AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **29** (2004), Pages 337–347

Some series of block designs with nested rows and columns

Kishore Sinha

Birsa Agricultural University
Ranchi-834006
India

Sanpei Kageyama

Hiroshima University
Higashi-Hiroshima 739-8524
Japan

Ashish Das

Indian Statistical Institute
New Delhi-110016
India

G. M. Saha

Indian Statistical Institute
Kolkata-700108
India

Abstract

Some new series of block designs with nested rows and columns have been constructed with some examples mainly from a combinatorial point of view. Two of the series are balanced (ternary), the others are partially balanced, based on rectangular and triangular association schemes.

1 Introduction

Balanced ternary designs with nested rows and columns have been studied by Bagchi et al. (1990), Bagchi (1996), and Tyagi and Rizwi (1995). Partially balanced designs with nested rows and columns have been studied by Street (1981), Agrawal and Prasad (1982a, 1982b, 1984), Morgan and Uddin (1990), Sinha and Kageyama (1990), and Gupta and Singh (1991). A useful review on nested row-column designs is provided in Morgan (1996). Partially balanced ternary designs were introduced in Sinha and Saha (1979). Useful surveys on construction of balanced ternary designs have been given by Billington (1984, 1989).

Under the combinatorial and/or statistical interest in the results of papers mentioned above, some series of such designs will be given in this paper. In particular, we shall present the constructions of a series of balanced ternary designs with nested rows and columns, and of some series of partially balanced designs with nested rows and columns, based on a rectangular or triangular association scheme, which are binary or ternary, along with some examples.

2 Preliminaries

Definitions of several designs and an association scheme are mentioned for easy discussions.

2.1 Block designs with nested rows and columns

A block design with nested rows and columns having parameters v, b, r, k, p, q is an arrangement of v treatments, each replicated r times, in b blocks of size $k = pq$ each which is a rectangular array of p rows and q columns (see Singh and Dey, 1979).

In a block design with nested rows and columns, let $\boldsymbol{N} = (n_{ij})$, $\boldsymbol{N}_r = (n_{ij}^{(r)})$ and $\boldsymbol{N}_c = (n_{ij}^{(c)})$ be the $v \times b$ treatment-block, $v \times p$ treatment-row and $v \times q$ treatment-column incidence matrices, respectively. Then under the usual additive model the coefficient matrix for intra-block estimation in the block design with nested rows and columns is given by

$$\boldsymbol{C} = r\boldsymbol{I} - \frac{1}{pq}\{p\boldsymbol{N}_r\boldsymbol{N}_r' + q\boldsymbol{N}_c\boldsymbol{N}_c' - \boldsymbol{N}\boldsymbol{N}'\},$$

where $\boldsymbol{I}$ is the identity matrix of appropriate order. Hence three coincidence structures $\boldsymbol{N}_r\boldsymbol{N}_r'$, $\boldsymbol{N}_c\boldsymbol{N}_c'$ and $\boldsymbol{N}\boldsymbol{N}'$ play a useful role for the analysis as well as combinatorial characterization of the design.

2.2 Balanced incomplete block (ternary) designs with nested rows and columns

These designs were considered by Bagchi et al. (1990). A ternary block design with nested rows and columns is said to be balanced if

$\boldsymbol{N}_r\boldsymbol{N}_r' = (r - \lambda_r)\boldsymbol{I} + \lambda_r\boldsymbol{J}, \quad n_{ij}^{(r)} = 0, 1;$

$\boldsymbol{N}_c\boldsymbol{N}_c' = (\Delta_c - \Lambda_c)\boldsymbol{I} + \Lambda_c\boldsymbol{J}, \quad n_{ij}^{(c)} = 0, 1, 2;$

$\boldsymbol{N}\boldsymbol{N}' = (\Delta - \Lambda)\boldsymbol{I} + \Lambda\boldsymbol{J}, \quad n_{ij} = 1, 2,$

where $\boldsymbol{J}$ is a matrix all of whose elements are unity, and, in particular, $\lambda_r = \sum_{j=1}^{p} n_{ij}^{(r)} n_{i'j}^{(r)}$ for any $i(\neq)i'$, $\Delta_c = \sum_{j=1}^{q}(n_{ij}^{(c)})^2$ and $\Delta = \sum_{j=1}^{b} n_{ij}^2$ for any i, $\Lambda_c = \sum_{j=1}^{q} n_{ij}^{(c)} n_{i'j}^{(c)}$ for any $i(\neq)i'$ and $\Lambda = \sum_{j=1}^{b} n_{ij} n_{i'j}$ for any $i(\neq)i'$. Note that symbols λ

and Λ are used when the corresponding incidence matrices have entries 0, 1 and 0, 1, 2 (or 1, 2), respectively.

In this case it follows that

$$\begin{aligned}\boldsymbol{C} &= r\boldsymbol{I} - \frac{1}{q}\boldsymbol{N}_r\boldsymbol{N}_r' - \frac{1}{p}\boldsymbol{N}_c\boldsymbol{N}_c' + \frac{1}{pq}\boldsymbol{N}\boldsymbol{N}' \\ &= \{r - \frac{1}{pq}[p(r-\lambda_r) + q(\Delta_c - \Lambda_c) - (\Delta - \Lambda)]\}\boldsymbol{I} \\ &\quad + \frac{1}{pq}(\Lambda - p\lambda_r - q\Lambda_c)\boldsymbol{J}.\end{aligned}$$

2.3 Association scheme

Given v treatments $1, 2, \ldots, v$, a relation satisfying the following conditions is said to form an association scheme with m associate classes:

(i) Any two treatments are either 1st, 2nd, ..., or mth associates, the relation of association being symmetrical, that is, if the treatment α is the ith associate of the treatment β, then β is the ith associate of α.

(ii) Each treatment α has n_i ith associates, the number n_i being independent of α, $i = 1, 2, \ldots, m$.

(iii) If any two treatments α and β are ith associates, then the number of treatments that are jth associates of α and ℓth associates of β is $p^i_{j\ell}$ and is independent of the pair of ith associates α and β.

The numbers v, n_i and $p^i_{j\ell}$ are called the parameters of the association scheme. Here we use a rectangular association scheme and a triangular association scheme (see Dey, 1986).

2.4 Partially balanced incomplete block designs with nested rows and columns

We shall recall the definition of partially balanced incomplete block (PBIB) designs with nested rows and columns, having parameters $v, b, r, k,\ p, q,\ \lambda_1, ..., \ \lambda_m$, from Gupta and Singh (1991) as follows.

An arrangement of v treatments in b blocks, each of p rows and q columns ($k = pq$), will be called an m-associate PBIB design with nested rows and columns (PBIB-RC) if the following conditions are satisfied:

(i) every treatment occurs at most once in a block;

(ii) every treatment occurs in exactly r blocks;

(iii) given any pair of treatments α and β which are ith associates, it holds that $p\lambda_{r(\alpha,\beta)} + q\lambda_{c(\alpha,\beta)} - \lambda_{b(\alpha,\beta)} = \lambda_i$, where $\lambda_{r(\alpha,\beta)}$, $\lambda_{c(\alpha,\beta)}$ and $\lambda_{b(\alpha,\beta)}$ denote the number of rows, columns and blocks in which α and β occur together, respectively, and λ_i is a constant for any pair of ith associates α and β chosen, $i = 1, 2, ..., m$.

For the PBIB-RC design it follows that

$$p\boldsymbol{N}_r\boldsymbol{N}_r' + q\boldsymbol{N}_c\boldsymbol{N}_c' - \boldsymbol{N}\boldsymbol{N}' = r(p+q-1)\boldsymbol{B}_0 + \sum_{i=1}^{m}\lambda_i\boldsymbol{B}_i$$

where $\boldsymbol{B}_0 = \boldsymbol{I}, \boldsymbol{B}_1, ..., \boldsymbol{B}_m$ denote the association matrices of an association scheme.

A rectangular association scheme is an arrangement of $v = mn$ treatments into m rows and n columns such that a pair of treatments in the same row are first associates, in the same column are second associates, otherwise they are third associates. A PBIB-RC design based on a rectangular association scheme is called a rectangular design with nested rows and columns. As a special case when $\lambda_1 = \lambda_2$ and $m = n$, we get a Latin square design with nested rows and columns.

2.5 Partially balanced (ternary) designs

A partially balanced ternary design is an arrangement of v treatments into b blocks such that

(i) the incidence matrix $\boldsymbol{N}_{v\times b} = (n_{ij})$ takes one of 0, 1, 2 as entries;

(ii) the column sum of $\boldsymbol{N}$ is k;

(iii) the row sum of $\boldsymbol{N}$ is r and the row sum of squares of entries of $\boldsymbol{N}$ is Δ;

(iv) the inner product of any two rows of $\boldsymbol{N}$ is Λ_i if treatments α and β corresponding to the two rows are ith associates, $i = 1, 2, ..., m$

(see Sinha and Saha, 1979).

In this case it follows that $\boldsymbol{N}\boldsymbol{N}' = \Delta\boldsymbol{B}_0 + \sum_{i=1}^{m}\Lambda_i\boldsymbol{B}_i$, where $\Delta = \sum_{j=1}^{b} n_{ij}^2$ for any i.

2.6 Partially balanced (ternary) designs with nested rows and columns

The definition in Section 2.4 is extended for the ternary case. A (ternary) block design with nested rows and columns is said to be partially balanced if, $\boldsymbol{N}$, $\boldsymbol{N}_r$ and $\boldsymbol{N}_c$ satisfy the condition

$$p\boldsymbol{N}_r\boldsymbol{N}_r' + q\boldsymbol{N}_c\boldsymbol{N}_c' - \boldsymbol{N}\boldsymbol{N}' = (p\Delta + q\Delta - r)\boldsymbol{B}_0 + \sum_{i=1}^{m}\Lambda_i\boldsymbol{B}_i,$$

where $\lambda_{i(r)} = \sum_{j=1}^{pb} n_{\alpha j}n_{\beta j}$ for any two treatments α and β being ith associates in rows, $\lambda_{i(c)} = \sum_{j=1}^{qb} n_{\alpha j}n_{\beta j}$ for any two treatments α and β being ith associates in columns, while $\lambda_i = \sum_{j=1}^{b} n_{\alpha j}n_{\beta j}$ for any two treatments α and β being ith associates occurring in λ_i blocks and $p\lambda_{i(r)} + q\lambda_{i(c)} - \lambda_i = \Lambda_i$ is a constant for any pair of ith associates α and β.

3 Balanced/universally optimal designs with nested rows and columns

A PBIB design with parameters $v, b, r, k,\ \lambda_1 = \lambda_2 = \cdots = \lambda_m$ ($= \lambda$, say) is called a balanced incomplete block (BIB) design with parameters $v, b, r, k,\ \lambda$.

Theorem 3.1. When $4t-1$ is a prime or a prime power, there exists a balanced ternary design with nested rows and columns, having parameters

(i) {blocks}(n_{ij} = 1,2); $v = 4t-1 = b, r = 4t = k, \Lambda = 4t+1, p = 2, q = 2t$;
(ii) {rows}($n_{ij}^{(r)}$ = 0, 1); $v = 4t-1, b = 2(4t-1), r = 4t, k = 2t = \lambda_r$;
(iii) {columns}($n_{ij}^{(c)}$ = 0, 1, 2); $v = 4t-1, b = 2t(4t-1), r = 4t, k = 2, \Lambda_c = 1$.

Proof. A BIB design with parameters $v = 4t-1 = b, r = 2t-1 = k, \lambda = t-1$ (and its complementary BIB design with parameters $v = 4t-1 = b, r = 2t = k, \lambda = t$) exists whenever $4t-1$ is a prime or a prime power (cf. Raghavarao, 1971, Theorem 5.7.4). The solutions to these BIB designs may be obtained respectively by developing the initial blocks $(x^0, x^2, ..., x^{4t-4})$ and $(0, x, x^3, ..., x^{4t-3})$, where x is a primitive element of GF$(4t-1)$. Then it can be shown that the initial blocks

$$\begin{pmatrix} 0 & x^0 & x^2 & \cdots & x^{4t-4} \\ 0 & x & x^3 & \cdots & x^{4t-3} \end{pmatrix}$$

generate a balanced ternary design with nested rows and columns (for detail, see Section 2.2), and with the parameters as stated above. □

Example 3.1. By Theorem 3.1 with $t = 2$ and $x = 3$, the initial blocks

$$\begin{pmatrix} 0 & 1 & 2 & 4 \\ 0 & 3 & 6 & 5 \end{pmatrix} \quad \text{mod } 7$$

form a balanced ternary design with nested rows and columns, having the parameters:

(i) {blocks}; $v = b = 7, r = k = 8, \Lambda = 9, p = 2, q = 4$;
(ii) {rows}; $v = 7, b = 14, r = 8, k = 4 = \lambda_r$;
(iii) {columns}; $v = 7, b = 28, r = 8, k = 2, \Lambda_c = 1$.

Remark 1. In the spirit of Bagchi et al. (1990), Theorem 3.1 can be revised with the following initial blocks

$$\begin{pmatrix} 0 & x^0 & x^2 & \cdots & x^{4t-4} \\ 0 & x^2 & x^4 & \cdots & x^0 \end{pmatrix}$$

having the same parameters as those in Theorem 3.1 except for $\Lambda = 4t$ in {blocks}, where the column design is balanced and the rows have the multiset property. Thus, for the usual analysis this design is uniformly better than the one in Theorem 3.1.

Also it may be possible that the design in Theorem 3.1 is better for the analysis recovering row, column and block information.

As a design close to balanced we can present the following.

Theorem 3.2. When $4t-1$ is a prime or a prime power, there exists a universally optimal ternary design with nested rows and columns, having parameters $v = 4t-1 = b, r = k = 4t, p = 2, q = 2t,\ \Lambda = 4t, \lambda_r = 2t, \lambda_c = 1, 2$.

Proof. Consider the initial blocks

$$\begin{pmatrix} 0 & x^0 & x^2 & \cdots & x^{4t-4} \\ x^{4t-4} & 0 & x^0 & \cdots & x^{4t-6} \end{pmatrix}$$

which yield the required design. It follows from noting that this design satisfies the row property given by Bagchi (1990, 1996). Note that the design here is not balanced in {columns}. □

Remark 2. A column of zeros can be appended to all initial blocks of any highly efficient cyclically generated design with nested rows and columns of $p \times q$ rectangles having the Bagchi row property to have a new highly efficient design with nested rows and columns of $p \times (q+1)$ rectangles.

4 Rectangular designs with nested rows and columns

Rectangular designs with nested rows and columns (see Section 2.4) have been studied by Agrawal and Prasad (1984), Morgan and Uddin (1990), and Bagchi (1996). A simple method of construction is given.

Theorem 4.1. For positive integers m, n (≥ 2), there exists a rectangular (ternary) design with nested rows and columns, having parameters

(i) {blocks}$(n_{ij} = 0, 1, 2)$; $v = mn = b, r = n(m-1) = k, \Lambda_1 = (m-1)(n-2), \Lambda_2 = (m-2)(n-1), \Lambda_3 = (m-2)(n-2) + 2(m-1), p = m-1, q = n$;

(ii) {rows}$(n_{ij}^{(r)} = 0, 1)$; $v = mn, b = mn(m-1), r = n(m-1), k = n, \lambda_1 = (m-1)(n-2), \lambda_2 = 0, \lambda_3 = 2$;

(iii) {columns}$(n_{ij}^{(c)} = 0, 1, 2)$; $v = mn, b = mn^2, r = n(m-1), k = m-1, \Lambda_1 = 0, \Lambda_2 = (m-2)(n-1), \Lambda_3 = 0$.

Proof. Let us form an $m \times n$ array of $v = mn$ treatments in m rows and n columns. Now corresponding to each of mn treatments, we obtain an $(m-1) \times (n-1)$ array by deleting the rows and columns containing the element. Thus, we get mn arrays of size $(m-1) \times (n-1)$. Without consideration of the rows and columns these mn arrays form blocks each of size $k = (m-1)(n-1)$ of a rectangular design [i] with parameters $v = mn = b, r = (m-1)(n-1) = k, \lambda_1 = (m-1)(n-2), \lambda_2 = (m-2)(n-1), \lambda_3 = (m-2)(n-2), p = m-1, q = n-1$. By deleting any two columns and the row containing any two treatments, we are left with $(m-1)(n-2) = \lambda_1$ treatments, since any two treatments in the same row are first associates.

Similarly, by deleting any two rows and the column containing any two treatments, we are left with $(m-2)(n-1) = \lambda_2$ treatments, since any two treatments in the same column are second associates.

Furthermore, since any two treatments which are not in the same row or column are third associates, by deleting rows and columns containing the two second associate treatments, we are left with $(m-2)(n-2) = \lambda_3$ treatments. By considering the rows and columns as blocks separately, we can get designs [ii] and [iii] with parameters $v = mn, b = mn(m-1), r = (m-1)(n-1), k = n-1, \lambda_1 = (m-1)(n-2), \lambda_2 = \lambda_3 = 0$, and with parameters $v = mn, b = mn(m-1), r = (m-1)(n-1), k = m-1, \lambda_1 = 0, \lambda_2 = (m-2)(n-1), \lambda_3 = 0$, respectively.

Finally, by adding the reference treatment itself in each row of the blocks of the designs [i], [ii] and [iii] constructed above and forming one column of reference treatments only, we get a partially balanced ternary design (see Sections 2.5 and 2.6), based on a rectangular association scheme, with nested rows and columns, having the parameters (i), (ii) and (iii), respectively (see also Example 4.1). □

As a special case, when $m = n$, by combining the first and second associates into one associate class in a rectangular association scheme, we get a Latin square design with parameters for blocks as $v = b = m^2, r = k = (m-1)^2, \lambda_1 = (m-2)(m-1), \lambda_2 = (m-2)^2$. The complement of this Latin square design is given in Dey (1986, Theorem 5.20).

Example 4.1. Given below is a plan of a rectangular (ternary) design with nested rows and columns, having an association scheme with $m = 3 = n$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix};$$

$$\left(\begin{matrix} 1 & 5 & 6 \\ 1 & 8 & 9 \end{matrix} \quad \begin{matrix} 2 & 4 & 6 \\ 2 & 7 & 9 \end{matrix} \quad \begin{matrix} 3 & 4 & 5 \\ 3 & 7 & 8 \end{matrix} \quad \begin{matrix} 4 & 2 & 3 \\ 4 & 8 & 9 \end{matrix} \quad \begin{matrix} 5 & 1 & 3 \\ 5 & 7 & 9 \end{matrix} \quad \begin{matrix} 6 & 1 & 2 \\ 6 & 7 & 8 \end{matrix}\right.$$

$$\left.\begin{matrix} 7 & 2 & 3 \\ 7 & 5 & 6 \end{matrix} \quad \begin{matrix} 8 & 1 & 3 \\ 8 & 4 & 6 \end{matrix} \quad \begin{matrix} 9 & 1 & 2 \\ 9 & 4 & 5 \end{matrix}\right),$$

and the parameters

(i) {blocks}; $v = b = 9, r = k = 6, \Lambda_1 = 2, \Lambda_2 = 2, \Lambda_3 = 5$;

(ii) {rows}; $v = 9, b = 18, r = 6, k = 3, \lambda_1 = 2, \lambda_2 = 0, \lambda_3 = 2$;

(iii) {columns}; $v = 9, b = 27, r = 6, k = 2, \Lambda_1 = 0, \Lambda_2 = 2, \Lambda_3 = 0$.

This design is disconnected in {columns}, whereas the others are connected.

Bagchi (1996) has constructed E-optimal designs with nested rows and columns for all even v with 2×4 rectangles as blocks.

Theorem 4.2. When s is a prime or a prime power, there exists an E-optimal rectangular design with nested rows and columns, having parameters $v = 2s, b =$

$s(s-1), p = 2 = q$, $\Lambda = 0, 4$ (i.e., $\Lambda_1 = 0, \Lambda_2 = 0, \Lambda_3 = 4$), $\lambda_r = 0, 2$ (i.e., $\lambda_{1(r)} = 0, \lambda_{2(r)} = 0, \lambda_{3(r)} = 2$), $\lambda_c = 0, 2$ (i.e., $\lambda_{1(c)} = 0, \lambda_{2(c)} = 0, \lambda_{3(c)} = 2$).

Proof. It is known (see Corollary 2.6.3, Sinha et al. 2002) that there exists a series of E-optimal rectangular design with parameters $v = 2s, b = s(s-1), r = s-1, k = 2, \lambda_1 = 0 = \lambda_2, \lambda_3 = 1$, based on a $2 \times s$ rectangular association scheme. Following Bagchi's procedure (p. 354, Bagchi 1996), the above design may be converted to the required E-optimal design with nested rows and columns. □

Srivastava (1981) pointed out that for a design with nested rows and columns it is desirable that the sizes of the rows as well as the columns be as small as possible so that the additive model is valid. The smallest possible designs would have 2×2 or 2×3 rectangles as blocks. Bagchi (1996) has constructed optimal designs with nested rows and columns for $v \equiv 0, 2 \pmod 4$ with 2×4 rectangles as blocks. Prasad et al. (2001) also constructed and tabulated optimal designs with nested rows and columns.

Given below is a table of E-optimal designs with nested rows and columns obtained from E-optimal rectangular designs with nested rows and columns having smallest possible rectangles 2×2 and 2×3 of blocks. SKS- and SSKS-numbers are from Sinha et al. (1993, 2002).

No	v	b	p	q	Source
1	6	6	2	2	SKS-1,2
2	6	6	2	3	SKS-3
3	8	18	2	2	SKS-16
4	10	20	2	2	SKS-20
5	12	30	2	2	SKS-25
6	14	42	2	2	SKS-35
7	16	56	2	2	SKS-42
8	18	72	2	2	SKS-48
9	20	90	2	2	SKS-52
10	22	110	2	2	SSKS-41

5 Triangular row-column designs

A construction of triangular designs with nested rows and columns is given by Agrawal and Prasad (1984). A series of triangular (ternary) designs with nested rows and columns (see Sections 2.3, 2.4 and 2.6) is presented here.

Theorem 5.1. For positive integers n (≥ 2), there exists a partially balanced ternary design, based on a triangular association scheme, with nested rows and columns, having parameters

(i) {blocks}($n_{ij} = 0, 1, 2$); $v = b = \binom{n}{2}, r = k = 2(n-1), \Lambda_1 = n+2, \Lambda_2 = 4$;

(ii) {rows}($n_{ij}^{(r)} = 0, 1$); $v = \binom{n}{2}, b = n(n-1), r = 2(n-1), k = n-1, \lambda_1 = n-1,$ $\lambda_2 = 0$;

(iii) {columns}($n_{ij}^{(c)} = 0,\ 1,\ 2$); $v = \binom{n}{2}, b = (n-1)\binom{n}{2}, r = 2(n-1), k = 2, \Lambda_1 = 1, \Lambda_2 = 0$.

Proof. Let X be a set of n elements $1, 2, ..., n$. The treatments are denoted by pairs $(ij) = (ji)$, $i \neq j, i, j \in X$. Each block corresponds to a treatment and contains all its first associates together with the treatment itself occurring twice, where two treatments are first associates of each other, if they have an element in common. Now a block is divided into two halves or rows such that all the treatments in a row have either i or j in common except the reference treatment which occurs in both the rows. Within a block, the columns are formed of pairs of treatments from different rows, with an element in common except one column consisting only of the reference treatment repeated twice. It is obvious from the construction that the resulting design is based on a triangular association scheme and has the parameters as stated above. $\square$

Example 5.1. By Theorem 5.1 with $n = 5$, a plan of a triangular (ternary) design with nested rows and columns, having a triangular association scheme:

$$\begin{bmatrix} & 1 & 2 & 3 & 4 \\ 1 & & 5 & 6 & 7 \\ 2 & 5 & & 8 & 9 \\ 3 & 6 & 8 & & 10 \\ 4 & 7 & 9 & 10 & \end{bmatrix},$$

where the above 10 treatments denote pairs (12), (13), (14), (15), (23), (24), (25), (34), (35), (45), respectively, is given by

$$\left(\begin{matrix} 12 & 13 & 14 & 15 \\ 12 & 23 & 24 & 25 \end{matrix}\quad \begin{matrix} 13 & 12 & 14 & 15 \\ 13 & 23 & 34 & 35 \end{matrix}\quad \begin{matrix} 14 & 12 & 13 & 15 \\ 14 & 24 & 34 & 45 \end{matrix}\quad \begin{matrix} 15 & 12 & 13 & 14 \\ 15 & 25 & 35 & 45 \end{matrix}\right.$$

$$\begin{matrix} 23 & 12 & 24 & 25 \\ 23 & 13 & 34 & 35 \end{matrix}\quad \begin{matrix} 24 & 12 & 23 & 25 \\ 24 & 14 & 34 & 45 \end{matrix}\quad \begin{matrix} 25 & 12 & 23 & 24 \\ 25 & 15 & 35 & 45 \end{matrix}\quad \begin{matrix} 34 & 13 & 23 & 35 \\ 34 & 14 & 24 & 45 \end{matrix}$$

$$\left.\begin{matrix} 35 & 13 & 23 & 34 \\ 35 & 15 & 25 & 45 \end{matrix}\quad \begin{matrix} 45 & 14 & 24 & 34 \\ 45 & 15 & 25 & 35 \end{matrix}\right),$$

having parameters

(i) {blocks}; $v = b = 10, r = k = 8, \Lambda_1 = 7, \Lambda_2 = 4$;

(ii) {rows}; $v = 10, b = 20, r = 8, k = 4, \lambda_1 = 4, \lambda_2 = 0$;

(iii) {columns}; $v = 10, b = 40, r = 8, k = 2, \Lambda_1 = 1, \Lambda_2 = 0$.

Remark 3. The triangular designs with $k = 2$ in columns, as n grows, use many fewer units than a balanced design with $k = 2$ would require as $v(v-1)$. These designs require minimum possible number of rows, i.e., 2 and also produce designs for odd v, whereas the designs in Bagchi (1996) are only for even v. Prasad et al. (2001) produce one optimal design for odd v $(= 9)$.

Acknowledgement. The authors are grateful to the referee for his/her constructive suggestions and comments to improve the paper.

References

[1] Agrawal, H. L. and Prasad, J. (1982a). Some methods of construction of GD-RC and rectangular-RC designs. *Austral. J. Statist.* 24, 191–200.

[2] Agrawal, H. L. and Prasad, J. (1982b). On nested row-column partially balanced incomplete block designs. *Calcutta Statist. Assoc. Bull.* 31, 131–136.

[3] Agrawal, H. L. and Prasad, J. (1984). Construction of partially balanced incomplete block designs with nested rows and columns. *Biom. J.* 26, 883–891.

[4] Bagchi, S. (1996). Two infinite series of E-optimal nested row-column designs. *J. Statist. Plann. Inference* 52, 353–357.

[5] Bagchi, S., Mukhopadhyay, A. C. and Sinha, B. K. (1990). A search for optimal nested row-column designs. *Sankhyā* 52B, 93–104.

[6] Billington, E. J. (1984). Balanced n-ary designs: a combinatorial survey and some new results. *Ars Combin.* 17A, 37–72.

[7] Billington, E. J. (1989). Designs with repeated elements in blocks: a survey and some recent results. *Congressus Numeratium* 68, 123–146.

[8] Dey, A. (1986). *Theory of Block Designs.* Wiley Eastern.

[9] Gupta, S. and Singh, M. (1991). Partially balanced incomplete block designs with nested rows and columns. *Utilitas Math.* 40, 291–302.

[10] Morgan, J. P. (1996). Nested designs. *Handbook of Statistics*, Vol. 13, Elsevier, Amsterdam, 939–976.

[11] Morgan, J. P. and Uddin, N. (1990). Some constructions for rectangular, Latin square and pseudocyclic nested row-column designs. *Utilitas Math.* 38, 43–51.

[12] Prasad, R., Gupta, V. K. and Voss, D. (2001). Optimal nested row-column designs. *J. Indian Soc. Agric. Statist.* LIV, 2, 244–257.

[13] Raghavarao, D. (1988). *Constructions and Combinatorial Problems in Design of Experiments.* Dover, New York.

[14] Singh, M. and Dey, A. (1979). Block designs with nested rows and columns. *Biometrika* 66, 321–326.

[15] Sinha, K. and Kageyama, S. (1990). A new series of triangular PBIB designs with nested rows and columns. *J. Statist. Plann. Inference* 24, 403–404.

[16] Sinha, K., Kageyama, S. and Singh, M. K. (1993). Construction of rectangular designs. *Statistics* 25, 63–70.

[17] Sinha, K. and Saha, G. M. (1979). On the construction of balanced and partially balanced ternary designs. *Biom. J.* 21, 767–772.

[18] Sinha, K., Singh, M. K., Kageyama, S. and Singh, R. S. (2002). Some series of rectangular designs. *J. Statist. Plann. Inference* 106, 39–46.

[19] Srivastava, J. N. (1981). Some problems with nested nuisance factors. *Bull. Int. Statist. Inst.* XLIX (Book 1), 547–565.

[20] Street, D. J. (1981). Graeco-Latin and nested row and column designs. *Combinatorial Math.*, Vol. 8, Lecture Notes in Math. 884, Springer, 304–313.

[21] Tyagi, B. N. and Rizwi, S. K. H. (1995). Balanced ternary designs with nested rows and columns. *Sankhyā* 57B, 456–459.

(Received 23 Jan 2003)