AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **30** (2004), Pages 85–93

On frame self-orthogonal Latin squares of type $h^m 1^n$

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Abstract

An SOLS (self orthogonal Latin square) of order v with n_i missing sub-SOLS (holes) of order h_i $(1 \leq i \leq k)$, which are disjoint and spanning (i.e. $\sum_{i=1}^{k} n_i h_i = v$), is called a frame SOLS and denoted by FSOLS$(h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k})$. In this paper, we prove that: if $h \geq 7$, $m \geq 4$, $n \geq 4$ and $n \neq 6$, then there exists an FSOLS$(h^m 1^n)$; an FSOLS$(h^m 1^6)$ exists if $h \geq 7$, $h \neq 8, 12, 28$ and $m \geq 17$.

1 Introduction

A *self-orthogonal* Latin square of order v, or SOLS(v), is a Latin square of order v which is orthogonal to its transpose. It is well known [3] that an SOLS(v) exists for all values of v, $v \neq 2$, 3 or 6.

Let S be a set and $\mathcal{H} = \{S_1, S_2, \cdots S_k\}$ be a set of nonempty subsets of S. A *holey Latin square* having *hole set* $\mathcal{H}$ is an $|S| \times |S|$ array, L, indexed by S, which satisfies the following properties:

(1) every cell of L is either empty or contains a symbol of S;

(2) every symbol of S occurs at most once in any row or column of L;

(3) the subarrays $S_i \times S_i$ are empty for $1 \leq i \leq k$ (these subarrays are referred to as *holes*);

(4) the symbol $x \in S$ occurs in row or column y if and only if $(x, y) \in (S \times S) \setminus \bigcup_{i=1}^{k} (S_i \times S_i)$.

* Research supported by NSFC 10071002

The *order* of L is $|S|$. Two holey Latin squares on symbol set S and hole set $\mathcal{H}$, say L_1 and L_2, are said to be *orthogonal* if their superposition yields every ordered pair in $(S \times S)\setminus \bigcup_{i=1}^{k}(S_i \times S_i)$. We shall use the notation IMOLS$(v; s_1, s_2, \cdots, s_k)$ to denote a pair of orthogonal holey Latin squares on symbol set S and hole set $\mathcal{H}= \{S_1, S_2, \cdots, S_k\}$, where $v = |S|$ and $s_i = |S_i|$ for $1 \leq i \leq k$. If $\mathcal{H} = \emptyset$, we obtain an MOLS(v). If $\mathcal{H}= \{S_1\}$, we simply write IMOLS(v, s_1) for the orthogonal pair of holey Latin squares.

If L_1 and L_2 form an IMOLS$(v; s_1, s_2, \cdots, s_k)$ such that L_2 is the transpose of L_1, then we call L_1 a *holey* SOLS, denoted by ISOLS$(v; s_1, s_2, \cdots, s_k)$. If $\mathcal{H}= \emptyset$, or $\{S_1\}$, then a holey SOLS is an SOLS(v), or ISOLS(v, s_1) respectively.

If $\mathcal{H}= \{S_1, S_2, \cdots, S_k\}$ is a partition of S, then an IMOLS is called a *frame* MOLS. The *type* of the frame MOLS is defined to be the multiset $\{|S_i| : 1 \leq i \leq k\}$. We shall use an "exponential" notation to describe types: Type $h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k}$ denotes n_i occurrences of $h_i, 1 \leq i \leq k$, in the multiset. We briefly denote a frame MOLS of type $h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k}$ FMOLS$(h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k})$.

If L_1 and L_2 form an FMOLS (frame MOLS) such that L_2 is the transpose of L_1, then we call L_1 an FSOLS.

We observe that the existence of an SOLS(v) is equivalent to the existence of an FSOLS(1^v), and the existence of an ISOLS(v, h) is equivalent to the existence of an FSOLS $(1^{v-h}h^1)$.

One of our recursive constructions in the following sections relies on information regarding the location of (holey) transversals in certain Latin squares. Suppose that L is a holey Latin square on symbol set S with hole S_1. A *holey transversal* with hole S_1 is a set T of $|S| - |S_1|$ (occupied) cells in L such that every symbol of $S\setminus S_1$ is contained in exactly one cell of T and the $|S| - |S_1|$ cells in T intersect each row and each column indexed by $S\setminus S_1$ in exactly one cell. $|S_1|$ is called the *size* of the hole of the holey transversal. A holey transversal T is *symmetric* if $(i, j) \in T$ implies $(j, i) \in T$. Two holey transversals T_1 and T_2 with the same hole are called a *symmetric pair of holey transversals* if $(i, j) \in T_1$ if and only if $(j, i) \in T_2$. If $S_1 = \emptyset$, then we call the holey transversal a (*complete*) transversal. A set of holey transversals are said to be *disjoint* if they have no cell in common.

FSOLS have been very useful in recursive constructions of various combinatorial designs, such as 2-perfect m-cycle systems [11], edge-colored designs [8], holey Schröder designs [1], intersections of transversal designs [7], and skew Room frames [5]. The following are known results concerning FSOLS(h^n), FSOLS$(h^n u^1)$ and FSOLS$(h^m 1^n)$.

Theorem 1.1

(1) [3] *There exists an FSOLS*(1^n) *if and only if* $n \geq 4, n \neq 6$.

(2) [13] *For* $h \geq 2$, *there exists an FSOLS*(h^n) *if and only if* $n \geq 4$.

(3) [15, Theorem 7.1] *Suppose that* h, n *and* u *are positive integers and* $h \neq u$. *Then there exists an FSOLS*($h^n u^1$) *if and only if* $n \geq 4$ *and* $n \geq 1 + 2u/h$, *except for* $(h, n, u) = (1, 6, 2)$ *and except possibly for* $(h, n, u) \in \{(t+2, 6, (5h-1)/2), (t, 14, (13h-1)/2), (t, 18, (17h-1)/2), (t, 22, (21h-1)/2) : t \text{ is odd}\}$.

(4) [15, Lemma 2.2 and Lemma 2.3] *There exist FSOLS of types* $2^2 1^5$, $2^3 1^4$, $2^4 1^2$, $2^2 1^6$, $2^2 1^7$, $2^3 1^5$, *and* $2^4 1^3$; *there do not exist FSOLS of types* $1^4 2^2$, $1^2 2^3$, *and* $1^3 2^3$.

In this paper, we prove that: if $h \geq 7$, $m \geq 4$, $n \geq 4$ and $n \neq 6$, then there exists an FSOLS($h^m 1^n$); an FSOLS($h^m 1^6$) exists if $h \geq 7$, $h \neq 8, 12, 28$ and $m \geq 17$.

2 Constructions

Construction 2.1 (Filling in holes) [14] *Suppose that there exist FSOLS of type* $\{s_i : 1 \leq i \leq n\}$ *and for* $1 \leq i \leq n, s_i = \sum_{j=1}^{t_i} s_{ij}$.

(1) *If there exist FSOLS of type* $\{s_{nj} : 1 \leq j \leq t_n\}$, *then there exist FSOLS of type* $\{s_i : 1 \leq i \leq n-1\} \bigcup \{s_{nj} : 1 \leq j \leq t_n\}$.

(2) *Let* $a \geq 0$ *be an integer. If there exist FSOLS of type* $\{a\} \bigcup \{s_{ij} : 1 \leq j \leq t_i\}$ *for all* $1 \leq i \leq n-1$, *then there exist FSOLS of type* $\{a + s_n\} \bigcup (\bigcup_{i=1}^{n-1} \{s_{ij} : 1 \leq j \leq t_i\})$.

The following recursive construction is referred to as *Inflation Construction*. It essentially "blows up" every occupied cell of an FSOLS into a Latin square such that if one cell is filled with a certain Latin square, then its symmetric cell is filled with the transpose of an orthogonal mate of the Latin square. We should mention the work of Brouwer and van Rees [4] and Stinson [12], which can be thought of as sources of Inflation Construction.

Construction 2.2 (Inflation Construction) *Suppose there exist an FSOLS*($h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k}$) *and an MOLS*($h$); *then there exists an FSOLS*($(hh_1)^{n_1} (hh_2)^{n_2} \cdots (hh_k)^{n_k}$). *In particular, the existence of FSOLS*(1^n) *and MOLS*(h) *implies the existence of an FSOLS*(h^n).

The following two lemmas are modifications of [9, Lemma 2.10].

Lemma 2.3 *If there exist an FSOLS*(1^d) *with* k *pairs of symmetric transversals (so that all* $2k$ *transversals are pairwise disjoint) and a symmetric holey transversal with a hole of size one such that the holey transversal intersect each of the* $2k$ *transversals in exactly one cell, there exist an MOLS*(h), *an IMOLS*($h+v_i$, v_i), *an IMOLS*($h+u$, u), *an IMOLS*($h + v_i + u$; v_i, u) $(1 \leq i \leq k)$, *then there is an FSOLS*($h^{d-1}(h+u)^1(2\sum_{i=1}^{k} v_i)^1$).

Lemma 2.4 *If there exist an SOLS*(h) *with* k *pairs of symmetric transversals (so that all* $2k$ *transversals are pairwise disjoint—and therefore cannot contain any cell of the main diagonal) and a symmetric transversal* T *which intersects each of the* $2k$ *transversals and the main diagonal in exactly one cell, there exist an FMOLS*(1^m)*, an FSOLS*$(1^m v^1)$*, an FSOLS*$(1^m v^1 u^1)$*, an FMOLS*$(1^m v_i^1)$*, an FMOLS*$(1^m v_i^1 u^1)$*,* $(1 \le i \le k)$*, then there is an FSOLS*$(h^m (v + 2\sum_{i=1}^{k} v_i)^1 u^1)$*.*

Given a set X of *points*, a *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$ which satisfies the following properties:

1. $\mathcal{G}$ is a partition of X and each member of $\mathcal{G}$ is called a *group* (also called a *point class*);
2. $\mathcal{A}$ is a set of subsets of X (each subset is called a *block*) such that a group and a block contain at most one common point;
3. every pair of points from distinct groups occurs in a unique block.

The *group type* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. A GDD $(X, \mathcal{G}, \mathcal{A})$ will be referred to as a K-GDD if $|A| \in$ K for every block A in $\mathcal{A}$.

Construction 2.5 (Weighting) [13, Lemma 2.5] *Suppose* $(X, \mathcal{G}, \mathcal{A})$ *is a GDD and let* w *be a map:* $X \to \mathbf{Z}^+ \bigcup \{0\}$*. Suppose there exist FSOLS of type* $\{w(x) : x \in A\}$ *for every* $A \in \mathcal{A}$*. Then there exist FSOLS of type* $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$*.*

To apply the above constructions we need some "ingredients" provided in the following theorems and lemmas.

Theorem 2.6 [2] *There exists an MOLS*(v) *for any positive integer* v*,* $v \ne 2, 6$*.*

Theorem 2.7 [10] *There exists an IMOLS*(v, n) *for all values of* v *and* n *satisfying* $v \ge 3n$ *except that IMOLS*$(6, 1)$ *does not exist.*

A *transversal design* TD(k, n) is a GDD of group type n^k and block size k. It is well known that a TD(k, n) is equivalent to $k - 2$ MOLS (mutually orthogonal Latin squares) of order n and that for any prime power p, there exist $p - 1$ MOLS of order p. So we have the following theorem.

Theorem 2.8 *If* p *is a prime power, then there exists a TD(k, p) for* $3 \le k \le p + 1$*.*

From [6] (Table 2.68 and 2.72; the cases of 39 and 54 are from `http://www.emba.uvm.edu/~dinitz/newresults.html`), we have the following theorem.

Theorem 2.9 *There is a TD*$(6, t)$ *if* $t \ge 5$ *and* $t \notin E_6 = \{6, 10, 14, 18, 22\}$*. There is a TD*$(7, t)$ *if* $t \ge 7$ *and* $m \notin E_7 = E_6 \cup \{15, 20, 26, 30, 34, 38, 46, 60, 62\}$*.*

3 Main Result

Lemma 3.1 *For $h \geq 7$, $m \geq 4$, $n \geq 4$ and $n \neq 6$, there exists an FSOLS$(h^m 1^n)$ if $4 \leq n \leq 2mh$.*

Proof. From Theorem 1.1 (3) we know that there exists an FSOLS$(h^{m+r}u^1)$ for $4 \leq u \leq (m+r-1)h/2$. Filling in r of the $m+r$ holes of size h with an FSOLS(1^h), and the hole of size u with an FSOLS(1^u) $(u \neq 6)$, we obtain an FSOLS$(h^m 1^n)$ for $rh+4 \leq n \leq rh+(m+r-1)h/2$ $(n \neq rh+6)$.

Take r from zero to $m+1$ we get FSOLSs of type $h^m 1^n$ for n in the ranges of the Table 3.1.

r	Range for n	
0	$[4, (m-1)h/2]$	$(n \neq 6)$
1	$[h+4, h+mh/2]$	$(n \neq h+6)$
2	$[2h+4, h+(m+1)h/2]$	$(n \neq 2h+6)$
$\vdots$	$\vdots$	
k	$[kh+4, kh+(m+k-1)h/2]$	$(n \neq kh+6)$
$k+1$	$[(k+1)h+4, (k+1)h+(m+k)h/2]$	$(n \neq (k+1)h+6)$
$\vdots$	$\vdots$	
$m+1$	$[(m+1)h+4, (m+1)h+mh]$	$(n \neq (m+1)h+6)$

Table 3.1

Notice that $(k+1)h+4 \leq kh+(m+k-1)h/2+1$, and $n = (k+1)h+6$ is also in the range of $kh+4 \leq n \leq kh+(m+k-1)h/2$ except for $k=0$, $m=4$ and $7 \leq h \leq 11$, so we have proved the lemma except for FSOLS of types $7^4 1^{13}$, $8^4 1^{14}$, $9^4 1^{15}$, $10^4 1^{16}$ and $11^4 1^{17}$.

	4	3	2	1
2		0	4	3
4	3		1	0
1	0	4		2
3	2	1	0	

Table 3.2 an FSOLS(1^5)

Table 3.2 is an FSOLS(1^5) with two pairs of disjoint symmetric transversals and a symmetric holey transversal. One pair is on cells (0,1), (1,2), (2,3), (3,4), (4,0) and (1,0), (2,1), (3,2), (4,3), (0,4) and the other pair is on cells (0,2), (1,3), (2,4), (3,0), (4,1) and (2,0), (3,1), (4,2), (0,3), (1,4). The symmetric holey one is on cells (1,4), (2,3), (3,2), (4,1). Apply Lemma 2.3 with $d=5$, $k=2$, $h \in \{7, 8, 9, 10, 11\}$, $v_1 = v_2 = 1$, $u = 2$, the input designs, MOLS(h) and IMOLS$(h+1, 1)$ are from

Theorems 2.6 and 2.7, IMOLS($h+3$; 1, 2) is from the existence of FSOLS($1^{h+1}2^1$) from Theorem 1.1 (3), we then get an FSOLS($h^4(h+2)^1 4^1$). Filling the hole of size $h+2$ with an FSOLS(1^{h+2}) and the hole of size 4 with an FSOLS(1^4), we obtain an FSOLS($h^4 1^{h+6}$) for $h \in \{7, 8, 9, 10, 11\}$. This completes the proof. ∎

Lemma 3.2 *For $h \geq 7$, $m \geq 4$ and $n \geq 4$, there exists an FSOLS($h^m 1^n$) if $n \geq 2mh+1$.*

Proof. From Theorem 1.1 (3) we know that there exists an FSOLS($1^n(mh)^1$) for $n \geq 2mh+1$. Filling the hole of size mh with an FSOLS(h^m), we obtain the desired FSOLS($h^m 1^n$). ∎

Lemma 3.3 *For $h > 10$, $h = 2t$ and $t \notin E_6 = \{6, 10, 14, 18, 22\}$, there exists an FSOLS($h^m 1^6$) if $m \geq 14$.*

Proof. Start with a TD(6, t) and apply Construction 2.5. Give each point of the first four groups weight 2. In the fifth group, give weight 1 to one point, weight 2 to two points, and weight zero to the remaining points. In the last group, give weight 1 to one point and weight zero the remaining points. The input designs, FSOLS of types 2^4, 2^5, $2^4 1^1$, $2^5 1^1$, and $2^4 1^2$, are from Theorem 1. We then get an FSOLS($(2t)^4 5^1 1^1$). Filling the holes of size 5 with an FSOLS(1^5), we obtain an FSOLS($(2t)^4 1^6$).

From Theorem 1 (3) we know that there exists an FSOLS($(2t)^k(8t+6)^1$) when $k \geq 10$ and $t > 5$. Filling the hole of size $8t+6$ with an FSOLS($(2t)^4 1^6$), we obtain an FSOLS($(2t)^m 1^6$) for $m \geq 14$. ∎

Lemma 3.4 *There exists an FSOLS($h^m 1^6$) for $h = 20, 32, 44$ and $m \geq 17$.*

Proof. Start with a TD(7, 7) and apply Construction 2.4. Take a block B, give each point of B in the first six groups weight 2, and the point of B in the last group weight 1. For the points of the TD not in B, give each point weight 3 in the first five groups, give one point weight 3 and five points weight zero in the sixth group, and give each point weight zero in the last group. The input designs, FSOLS of types $2^6 1^1$, $3^4 2^1$, $3^5 2^1$, $3^5 1^1$, 3^5, 3^6, $3^6 1^1$, are from Theorem 1. We then get an FSOLS($20^5 5^1 1^1$). Filling the hole of size five with an FSOLS(1^5), we obtain an FSOLS($20^5 1^6$).

Apply Construction 2.4 with a TD(7, 7). Take a block B, give each point of B in the first five groups weight 2, the point of B in the sixth group weight zero and the point of B in the last group weight 1. For the remaining points not in B, give each point weight 5 in the first five groups, give one point weight 5 and five points weight zero in the sixth group, and give each point weight zero in the last group. The input designs, FSOLS of types $2^5 1^1$, 5^5, 5^6, $5^5 1^1$, $5^6 1^1$, $5^4 2^1$, $5^5 2^1$, are from Theorem 1. We then get an FSOLS($32^5 5^1 1^1$). Filling the hole of size five with an FSOLS(1^5), we obtain an FSOLS($32^5 1^6$).

Apply Construction 2.4 with a TD(7, 9). Take a block B, give each point of B in the first five groups weight 4, the point of B in the sixth group weight zero and the point of B in the last group weight 1. For the remaining points of the TD, give each point weight 5 in the first five groups, give one point weight 5 and five points weight zero in the sixth group, give each point weight zero in the last group. The input designs, FSOLS of types $4^5 1^1$, 5^5, 5^6, $5^5 1^1$, $5^6 1^1$, $5^4 4^1$, $5^5 4^1$, are from Theorem 1. We then get an FSOLS($44^5 5^1 1^1$). Filling the hole of size five with an FSOLS(1^5), we obtain an FSOLS($44^5 1^6$).

For $h \in \{20, 32, 44\}$, from Theorem 1 (3) we know that there exists an FSOLS($h^k(5h{+}6)^1$) when $k \geq 12$. Filling the hole of size $5h{+}6$ with an FSOLS($h^5 1^6$), we obtain an FSOLS($h^m 1^6$) for $m \geq 17$. ▯

Lemma 3.5 *For $h = 2t - 1$ and $h \geq 13$, there exists an FSOLS($h^m 1^6$) if $m \geq 17$.*

Proof. Apply Construction 2.4 with a TD(7, t), where $t \geq 7$ and $t \notin E_7 = E_6 \cup \{15,$ 20, 26, 30, 34, 38, 46, 60, 62$\}$. Take a block B, give each of its points weight 1. For the points not in B, give each point weight 2 in the first five groups, give weight 2 to two points and weight zero to the remaining $t - 3$ points in the sixth group, give weight zero to each point in the last group. The input designs, FSOLS of types 1^7, 2^5, 2^6, $2^4 1^1$, $2^5 1^1$, and $2^6 1^1$, are from Theorem 1. We then get an FSOLS($(2t-1)^5 5^1 1^1$). Filling the holes of size 5 with an FSOLS(1^5), we obtain an FSOLS($(2t-1)^5 1^6$).

Apply Construction 2.4 with a TD(7, $t-1$) for $t \in \{$10, 14, 18, 20, 22, 26, 30, 34, 38, 46, 60, 62$\}$. Take a block B, give each point in the first six groups weight 3, give the point in the last group weight 1. For the points not in B, give each point weight 2 in the first five groups, give one point weight 2 and the remaining points weight zero in the sixth group, give each point weight zero in the last group. The input designs are $3^6 1^1$, 2^5, 2^6, $2^5 3^1$, $2^4 3^1$, $2^5 1^1$, $2^6 1^1$. Then we get an FSOLS of type $(2t-1)^5 5^1 1^1$. Filling the holes of size 5 with an FSOLS(1^5), we obtain an FSOLS($(2t-1)^5 1^6$).

For $t = 15$, start with a TD(7, 8) and apply Construction 2.4. Take a block B, give each of its points weight 1. For the points not in B, give each of the points weight 4 in the first five groups, give one point weight 4 and the remaining six points weight zero in the sixth group, give each point weight zero in the last group. The input designs, FSOLS of types 1^7, 4^5, 4^6, $4^4 1^1$, $4^5 1^1$, and $4^6 1^1$, are from Theorem 1. We then get an FSOLS($(2t-1)^5 5^1 1^1$). Filling the holes of size 5 with an FSOLS(1^5), we obtain an FSOLS($(2t-1)^5 1^6$).

From Theorem 1 (3) we know that there exists an FSOLS($(2t-1)^k (10t+1)^1$) when $k \geq 12$ and $t \geq 7$. Filling the hole of size $10t + 1$ with an FSOLS($(2t-1)^5 1^6$), we obtain an FSOLS($(2t-1)^m 1^6$) for $m \geq 17$. ▯

Lemma 3.6 *There exists an FSOLS($h^m 1^6$) for $h \in \{9, 10, 11\}$ and $m \geq 15$.*

Proof. Apply Construction 2.4 with a TD(10, h), where $h \in \{9, 11\}$. Take a block B, give weight 1 to each point. For the points not in B, give weight 1 to each point in the first four groups and weight zero to each point in the last six groups. The input designs, 1^{10}, 1^4 and 1^5, are from Theorem 1.1. We then obtain an FSOLS($h^4 1^6$).

Start with a TD(6, 5) and apply Construction 2.4. Give each point of the first four groups weight 2. In the fifth group, give weight 1 to one point, weight 2 to two points, and weight zero to the remaining points. In the last group, give weight 1 to one point and weight zero the remaining points. The input designs, FSOLS of types 2^4, 2^5, $2^4 1^1$, $2^5 1^1$, and $2^4 1^2$, are from Theorem 1. We then get an FSOLS($10^4 5^1 1^1$). Filling the holes of size 5 with an FSOLS(1^5), we obtain an FSOLS($10^4 1^6$).

From Theorem 1 (3) we know that there exists an FSOLS($h^k(4h+6)^1$) when $k \geq 11$ and $h \in \{9, 10, 11\}$. Filling the hole of size $4h+6$ with an FSOLS($h^4 1^6$), we obtain an FSOLS($(2t)^m 1^6$) for $m \geq 15$. ∎

0	2	4	6	1	3	5
6	1	3	5	0	2	4
5	0	2	4	6	1	3
4	6	1	3	5	0	2
3	5	0	2	4	6	1
2	4	6	1	3	5	0
1	3	5	0	2	4	6

Table 3.3 an SOLS(7)

Lemma 3.7 *There exists an FSOLS*($7^m 1^6$) *for* $m \geq 7$.

Proof. Table 3.3 is an SOLS(7) with 2 pairs of symmetric transversals off the main diagonal and they are pairwise disjoint, and also with a symmetric transversal T on cells (0,6), (1,5), (2,4), (3,3), (4,2), (5,1), (6,0) which intersect each of the four transversals and the main diagonal in exactly one cell. Apply Lemma 2.4 with $h = 7$, $k = 2$, $m \geq 7$, $u = v = v_1 = v_2 = 1$, the input designs, FSOLS of types 1^m, 1^{m+1} and 1^{m+2}, are from Theorem 1.1 (1), we get an FSOLS of type $7^m 5^1 1^1$. Filling the hole of size five with an FSOLS(1^5), we obtain an FSOLS($7^m 1^6$). ∎

Combining Lemmas 3.1–3.7 we have the main result of this article.

Theorem 3.8 *(1) If* $h \geq 7$, $m \geq 4$, $n \geq 4$ *and* $n \neq 6$, *then there exists an FSOLS*($h^m 1^n$). *(2) An FSOLS*($h^m 1^6$) *exists if* $h \geq 7$, $h \neq 8, 12, 28$ *and* $m \geq 17$.

References

[1] F.E. Bennett, R. Wei and H. Zhang, Holey Schröder designs of type $2^n u^1$, *J. Combin. Designs* **6** (1998), 131–150.

[2] R.C. Bose, S.S. Shrikhande and E.T. Parker, Further results on the construction of mutually orthogonal Latin squares and falsity of Euler's conjecture, *Canad. J. Math.* **12** (1960), 189–203.

[3] R.K. Brayton, D. Coppersmith and A.J. Hoffmam, Self-orthogonal Latin squares, *Teorie Combinatorie*, Proc. Rome Conf., 1976, 509–517.

[4] A.E. Brouwer and G.H.J. van Rees, More mutually orthogonal Latin squares, *Discrete Math.* **39** (1982), 263–281.

[5] Chen Kejun and Zhu Lie, On the existence of skew Room frames of type t^u, *Ars Combinatoria* **43** (1996), 65–79.

[6] C.J. Colbourn and J.H. Dinitz (eds), *The CRC Handbook of Combinatorial Designs*, CRC Press, Boca Raton, 1996. (New results are reported at `http://www.emba.uvm.edu/~dinitz/newresults.html`).

[7] C.J. Colbourn and G.F. Royle, Supports of $(v, 4, 2)$ designs, *Le. Matematiche* XLV (1990), 39–60.

[8] C.J. Colbourn and D.R. Stinson, Edge-coloured designs with block size four, *Aequationes Math.* **36** (1988), 230–245.

[9] K. Heinrich, L. Wu and L. Zhu, Incomplete self-orthogonal Latin squares ISOLS$(6m + 6, 2m)$ exist for all m, *Discrete Math.* **87** (1991), 281–290.

[10] K. Heinrich and L. Zhu, Existence of orthogonal Latin squares with aligned subsquares, *Discrete Math.* **59** (1986), 69–87.

[11] C.C. Lindner and C.A. Rodger, "Decomposition into cycles II: Cycle systems", *Contemporary Design Theory*, Wiley, New York, 1992, pp. 325–369.

[12] D.R. Stinson, A general construction for group divisible designs, *Discrete Math.* **33** (1981), 89–94.

[13] D.R. Stinson and L. Zhu, On the existence of certain SOLS with holes, *J. Combin. Math. Combin. Computing* **15** (1994), 33–45.

[14] Xu Yunqing and Zhu Lie, Existence of frame SOLS of type $2^n u^1$, *J. Combin. Designs* **3** (1995), 115–133.

[15] Y. Xu, H. Zhang and L. Zhu, Existence of frame SOLS of type $a^n b^1$, *Discrete Math.* **250** (2002), 211–230.

(Received 11 Dec 2002; revised 23 July 2003)