathcal{D}'_{n+2})}{n} \\ &\le \lim_{n\to\infty} \frac{tcr(\mathcal{D}_n) + C(\pi, \pi')}{n} = acr(T_\pi). \end{aligned}$$

The theorem follows since π and π' are arbitrary. □

Note that, by taking the identity permutation for π' in Theorem 4, we have $acr(T_\pi) = acr(T)$.

The following is an immediate consequence of Theorem 3 and the definition of $\circ(T^n)$.

Lemma 5. *Let T be any self-compatible tile. Then, for each $n \ge 1$,*

$$tcr(T^n) \ge n \cdot acr(T) \text{ and } cr(\circ(T^n)) \le n \cdot tcr(T).$$

4 The relationship between cr and acr.

In this section we prove that $\lim_{n\to\infty} cr(\circ(T^n))/n = acr(T)$.

Lemma 6. *Let G be a connected graph embedded without crossings in the plane. Let $W \subseteq V(G)$ and let $\mathcal{T}$ be a tree in G containing W. Then there is a simple closed curve γ in the plane such that:*

1. *$W = V(G) \cap \gamma$;*
2. *$V(G) \cap int(\gamma) = \emptyset$;*
3. *if e is an edge of G, then $e \cap \overline{int(\gamma)}$ consists of at most two arcs, each having its ends in γ; and*
4. *if an edge e of G crosses γ, then e is incident with a vertex of $\mathcal{T}$.*

***Proof*:** Consider the embedding of $\mathcal{T}$ induced by G. This has one face F; let P be a subwalk of the boundary walk of F that starts and ends at vertices v and w of W and contains all the vertices of W. In the perimeter of F, there is an arc σ joining v and w that follows P and is contained in F except for meeting the boundary of F often enough to contain all the vertices of W. There is a second arc σ' contained in F, parallel to σ, joining v and w, but otherwise disjoint from σ. Set $\gamma = \sigma \cup \sigma'$. □

Theorem 7. *Let T be a tile. There is a constant c_T such that, for every $n \geq 1$, there is a permutation π_n for which*

$$cr(\circ(T^n)) \leq tcr(T^n_{\pi_n}) \leq cr(\circ(T^n)) + c_T.$$

***Proof*:** For any permutation π, $\circ(T^n) = \circ(T^n_\pi)$, so the first inequality follows directly from Lemma 1. We only need to prove the second inequality. We do this by turning an optimal drawing of $\circ(T^n)$ into a tile drawing of $T^n_{\pi_n}$ by adding relatively few crossings.

Consider an optimal drawing $\mathcal{D}$ of $\circ(T^n)$ with $cr(\circ(T^n))$ crossings. Let $\mathcal{T}$ be a tree in T containing all the right-wall vertices of T and let $\mathcal{T}_j$ be the copy of $\mathcal{T}$ in the j-th ($1 \leq j \leq n$) copy of T in $\circ(T^n)$. Let c_j be the number of crossings of $\mathcal{T}_j$. Then

$$\sum_{j=1}^{n} c_j \leq 2cr(\circ(T^n))$$

and so there is some i such that

$$c_i \leq \frac{2cr(\circ(T^n))}{n}.$$

By Lemma 5, $c_i \leq 2 \cdot tcr(T)$.

Turn all the crossings of $\circ(T^n)$ into vertices of degree 4, yielding a planar graph G^+. Then $\mathcal{T}_i$ has been transformed into a connected subgraph H of G^+. Let W, with size $|W| = h$, be the set of the right-wall vertices of the i-th copy T_i of T in $\circ(T^n)$ (these are contained in the tree $\mathcal{T}_i$, which has at most $2tcr(T)$ crossings). Let $\mathcal{T}_i^+$ be a tree in H containing W.

By Lemma 6, we can find a simple closed curve γ containing the vertices of W. Let $v_1, v_2, \ldots, v_h$ be the cyclic order in which the vertices of W occur in γ. The edges of G^+ that cross γ are those incident with a vertex of $\mathcal{T}_i^+$; such a vertex is either a vertex of $\mathcal{T}_i$ or a crossing involving an edge of $\mathcal{T}_i$. Thus, an edge of G that crosses γ is either incident with a vertex in W, a vertex in $\mathcal{T}_i$ that is not in W, or crosses an edge of $\mathcal{T}_i$. Edges incident with vertices in W come in two types: those in T_i (we shall refer to these as "blue edges") and those in T_{i+1} ("red edges"). Every other edge that crosses γ is either in T_i or crosses $\mathcal{T}_i$ (and is a "green edge").

Now we will split the vertices of W and reroute edges that cross γ to go around γ as follows:

- view γ as a rectangle, with $v_1, v_2, \dots, v_h$ in this order down the right-hand side; relabel v_j as v_j^R; and place new vertices v_j^L, $j = 1, 2, \dots, h$, down the left-hand side, in this order;

- for every blue edge that crosses γ, reroute it beside γ until we come to its end v_j^R;

- for every red edge, reroute it beside γ until we come to its end v_j^L;

- for every green edge, reroute any portion inside γ to be beside γ to join up the appropriate parts.

Figure 1 shows an example of splitting and rerouting. The curve γ is indicated with stipple lines, the blue edges b's and the green edges g's with narrow lines, and the red edges r's with bold lines.

After the splitting and rerouting, we see that there is one face of the new drawing which has all the vertices v_j^L and v_j^R on the same face. Note that the order of the vertices on both sides of the rectangles is a permutation π_n of the order of wall vertices of T. So the new drawing induces a tile drawing $\mathcal{D}'$ of $T^n_{\pi_n}$.

The proof is completed by estimating the number of crossings that $\mathcal{D}'$ has. The estimate does not have to be very accurate: it is conceivable that every edge of T_i is rerouted, that every edge of T_{i+1} incident with a vertex in W is rerouted, and every crossing of $\mathcal{T}_i$ is rerouted and that every two of these rerouted edges now cross. Thus, $tcr(T^n_{\pi_n}) \le cr(\mathcal{D}') \le cr(\mathcal{D}) + (2|E(T)| + 2tcr(T))^2 = cr(\circ(T^n)) + (2|E(T)| + 2tcr(T))^2$, as required. $\square$

Corollary 8. *Let T be a tile. Then*

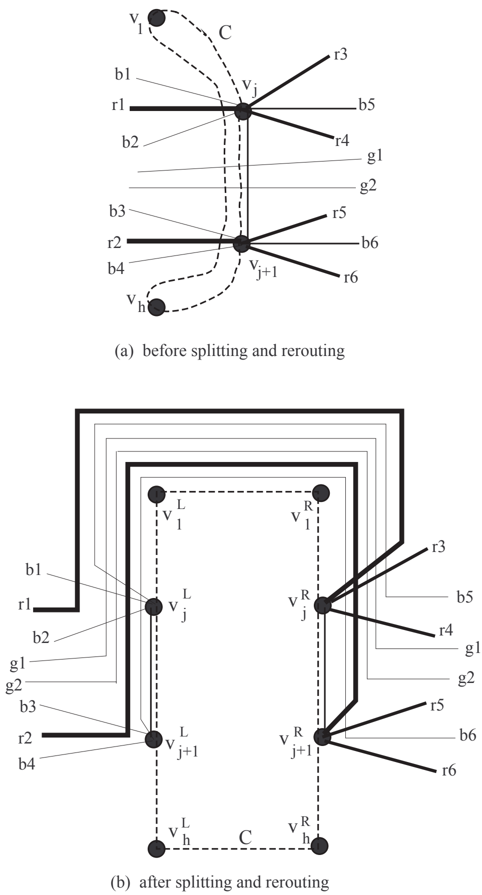
$$\lim_{n\to\infty} \frac{cr(\circ(T^n))}{n} = acr(T).$$

***Proof*:** By Theorem 7, for a given n there is a permutation π_n so that $cr(\circ(T^n)) \le tcr(T^n_{\pi_n}) \le cr(\circ(T^n)) + c_T$. There are only finitely many possibilities for π_n.

For a given π for which $\pi_n = \pi$ infinitely often, on the corresponding subsequence n_k we find

$$\lim_{k\to\infty} \frac{cr(\circ(T^{n_k}))}{n_k} \le \lim_{k\to\infty} \frac{tcr(T^{n_k}_\pi)}{n_k} \le \lim_{k\to\infty} \frac{cr(\circ(T^{n_k})) + c_T}{n_k},$$

which implies, for that subsequence, that $\lim_{k\to\infty} cr(\circ(T^{n_k}))/n_k = acr(T_\pi) = acr(T)$. Since there are only finitely many such subsequences, we deduce that $\lim_{n\to\infty} cr(\circ(T^n))/n = acr(T)$. $\square$



(a) before splitting and rerouting

(b) after splitting and rerouting

Figure 1: *Splitting and rerouting.*

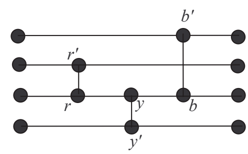


Figure 2: *Tile for* $P(3n, 3)$.

5 Examples

Let n and k be positive integers. The Generalized Petersen Graph $P(n, k)$ is the graph consisting of an n-cycle $a_0\, a_1 \ldots a_{n-1}\, a_0$, and n other vertices $b_0, b_1, \ldots, b_{n-1}$, with the edges a_ib_i (the *spokes*), and b_ib_{i+k}, with the subscripts taken modulo n. The (classical) Petersen Graph is $P(5, 2)$.

In [8], Richter and Salazar show that $cr(P(3k + h, 3)) = k + h$, if $h \in \{0, 2\}$, and $cr(P(3k + 1, 3)) = k + 3$, for each $k \geq 3$ with the single exception for $P(9, 3)$, whose crossing number is 2. This is a long and complicated induction (and actually the base case $P(10, 3)$ is now in doubt). We illustrate our theory by showing (relatively easily) that there is a constant c such that $cr(P(n, 3)) \geq (n/3) - c$.

Let $T = (G, L, R)$ be the tile in Figure 2. We note that $\circ(T^n)$ is a subdivision of $P(3n, 3)$. It is a fairly easy induction to show that $tcr(T^n) \geq n - 2$. First, if some spoke is crossed in an optimal drawing, then the inductive assumption gives the result easily. Now suppose no spoke is crossed. Note that any three consecutive copies of T in T^n have a subdivision of $K_{3,3}$. (See Figure 3.) Three copies of $K_{3,3}$ have at most one common edge, which is a spoke and therefore not crossed. Thus, we have only to worry that the crossings in consecutive $K_{3,3}$'s are counted twice. But if consecutive $K_{3,3}$'s have a common crossing, it is easy to find disjoint cycles in those $K_{3,3}$'s that must cross, and therefore there are at least two crossings between them.

Hence $acr(T) = \lim_{n\to\infty} tcr(T^n)/n \geq \lim_{n\to\infty} (n - 2)/n = 1$. Thus, for all n, $tcr(T^n) \geq n \cdot acr(T) \geq n$. It is easy to find, for some permutation π, a drawing that shows $tcr(T^n_\pi) \leq n$. Therefore, $acr(T) = 1$ and so Theorem 7 implies there is a constant c such that $cr(P(3n, 3)) \geq n - c$. Each of $P(3n + 1, 3)$ and $P(3n + 2, 3)$ can be obtained from T^n by the addition of a single tile to adjust the number of vertices and the connections between the paths. Thus, each of these graphs also has crossing number $n - c$.

The crossing number of $P(n, 2)$ is easy (c.f., [2]), while for $k \geq 4$, very little is known. For $k \geq 6$ and $n \geq 2k + 1$, a referee of an earlier version of this article gave simple arguments to show $nk/3 - c_k \leq cr(P(n, k)) \leq (2 - (4/k))n + c_k$. It would be interesting if simple arguments could be used to compute $acr(T_k)$, where T_k is the tile that gives (as T_3 did for $P(3n, 3)$) $P(nk, k)$. This would give a nice general estimate on $cr(P(n, k))$.

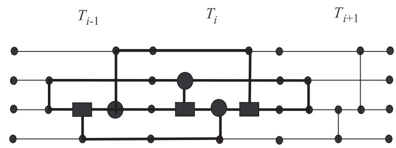


Figure 3: *Subdivision of* $K_{3,3}$ *in* $T_{i-1}T_iT_{i+1}$.

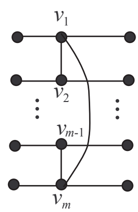


Figure 4: *Tile for* $C_m \times C_n$.

As another point for discussion, we consider the Cartesian product $C_m \times C_n$ of an m-cycle with an n-cycle.

In 1973, Harary, Kainen and Schwenk [4] conjectured that the crossing number of $C_m \times C_n$ is, for $3 \le m \le n$, $(m-2)n$. This had been proved for $n \le 6$ (see [7] for a bibliography) and for $m = n = 7$. The case $m = 7$ and $n > 7$ is resolved in [1].

Let $T_m = (G_m, L_m, R_m)$ be the tile in Figure 4, consisting of an m-cycle $C = v_1v_2 \ldots v_mv_1$, where $v_i \notin L_m \cup R_m$, and m disjoint paths $p_i = L_m[i]\, v_i\, R_m[i]$, $1 \le i \le m$.

The following is a simple consequence of [1].

Lemma 9. $acr(T_m) = m - 2$.

We have the following immediate consequence of Theorem 7 and Lemma 9.

Theorem 10. *For each integer* $m \ge 3$, *there is a constant* c_m *such that, for all integers* n, $cr(C_m \times C_n) \ge (m-2)n - c_m$. □

Glebsky and Salazar [3] have recently adapted the arguments of [1] to obtain the following, currently best, result for $C_m \times C_n$.

Theorem 11. *Let* $m \ge 3$ *be an integer and let* $n \ge m(m+1)$. *Then* $cr(C_m \times C_n) = (m-2)n$.

This example shows that it is not clear whether our general theory will ever be of use. It is reasonable to wonder if being able to compute $acr(T)$ necessarily shows how to compute $cr(\circ(T^n))$. This is what happened in the case of $C_m \times C_n$. As a specific problem in this area, we put forward the following.

Conjecture 12. *Let T be a self-compatible tile for which $acr(T) = tcr(T)$. Then there is an $N = N(T)$ so that, if $n \geq N$, then $cr(\circ(T^n)) = n \cdot tcr(T)$.*

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(Received 31 Jan 2003)