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Labelings of unions of up to four uniform cycles

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Abstract

We show that every 2-regular graph consisting of at most four uniform components has a ρ-labeling (or a more restricted labeling). This has an application in the cyclic decomposition of certain complete graphs into the disjoint unions of cycles.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a < b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. Let $V(K_v) = \mathbb{Z}_v$ and let G be a subgraph of K_v. By *clicking* G, we mean applying the isomorphism $i \to i + 1$ to $V(G)$. Let K and G be graphs such that G is a subgraph of K. A G*-decomposition* of K is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that the edge sets of the graphs G_i form a partition of the edge set of K. If K is K_v, a G-decomposition Γ of K is *cyclic* if clicking is a permutation of Γ. If G is a graph and r is a positive integer, rG denotes the vertex disjoint union of r copies of G.

For any graph G, an injective function $h : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [14], Rosa introduced a hierarchy of labelings. We add a few items to this hierarchy. Let G be a graph with n edges and no isolated vertices and let h be a labeling of G. Let $h(V(G)) = \{h(u) : u \in V(G)\}$. Define a function $\bar{h} : E(G) \to \mathbb{Z}^+$ by $\bar{h}(e) = |h(u) - h(v)|$, where $e = \{u, v\} \in E(G)$. Let $\bar{E}(G) = \{\bar{h}(e) : e \in E(G)\}$. Consider the following conditions:

(a) $h(V(G)) \subseteq [0, 2n]$,

(b) $h(V(G)) \subseteq [0, n]$,

(c) $\bar{E}(G) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n+1-i$,

(d) $\bar{E}(G) = [1, n]$.

If in addition G is bipartite, then there exists a bipartition (A, B) of $V(G)$ (with every edge in G having one endvertex in A and the other in B) such that

(e) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $h(a) < h(b)$,

(f) there exists an integer λ such that $h(a) \leq \lambda$ for all $a \in A$ and $h(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:
(a), (c) is called a *ρ-labeling*;
(a), (d) is called a *σ-labeling*;
(b), (d) is called a *β-labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. If G is bipartite and a ρ, σ or β-labeling of G also satisfies (e), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition (f) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an *α-labeling* as introduced in [14].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results by Rosa [14].

Theorem 1 *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 2 *Let G be a graph with n edges that has an α-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

Clearly if G is bipartite, then an α-labeling of G is the most desired labeling. However, there exist numerous classes of bipartite graphs (including some classes of trees) which do not admit α-labelings (see [14]). Hence the need to introduce the variations on the theme of α-labelings. In [6] it was shown that Theorem 2 extends to graphs with ρ^+-labelings.

Theorem 3 *Let G be a graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

Let G be a graph with n edges and Eulerian components and let h be a β-labeling of G. It is well-known (see [14]) that we must have $n \equiv 0$ or $3 \pmod 4$. Moreover, if such a G is bipartite, then $n \equiv 0 \pmod 4$. These conditions hold since for such a G, $\sum_{e \in E(G)} \bar{h}(e) = n(n+1)/2$. This sum must in turn be even, since each vertex is incident with an even number of edges and $\bar{h}(e) = |h(u) - h(v)|$, where u and v are the endvertices of e. Thus we must have $4|n(n+1)$. Clearly, the same will hold if such a G admits a σ-labeling. We shall refer to this restriction as the *parity condition*. There are no such restrictions on $|E(G)|$ if h is a ρ-labeling.

Theorem 4 (Parity Condition) *If a graph G with Eulerian components and n edges has a σ-labeling, then $n \equiv 0$ or $3 \pmod 4$. If such a G is bipartite, then $n \equiv 0 \pmod 4$.*

In [14], Rosa presented α- and β-labelings of C_{4m} and of C_{4m+3}, respectively. It is also known that both C_{4m+1} and C_{4m+2} admit ρ-labelings. It was also shown in [6] that there exists a ρ^+-labeling of C_{4m+2}, for all positive integers m. It can be easily checked that this labeling is actually a ρ^{++}-labeling.

In this manuscript, we will focus on labelings of 2-regular graphs (i.e., the vertex-disjoint union of cycles). If a 2-regular graph G is bipartite, then it is shown in [3] that G necessarily admits a ρ^{++}-labeling. Such a G need not admit an α-labeling, even if the parity condition is satisfied. It is well-known for example that $3C_4$ does not have an α-labeling (see [11]). Similarly, if G is not bipartite, then G need not admit a β-labeling even if the parity condition is satisfied. For example, it is shown in [12] that rC_3 does not admit a β-labeling for all $r > 1$ and rC_5 never admits a β-labeling. It is thus reasonable to focus on labelings that are less restrictive than β-labelings when studying 2-regular graphs.

Here, we shall show that every 2-regular graph consisting of at most four uniform components has a ρ-labeling (or a more restricted labeling). This has an application in the cyclic decomposition of certain complete graphs into the disjoint unions of cycles. Moreover, it provides further evidence in support of a conjecture that every 2-regular graph admits a ρ-labeling.

2 Summary of Some of the Known Results

As stated in the previous section, the following is known for cycles (see [13], [14] and [6]).

Theorem 5 *Let $m \geq 3$ be an integer. Then, C_m admits an α-labeling if $m \equiv 0 \pmod 4$, a ρ-labeling if $m \equiv 1 \pmod 4$, a ρ^{++}-labeling if $m \equiv 2 \pmod 4$, and a β-labeling if $m \equiv 3 \pmod 4$.*

For 2-regular graphs with two components, we have the following from Abrham and Kotzig [2].

Theorem 6 *Let $m \geq 3$ and $n \geq 3$ be integers. Then the graph $C_m \cup C_n$ has a β-labeling if and only if $m+n \equiv 0$ or $3 \pmod 4$. Moreover, $C_m \cup C_n$ has an α-labeling if and only if both m and n are even and $m+n \equiv 0 \pmod 4$.*

Thus $2C_m$ has an α-labeling if $m \geq 4$ is even. In the next section, we show that $2C_m$ admits a ρ-labeling if $m \geq 3$ is odd.

For 2-regular graphs with more than two components, the following is known. In [11], Kotzig shows that if $r > 1$, then rC_3 does not admit a β-labeling. Similarly, he shows that rC_5 does not admit a β-labeling for any r. In [12], Kotzig shows that $3C_{4k+1}$ admits a β-labeling for all $k \geq 2$. In [5], it is shown that rC_3 admits a ρ-labeling for all $r \geq 1$. In [8], Eshghi shows that $C_{2m} \cup C_{2n} \cup C_{2k}$ has an α-labeling for all m, n, and $k \geq 2$ with $m+n+k \equiv 0 \pmod 2$ except when $m = n = k = 2$. Thus $3C_{4m}$ has an α-labeling for all $m > 1$. In [1], Abrham and Kotzig show that rC_4 has an α-labeling for all positive integers $r \neq 3$. An additional result follows by combining results from [6] and from [3].

Theorem 7 *Let G be a 2-regular bipartite graph of order n. Then G has a σ^{++}-labeling if $n \equiv 0 \pmod 4$ and a ρ^{++}-labeling if $n \equiv 2 \pmod 4$.*

3 Main results

We shall show that $2C_m$ has a ρ-labeling when m is odd, $3C_5$ has a σ-labeling, $3C_m$ has a ρ-labeling when $m \equiv 3 \pmod 4$, and $4C_m$ has a σ-labeling when m is odd. This along with some of the known results shows that rC_m has a ρ-labeling (or a more restricted labeling) when $r \leq 4$. Some additional definitions and notational conventions are necessary.

We denote the path with consecutive vertices $a_1, a_2, \ldots, a_k$ by $(a_1, a_2, \ldots, a_k)$. By $(a_1, a_2, \ldots, a_k) + (b_1, b_2, \ldots, b_j)$, where $a_k = b_1$, we mean the path $(a_1, \ldots, a_k, b_2, \ldots, b_j)$.

To simplify our consideration of various labelings, we will sometimes consider graphs whose vertices are named by distinct nonnegative integers, which are also their labels.

Let a, b, and h be integers with $0 \leq a \leq b$ and $h > 0$. Set $d = b - a$. We define the path

$$P(a,h,b) = (a, a+h+2d-1, a+1, a+h+2d-2, a+2, \ldots, b-1, b+h, b).$$

It is easily checked that $P(a,h,b)$ is simple and

$$V(P(a,h,b)) = [a,b] \cup [b+h, b+h+d-1].$$

Furthermore, the edge labels of $P(a,h,b)$ are distinct and

$$\bar{E}(P(a,h,b)) = [h, h+2d-1].$$

These formulas will be used extensively in the proofs that follow.

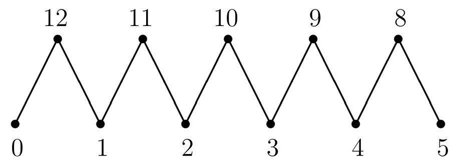

Figure 1: The path $P(0,3,5)$.

Theorem 8 *Let the graph G consist of two vertex-disjoint cycles, each of the same odd length. Then G has a ρ-labeling.*

Proof. First we consider cycles of length $4x+1$, x a positive integer. The two cycles will be G_1 and G_2, defined as follows:

$$\begin{aligned} G_1 &= P(0,6x+4,x-1)+P(x-1,4x+3,2x-1)+(2x-1,2x,8x+3,0), \\ G_2 &= P(8x+4,2x+2,9x+4)+P(9x+4,3,10x+3) \\ &+ (10x+3,10x+5,12x+6,8x+4). \end{aligned}$$

Now we compute

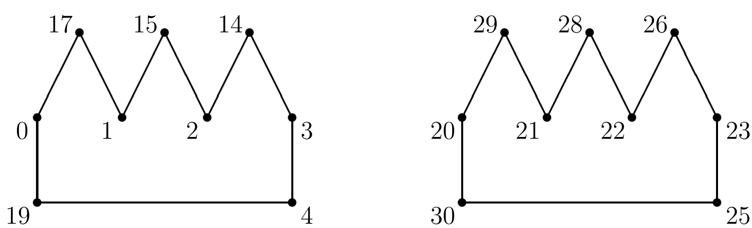

Figure 2: A ρ-labeling of $C_9 \cup C_9$.

$$\begin{aligned} V(G_1) &= [0,2x-1]\cup[7x+3,8x+1]\cup[6x+2,7x+1]\cup\{2x,8x+3\} \\ V(G_2) &= [8x+4,10x+3]\cup[11x+6,12x+5]\cup[10x+6,11x+4] \\ &\quad \cup\{10x+5,12x+6\}. \end{aligned}$$

We can order these as

$$[0,2x-1],2x,[6x+2,7x+1],[7x+3,8x+1],8x+3$$

from G_1, and

$$[8x+4,10x+3],10x+5,[10x+6,11x+4],[11x+6,12x+5],12x+6$$

from G_2. We see that the vertices of the two cycles are distinct and contained in $[0, 2(8x+2)] = [0, 16x+4]$. (If $x = 1$ the sets $[7x+3, 8x+1]$ and $[10x+6, 11x+4]$ are empty, but this does not change the proof.)

Likewise we compute

$$\begin{aligned}
\bar{E}(G_1) &= [6x+4, 8x+1] \cup [4x+3, 6x+2] \cup \{1, 6x+3, 8x+3\}, \\
\bar{E}(G_2) &= [2x+2, 4x+1] \cup [3, 2x] \cup \{2, 2x+1, 4x+2\}.
\end{aligned}$$

We can order these as the edge label 1 from G_1,

$$2, [3, 2x], 2x+1, [2x+2, 4x+1], 4x+2$$

from G_2, and

$$[4x+3, 6x+2], 6x+3, [6x+4, 8x+1], 8x+3$$

from G_1. Thus $\bar{E}(G) = [1, 8x+1] \cup \{8x+3\}$. Since $2(8x+2)+1-(8x+3) = 8x+2$, we have a ρ-labeling. (If $x = 1$ the sets $[3, 2x]$ and $[6x+4, 8x+1]$ are empty, but this does not change the proof.)

Now suppose the cycles have length $4x+3$, x a nonnegative integer. The two cycles will be defined as follows:

$$\begin{aligned}
G_1 &= P(0, 6x+6, x) + P(x, 4x+5, 2x) + (2x, 2x+2, 8x+7, 0), \\
G_2 &= P(8x+8, 2x+4, 9x+8) + P(9x+8, 3, 10x+8) \\
&+ (10x+8, 10x+9, 12x+12, 8x+8).
\end{aligned}$$

Now we compute

$$\begin{aligned}
V(G_1) &= [0, 2x] \cup [7x+6, 8x+5] \cup [6x+5, 7x+4] \cup \{2x+2, 8x+7\} \\
V(G_2) &= [8x+8, 10x+8] \cup [11x+12, 12x+11] \cup [10x+11, 11x+10] \\
&\quad \cup \{10x+9, 12x+12\}.
\end{aligned}$$

We can order these as

$$[0, 2x], 2x+2, [6x+5, 7x+4], [7x+6, 8x+5], 8x+7$$

from G_1, and

$$[8x+8, 10x+8], 10x+9, [10x+11, 11x+10], [11x+12, 12x+11], 12x+12$$

from G_2. We see that the vertices of the two cycles are distinct and contained in $[0, 2(8x+6)] = [0, 16x+12]$. (If $x = 0$ the sets $[6x+5, 7x+4]$, $[7x+6, 8x+5]$, $[10x+11, 11x+10]$ and $[11x+12, 12x+11]$ are empty, but this does not change the proof.) Likewise we compute

$$\begin{aligned}
\bar{E}(G_1) &= [6x+6, 8x+5] \cup [4x+5, 6x+4] \cup \{2, 6x+5, 8x+7\}, \\
\bar{E}(G_2) &= [2x+4, 4x+3] \cup [3, 2x+2] \cup \{1, 2x+3, 4x+4\}.
\end{aligned}$$

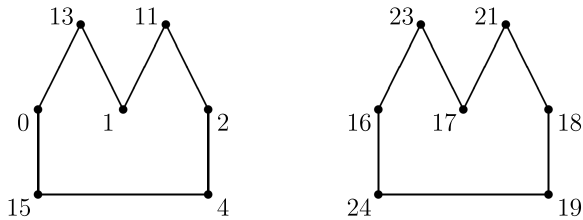

Figure 3: A ρ-labeling of $C_7 \cup C_7$.

We can order these as the edge label 1 from G_2, 2 from G_1,

$$[3, 2x+2], 2x+3, [2x+4, 4x+3], 4x+4$$

from G_2, and

$$[4x+5, 6x+4], 6x+5, [6x+6, 8x+5], 8x+7$$

from G_1. Thus $\bar{E}(G) = [1, 8x+5] \cup \{8x+7\}$. Since $2(8x+6)+1-(8x+7) = 8x+6$, we have a ρ-labeling. (If $x = 0$ the sets $[3, 2x+2], [2x+4, 4x+3], [4x+5, 6x+4]$ and $[6x+6, 8x+5]$ are empty, but this does not change the proof.) □

It is known that $3C_5$ does not have a β-labeling (see [11]) and that $3C_{4x+1}$ has a β-labeling for $x \geq 2$ (see [12]).

Lemma 9 *The graph consisting of the vertex-disjoint union of three C_5's has a σ-labeling.*

Proof. Take the cycles $(0, 14, 1, 4, 15, 0)$, $(16, 25, 17, 18, 28, 16)$ and $(5, 11, 6, 8, 12, 5)$. □

Theorem 10 *Let x be a nonnegative integer, and let a graph consist of three vertex-disjoint cycles, each of length $4x+3$. Then the graph has a ρ-labeling.*

Proof. First assume $x > 0$. The three cycles will be G_1, G_2, and G_3, defined as follows:

$$\begin{aligned}
G_1 &= P(0, 10x+9, x) + P(x, 8x+7, 2x) + (2x, 2x+3, 12x+10, 0),\\
G_2 &= P(12x+11, 4x+6, 14x+11) + (14x+11, 14x+13, 22x+19, 12x+11),\\
G_3 &= P(2x+4, 2x+5, 3x+4) + P(3x+4, 4, 4x+4)\\
&+ (4x+4, 4x+5, 6x+9, 2x+4).
\end{aligned}$$

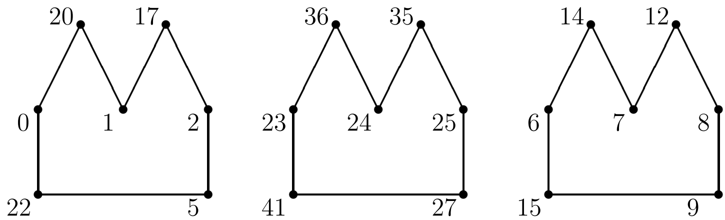

Figure 4: A ρ-labeling of $C_7 \cup C_7 \cup C_7$.

Now we compute

$$\begin{aligned}
V(G_1) &= [0,2x] \cup [11x+9, 12x+8] \cup [10x+7, 11x+6] \cup \{2x+3, 12x+10\},\\
V(G_2) &= [12x+11, 14x+11] \cup [18x+17, 20x+16] \cup \{14x+13, 22x+19\},\\
V(G_3) &= [2x+4, 4x+4] \cup [5x+9, 6x+8] \cup [4x+8, 5x+7] \cup \{4x+5, 6x+9\}.
\end{aligned}$$

We can order these as

$$[0,2x], 2x+3$$

from G_1,

$$[2x+4, 4x+4], 4x+5, [4x+8, 5x+7], [5x+9, 6x+8], 6x+9$$

from G_3,

$$[10x+7, 11x+6], [11x+9, 12x+8], 12x+10$$

from G_1, and

$$[12x+11, 14x+11], 14x+13, [18x+17, 20x+16], 22x+19$$

from G_2. We see that for $x > 0$ the vertices of the three cycles are distinct and contained in $[0, 2(12x+9)] = [0, 24x+18]$.

Likewise we compute

$$\begin{aligned}
\bar{E}(G_1) &= [10x+9, 12x+8] \cup [8x+7, 10x+6] \cup \{3, 10x+7, 12x+10\},\\
\bar{E}(G_2) &= [4x+6, 8x+5] \cup \{2, 8x+6, 10x+8\},\\
\bar{E}(G_3) &= [2x+5, 4x+4] \cup [4, 2x+3] \cup \{1, 2x+4, 4x+5\}.
\end{aligned}$$

We can order these as the edge label 1 from G_3, 2 from G_2, 3 from G_1,

$$[4, 2x+3], 2x+4, [2x+5, 4x+4], 4x+5$$

from G_3,

$$[4x+6, 8x+5], 8x+6$$

from G_2,

$$[8x+7, 10x+6], 10x+7$$

from G_1, $10x+8$ from G_2, and

$$[10x+9, 12x+8], 12x+10$$

from G_1. Thus $E(G) = [1, 12x+8] \cup \{12x+10\}$. Since $2(12x+9)+1-(12x+10) = 12x+9$, we have a ρ-labeling.

Finally if $x = 0$ we take our cycles to be $(0, 3, 10, 0)$, $(1, 2, 6, 1)$, and $(5, 7, 13, 5)$. $\square$

Theorem 11 *Let the graph G consist of four vertex-disjoint cycles, each of the same odd length. Then G has a σ-labeling.*

Proof. First we consider the case when the cycles have length $4x+1$, where x is a positive integer. Our four cycles will be G_1, G_2, G_2, and G_4, defined as follows:

$$\begin{aligned}
G_1 &= P(0, 12x+6, 2x-1) + (2x-1, 4x-1, 16x+4, 0), \\
G_2 &= P(16x+5, 8x+6, 18x+4) + (18x+4, 20x+5, 28x+9, 16x+5), \\
G_3 &= P(4x, 4x+6, 6x-1) + (6x-1, 8x+1, 12x+5, 4x), \\
G_4 &= P(20x+6, 2x+6, 21x+5) + (21x+5, 23x+9, 21x+6) \\
&+ P(21x+6, 2, 22x+5) + (22x+5, 22x+6, 24x+11, 20x+6)
\end{aligned}$$

Now we compute

$$\begin{aligned}
V(G_1) &= [0, 2x-1] \cup [14x+5, 16x+3] \cup \{4x-1, 16x+4\}, \\
V(G_2) &= [16x+5, 18x+4] \cup [26x+10, 28x+8] \cup \{20x+5, 28x+9\}, \\
V(G_3) &= [4x, 6x-1] \cup [10x+5, 12x+3] \cup \{8x+1, 12x+5\}, \\
V(G_4) &= [20x+6, 21x+5] \cup [23x+11, 24x+9] \cup \{23x+9\} \\
&\cup [21x+6, 22x+5] \cup [22x+7, 23x+5] \cup \{22x+6, 24x+11\}.
\end{aligned}$$

We can order these sets as follows.

cycle	vertices
G_1	$[0, 2x-1]$
G_1	$4x-1$
G_3	$[4x, 6x-1]$
G_3	$8x+1$
G_3	$[10x+5, 12x+3]$
G_3	$12x+5$
G_1	$[14x+5, 16x+3]$
G_1	$16x+4$
G_2	$[16x+5, 18x+4]$
G_2	$20x+5$

cycle	vertices
G_4	$[20x+6, 21x+5]$
G_4	$[21x+6, 22x+5]$
G_4	$22x+6$
G_4	$[22x+7, 23x+5]$
G_4	$23x+9$
G_4	$[23x+11, 24x+9]$
G_4	$24x+11$
G_2	$[26x+10, 28x+8]$
G_2	$28x+9$

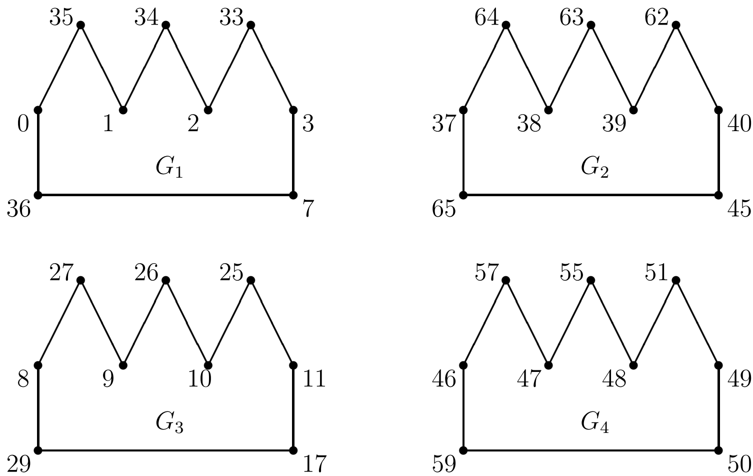

Figure 5: A σ-labeling of $4C_9$.

From this we can see that the vertices of the four cycles are distinct and contained in $[0, 2(16x+4)] = [0, 32x+8]$. (If $x = 1$ the sets $[22x+7, 23x+5]$ and $[23x+11, 24x+9]$ are empty, but this does not change the proof.)

Likewise we compute

$$\begin{aligned}
\bar{E}(G_1) &= [12x+6, 16x+3] \cup \{2x, 12x+5, 16x+4\},\\
\bar{E}(G_2) &= [8x+6, 12x+3] \cup \{2x+1, 8x+4, 12x+4\},\\
\bar{E}(G_3) &= [4x+6, 8x+3] \cup \{2x+2, 4x+4, 8x+5\},\\
\bar{E}(G_4) &= [2x+6, 4x+3] \cup \{2x+4, 2x+3\} \cup [2, 2x-1] \cup \{1, 2x+5, 4x+5\}.
\end{aligned}$$

We can order these sets as follows.

cycle	edge labels
G_4	1
G_4	$[2, 2x-1]$
G_1	$2x$
G_2	$2x+1$
G_3	$2x+2$
G_4	$2x+3$
G_4	$2x+4$
G_4	$2x+5$
G_4	$[2x+6, 4x+3]$
G_3	$4x+4$

cycle	edge labels
G_4	$4x+5$
G_3	$[4x+6, 8x+3]$
G_2	$8x+4$
G_3	$8x+5$
G_2	$[8x+6, 12x+3]$
G_2	$12x+4$
G_1	$12x+5$
G_1	$[12x+6, 16x+3]$
G_1	$16x+4$

We see that the edge labels are exactly the set $[1, 16x+4]$. (If $x = 1$, the sets $[2, 2x-1]$ and $[2x+6, 4x+3]$ are empty, but this does not change the proof.)

Now we consider the case when the cycles have length $4x+3$, where x is a positive integer. The case $x = 0$ will be considered separately. Our four cycles will be defined as follows:

$$
\begin{aligned}
G_1 &= P(0, 14x+12, x) + P(x, 12x+11, 2x) + (2x, 2x+1, 16x+12, 0),\\
G_2 &= P(16x+13, 10x+8, 17x+14) + P(17x+14, 8x+9, 18x+13)\\
&+ (18x+13, 18x+16, 28x+23, 16x+13),\\
G_3 &= P(2x+2, 6x+8, 3x+2) + P(3x+2, 4x+6, 4x+2)\\
&+ (4x+2, 4x+4, 10x+10, 2x+2),\\
G_4 &= (18x+17, 22x+22, 18x+18) + P(18x+18, 5, 20x+17)\\
&+ (20x+17, 20x+21, 24x+24, 18x+17).
\end{aligned}
$$

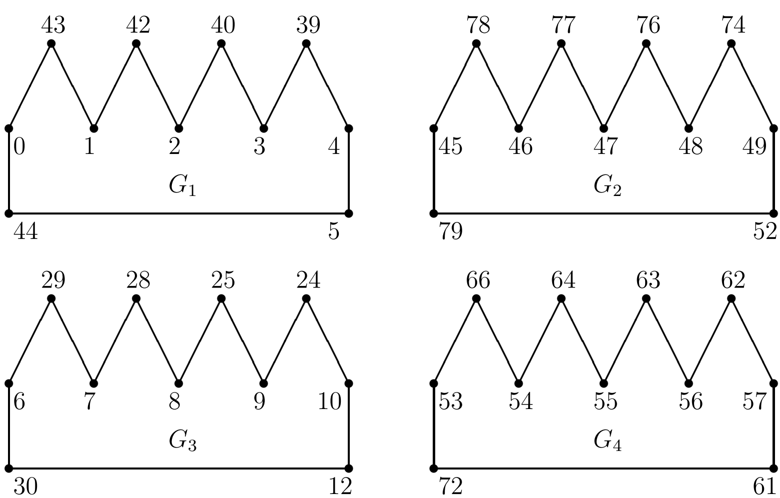

Figure 6: A ρ-labeling of $4C_{11}$.

Now we compute

$$
\begin{aligned}
V(G_1) &= [0, 2x] \cup [15x+12, 16x+11] \cup [14x+11, 15x+10] \cup \{2x+1, 16x+12\},\\
V(G_2) &= [16x+13, 18x+13] \cup [27x+22, 28x+22] \cup [26x+22, 27x+20]\\
&\quad \cup \{18x+16, 28x+23\},\\
V(G_3) &= [2x+2, 4x+2] \cup [9x+10, 10x+9] \cup [8x+8, 9x+7] \cup \{4x+4, 10x+10\},\\
V(G_4) &= \{18x+17, 22x+22\} \cup [18x+18, 20x+17] \cup [20x+22, 22x+20]\\
&\quad \cup \{20x+21, 24x+24\}.
\end{aligned}
$$

We can order these sets as follows.

cycle	vertices
G_1	$[0, 2x]$
G_1	$2x + 1$
G_3	$[2x + 2, 4x + 2]$
G_3	$4x + 4$
G_3	$[8x + 8, 9x + 7]$
G_3	$[9x + 10, 10x + 9]$
G_3	$10x + 10$
G_1	$[14x + 11, 15x + 10]$
G_1	$[15x + 12, 16x + 11]$
G_1	$16x + 12$
G_2	$[16x + 13, 18x + 13]$

cycle	vertices
G_2	$18x + 16$
G_4	$18x + 17$
G_4	$[18x + 18, 20x + 17]$
G_4	$20x + 21$
G_4	$[20x + 22, 22x + 20]$
G_4	$22x + 22$
G_4	$24x + 24$
G_2	$[26x + 22, 27x + 20]$
G_2	$[27x + 22, 28x + 22]$
G_2	$28x + 23$

From this we can see that the vertices of the four cycles are distinct and contained in $[0, 2(16x + 12)] = [0, 32x + 24]$ for $x > 0$. (If $x = 1$ the set $[26x + 22, 27x + 20]$ is empty, but this does not change the proof.)

Likewise we compute

$$
\begin{aligned}
\bar{E}(G_1) &= [14x + 12, 16x + 11] \cup [12x + 11, 14x + 10] \cup \{1, 14x + 11, 16x + 12\}, \\
\bar{E}(G_2) &= [10x + 8, 12x + 9] \cup [8x + 9, 10x + 6] \cup \{3, 10x + 7, 12x + 10\}, \\
\bar{E}(G_3) &= [6x + 8, 8x + 7] \cup [4x + 6, 6x + 5] \cup \{2, 6x + 6, 8x + 8\}, \\
\bar{E}(G_4) &= \{4x + 5, 4x + 4\} \cup [5, 4x + 2] \cup \{4, 4x + 3, 6x + 7\}.
\end{aligned}
$$

We can order these sets as follows.

cycle	edge labels
G_1	1
G_2	3
G_3	2
G_4	4
G_4	$[5, 4x + 2]$
G_4	$4x + 3$
G_4	$4x + 4$
G_4	$4x + 5$
G_3	$[4x + 6, 6x + 5]$
G_3	$6x + 6$
G_4	$6x + 7$

cycle	edge labels
G_3	$[6x + 8, 8x + 7]$
G_3	$8x + 8$
G_2	$[8x + 9, 10x + 6]$
G_2	$10x + 7$
G_2	$[10x + 8, 12x + 9]$
G_2	$12x + 10$
G_1	$[12x + 11, 14x + 10]$
G_1	$14x + 11$
G_1	$[14x + 12, 16x + 11]$
G_1	$16x + 12$

We see that for $x \geq 2$ the edge labels are exactly the set $[1, 16x + 12]$. (If $x = 1$ the set $[8x + 9, 10x + 6]$ is empty, but this does not change the proof.)

Finally, if $x = 0$ we take $G_1 = (0, 1, 12, 0)$, $G_2 = (13, 16, 23, 13)$, $G_3 = (2, 4, 10, 2)$, and $G_4 = (5, 9, 14, 5)$. $\square$

The results for labelings of rC_{4x+k}, $1 \leq r \leq 4$, $3 \leq k \leq 6$ and $x \geq 0$ are summarized in the table below.

	$k=3$	$k=4$	$k=5$	$k=6$
$r=1$	β	α	ρ	ρ^{++}
$r=2$	ρ	α	ρ	α
$r=3$	ρ	σ^{++} if $x=0$ α if $x>0$	σ if $x=0$ β if $x>0$	ρ^{++}
$r=4$	σ	α if $x=0$ σ^{++} if $x>0$	σ	σ^{++}

Table 1. Labelings of rC_{4x+k}, $1 \le r \le 4$, $3 \le k \le 6$ and $x \ge 0$.

We can offer the following corollary.

Corollary 12 *Let G be a 2-regular graph with n edges and at most 4 components. Then there exists a cyclic G-decompositions of K_{2n+1}.*

4 Concluding Remarks

The study of graph decompositions is a popular branch of modern combinatorial design theory (see [4] for an overview). In particular, the study of G-decompositions of K_{2n+1} (and of K_{2nx+1}) when G is a graph with n edges (and x is a positive integer) has attracted considerable attention. The study of graph labelings is also quite popular (see Gallian [9] for a dynamic survey). Theorem 1 provides a powerful link between the two areas. Much of the attention on labelings has been on graceful labelings (i.e., β-labelings). Unfortunately, the parity condition "disqualifies" large classes of graphs from admitting graceful labelings. This difficulty is compounded by the fact that certain classes of graphs with ρ-labelings meet the parity condition, yet fail to be graceful.

In conclusion, we note that our results here, along with results from [5] and [10], provide further evidence in support of the following conjecture which is presented in a forthcoming survey [7].

Conjecture 13 *Every 2-regular graph has a ρ-labeling.*

Evidence suggests that the above conjecture can be strengthened to predict a σ-labeling if the parity condition is satisfied.

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