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Color-induced subgraphs of Grünbaum colorings of triangulations of the sphere

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Abstract

A Grünbaum coloring of a triangulation is an assignment of colors to edges so that the edges about each face are assigned unique colors. In this paper we examine the color induced subgraphs given by a Grünbaum coloring of a triangulation and show that the existence of connected color induced subgraphs is equivalent to the Three Color Theorem.

A triangulation of an orientable surface is an embedding of a simple graph in that surface such that the boundary of every face is a 3-cycle. Grünbaum [3] conjectured the following:

Conjecture. *For every triangulation of each orientable surface it is possible to color the edges by three colors in such a fashion that the edges of each triangle have three different colors.*

Following Archdeacon [1], we will call such colorings *Grünbaum colorings.* We will use the colors R, S, and B for Red, Silver and Black.

Definition 1. A *Grünbaum coloring* of a triangulation G is a map γ from the set of edges of G to the set $\{R, S, B\}$ such that the edges around a face of G are each given distinct colors.

A *Tait coloring* of a 3-regular graph is a 3-coloring of the edges so that no two adjacent edges share the same color. It is not hard to see that a Grünbaum coloring of a plane triangulation corresponds to a Tait coloring of the dual graph. In fact, by using the equivalence of the Four Color Theorem with the statement that every simple 2-edge connected 3-regular planar graph has a Tait coloring, it can be shown that Grünbaum's conjecture implies the Four Color Theorem for triangulations of the sphere. Thus Grünbaum's conjecture, if true, would be a strengthening of the Four Color Theorem. See [2], pp. 158–159, [4], pp. 103–104, and [1].

In this note we will be concerned with Grünbaum colorings of triangulations of the sphere (corresponding to graphs that are plane triangulations). In the first section we present examples of Grünbaum colorings of various triangulations. In the second section we consider the color-induced subgraphs (CISGs) of a given Grünbaum coloring of a graph and show that there exists a Grünbaum coloring for which each color-induced subgraph is connected if and only if the degree of each vertex of the graph is even.

If a given plane triangulation G has two Grünbaum colorings then we will consider them isomorphic if there is a permutation of the colors and a graph automorphism that takes one coloring to the other.

Definition 2. Let G be a plane triangulation and γ_1 and γ_2 two Grünbaum colorings of G. We say that γ_1 and γ_2 are *isomorphic* if there is a G-automorphism φ and a permutation σ of $\{R, S, B\}$ such that for each edge e of G we have $\gamma_1(e) = \sigma(\gamma_2(\varphi(e)))$.

1 Examples of Grünbaum Colorings

It is easy to see that the tetrahedron has exactly one Grünbaum coloring (up to isomorphism), shown in Figure 1.

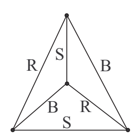


Figure 1: The Grünbaum coloring of the tetrahedron

The octahedron has two non-isomorphic Grünbaum colorings: one in which one CISG is a 4-cycle while the other two are made up of two 2-paths and another in which the CISGs are all 4-cycles. The CISGs for each of these colorings are shown in Figure 2.

We now show that the icosahedron has a unique Grünbaum coloring up to isomorphism. Note that each vertex v of the icosahedron is the center vertex of a 5-wheel. We show first there is essentially one way to arrange the three colors on these 5-wheels. If C is a color used in a Grünbaum coloring of a graph with vertex v, let $d_C(v)$ denote the C-degree of v, that is, the number of edges of color C incident to v.

Lemma 3. *In a Grünbaum coloring of the icosahedron, the C-color degree of each vertex is either one or two.*

Proof. No three edges colored C can be incident to v or else two of them would be edges of the same face. If $d_C(v) = 0$ then there are five edges of the other two colors incident to v. It follows that there is a color C' so that $d_{C'}(v) \geq 3$, contradicting the previous conclusion. □

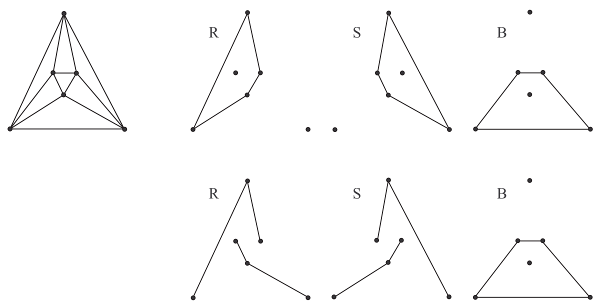


Figure 2: The two Grünbaum colorings of the octahedron

Lemma 4. *In a Grünbaum coloring of a triangulation* G*, each* 5*-wheel in* G *receives a coloring isomorphic to the graph depicted in Figure 3.*

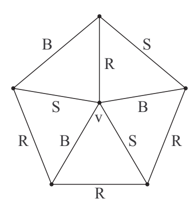


Figure 3: An R-Sailboat

Proof. Consider the center vertex v for a 5-wheel. By Lemma 3, there must be some color, say R, such that $d_R(v) = 1$ while $d_B(v) = d_S(v) = 2$. Once those colors are placed on the edges incident to v (there is essentially one way to do this), the outside edges of the wheel are determined and we are done. □

The graph shown in Figure 3 will be called an R*-sailboat.* Similarly, there exist B-sailboats and S-sailboats.

Theorem 5. *There exists only one Grünbaum coloring of the icosahedron, up to isomorphism.*

Proof. Suppose we have a Grünbaum coloring of the icosahedron and that we have an R-sailboat, centered at some vertex v as shown in Figure 4. Note that the R-sailboat at v implies that edge tz will also be colored R.

If $d_R(w) = 1$ then the other edges incident to w must be colored either B or S. This forces the following colorings: wx is colored B, wy is colored S, xy is colored

R, and sy is colored R. Then vertex z must be the center of an R-sailboat, and edge uy must be colored R, a contradiction.

Thus by Lemma 3, we must have $d_R(w) = 2$. There are two ways to extend the path of R edges through vertex w, but once we have made that choice, the other edges are forced and we obtain two possible Grünbaum colorings. However, it can be shown that these colorings are isomorphic. Therefore there is only one Grünbaum coloring of the icosahedron. The CISGs for this coloring are shown in Figure 5. □

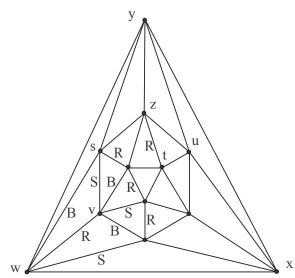


Figure 4: Icosahedron with an R-sailboat at v

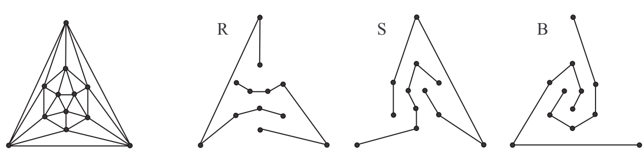


Figure 5: The Grünbaum coloring of the icosahedron

Included below are two example triangulations with their non-isomorphic Grünbaum colorings (Figures 6 and 7) and a table of example triangulations with the number of non-isomorphic Grünbaum colorings for each (Table 1).

2 Connected Color Induced Subgraphs

In this section we look more closely at the cases in which the color induced subgraphs for a Grünbaum coloring of a plane triangulation G are connected. Notice that in the case of the octahedron, one of the Grünbaum colorings produced CISGs that were each connected (Figure 2). Similarly, the triangulation in Figure 7 has a Grünbaum coloring in which each CISG is connected. In both of these examples, the degree of each vertex is even while for the other example triangulations there exist vertices of

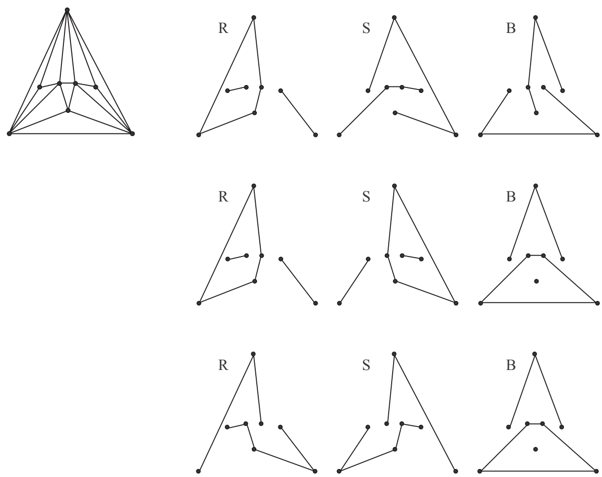


Figure 6: An example triangulation with its non-isomorphic Grünbaum colorings

odd degree and there is no coloring for which each CISG is connected. We will show that this property serves to characterize those triangulations that have Grünbaum colorings with connected CISGs.

We will use the Three Color Theorem, stated here without proof. For more information see [5].

Three Color Theorem. *A plane triangulation is vertex* 3*-colorable if and only if each vertex has even degree.*

Given a graph G and a Grünbaum coloring of G, let G_C be the subgraph of G induced by the edges colored C. We say that two C-colored edges in G are *C-connected* if there exists a path in G_C connecting the edges.

Theorem 6. *Let G be a plane triangulation. There exists a Grünbaum coloring of G such that each color induced subgraph of G is connected if and only if the degree of each vertex of G is even.*

Proof. If G has only three vertices then the theorem holds trivially. Assume that G has more than three vertices.

Suppose first that we have a Grünbaum coloring of G for which each CISG of G is connected. Consider a vertex v of G. The vertex is the center of a wheel whose vertices we may label $x_0, x_1, x_2, \ldots, x_n = x_0$, $n \geq 3$, as shown in Figure 8. Without loss of generality, we may assume that edges of the triangle vx_0x_1 are colored as shown. We will show that edge vx_2 must be colored R.

Let V' be the set of vertices obtained by subdividing each edge e of G not colored R. For vertices s and t in V', add an edge connecting s and t if and only if their

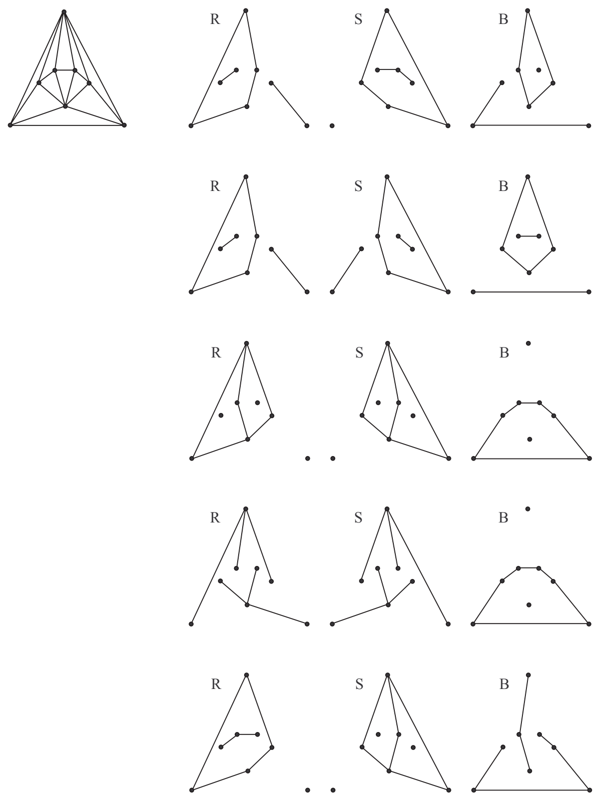


Figure 7: An example triangulation with its non-isomorphic Grünbaum colorings

corresponding edges in G are incident to a common face of G. Color these edges W. (For example, Figure 9 shows the resulting graph after subdividing the Grünbaum coloring of the tetrahedron given in Figure 1.)

Each vertex in V' will have W-degree 2 (as exactly two edges of each triangle will not be colored R) so the edges colored W will form disjoint cycles and in particular Jordan curves. Thus if edge x_1x_2 is colored R then the connected component of G_R containing it must lie in a different region than the connected component of G_R containing vx_0, contradicting our assumption that each CISG is connected. Therefore x_1x_2 must be colored B and so vx_2 must be colored R.

By continuing this reasoning around the wheel we can conclude that vx_3 is colored S, vx_4 is colored R, and so on—edges vx_i for i odd will be colored S while edges vx_i for i even will be colored R. To avoid contradiction with the initial coloring R of vx_0, we must have that n is even and therefore the degree of v is even. The choice of v was arbitrary so every vertex of G has even degree.

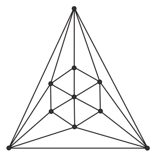
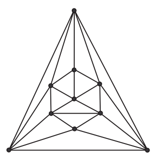
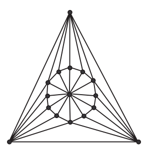
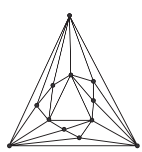
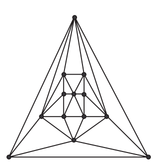
Triangulation	Number of Non-Isomorphic Grünbaum colorings
	1
	2
	4
	15
	23

Table 1: Number of non-isomorphic Grünbaum colorings for certain triangulations

Now suppose that each vertex of G has even degree. By the Three Color Theorem, G has a vertex 3-coloring, say by "colors" 1, 2, and 3. The vertex 3-coloring immediately provides a Grünbaum coloring of G: take edges between vertices colored 1 and 2 to be colored R, edges between vertices colored 2 and 3 to be B and edges between vertices colored 1 and 3 to be colored S. For each face, the vertices will be colored 1, 2 and 3 so the edges of each face will have different colors.

It remains to show that this Grünbaum coloring gives connected color induced subgraphs. If f_1 and f_2 are two faces of G that share an edge and v_1 is the vertex in f_1 not in f_2 and v_2 is the vertex in f_2 not in f_1, then v_1 and v_2 have the same color. Thus for each color C, the set of C-colored edges in $f_1 \cup f_2$ must be C-connected (see Figure 10).

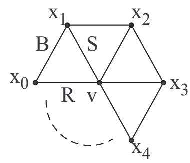


Figure 8:

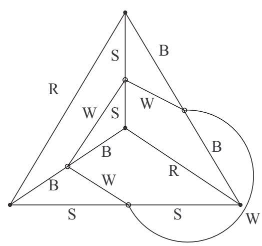


Figure 9: Subdivision of the tetrahedron

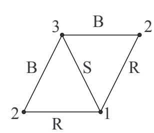


Figure 10: A vertex 3-coloring and Grünbaum coloring of the union of two faces

Let $n \geq 2$. Suppose $f_1, f_2, \ldots, f_n$ is a sequence of faces such that for $1 \leq i \leq n-1$, face f_i shares an edge with f_{i+1}. Suppose further that for each color C each set of C-colored edges in $f_1 \cup f_2 \cup \cdots \cup f_n$ is C-connected. Let f be a face of G that shares an edge with f_n. By the above, for each color C the edge of f colored C is C-connected to the edge of f_n colored C. Thus the edges colored C in $f_1 \cup f_2 \cup \cdots \cup f_n \cup f$ are C-connected. Therefore, for $n \geq 2$, given any sequence of faces $f_1, f_2, \ldots, f_n$, such that f_i shares an edge with f_{i+1} for $1 \leq i \leq n-1$, the set of C-colored edges in $f_1 \cup f_2 \cup \cdots \cup f_n$ is C-connected.

Now, given two edges e and e' colored C, there exists a sequence of faces $f_1, f_2, \ldots,$ f_n of G such that e is contained in f_1, e' is contained in f_n and f_i and f_{i+1} share an edge for $1 \leq i \leq n-1$. By the above, e and e' are C-connected. As the choice of color was arbitrary, we have that for the Grünbaum coloring induced by the vertex 3-coloring, every CISG of G is connected. □

Remark 7. The proof of Theorem 6 shows that if a plane triangulation has a Grünbaum coloring for which each CISG is connected then with this coloring no CISG has cycles of odd length. (A graph is vertex 2-colorable if and only if it contains no odd cycles; the vertex 3-coloring of the graph restricted to a CISG is a vertex 2-coloring of the CISG.)

It is not difficult to show that in fact Theorem 6 is equivalent to the Three Color Theorem.

Theorem 8. *Theorem 6 implies the Three Color Theorem.*

Proof. Let G be a plane triangulation. We want to show that the degree of each vertex of G is even if and only if G has a vertex 3-coloring. Suppose first that there exists a vertex 3-coloring of G. Then this coloring provides a Grünbaum coloring of G and, by the proof of Theorem 6, the CISGs for this coloring will be connected. Thus by Theorem 6 the degree of each vertex will be even.

Now suppose that the degree of each vertex of G is even. By Theorem 6 there exists a Grünbaum coloring in which each CISG is connected. Consider the R-induced subgraph G_R. Select a vertex v in G_R and assign it the color 1. Each vertex w in G_R which is adjacent to v may be assigned the color 2. We then assign color 1 to every vertex of G_R adjacent to a vertex colored 2. Since G_R contains no odd cycles, we can continue assigning alternate colors in a consistent way until all vertices of G_R are assigned a color. Then color the remaining vertices of G the color 3. This provides a vertex 3-coloring of G. □

Remark 9. A given plane triangulation may have many non-isomorphic Grünbaum colorings. However, since Grünbaum colorings with connected CISGs induce vertex 3-colorings of the graph and vertex 3-colorings of plane triangulations are unique (up to permutation of the vertex colors), there can be at most one Grünbaum coloring of a plane triangulation with connected CISGs.

Remark 10. Since Grünbaum's conjecture concerns all orientable surfaces, it is natural to ask if the result in Theorem 6 extends to other surfaces. Unfortunately it does not extend to the torus—the triangulation of the torus shown in Figure 11 (with the usual identifications on the square) does not admit a Grünbaum coloring with connected CISGS.

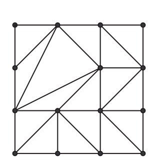


Figure 11: A triangulation of the torus having no Grünbaum coloring with connected CISGs

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