AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **29** (2004), Pages 301–308

Bhaskar Rao designs and the groups of order 12

R.J.R. Abel *   D. Combe†

School of Mathematics
The University of New South Wales
NSW 2052, Australia
julian@maths.unsw.edu.au   diana@maths.unsw.edu.au

W.D. Palmer‡

School of Mathematics and Statistics
The University of Sydney
NSW 2006, Australia
billp@maths.usyd.edu.au

Abstract

We complete the solution of the existence problem for generalized Bhaskar Rao designs of block size 3 over groups of order 12. In particular we prove that if $\mathbb{G}$ is a group of order 12 which is cyclic or dicyclic, then a generalized Bhaskar Rao design, GBRD$(v, 3, \lambda = 12t; \mathbb{G})$ exists for all $v \geq 3$ when t is even and for all $v \geq 4$ when t is odd.

1 Introduction

There are five groups of order 12. We denote the *cyclic group* of order n by C_n. The abelian groups of order 12 are C_{12} and $C_2 \times C_3 \times C_2$. The non-abelian groups are the dihedral group D_6, the alternating group A_4 and the dicyclic group Q_6.

The problem of the existence of generalized Bhaskar Rao designs of block size 3 over a group $\mathbb{G}$ means determining for which values of the parameters v, b, r, λ there exists a GBRD$(v, b, r, 3, \lambda; \mathbb{G})$. A solution usually takes the form of a set of necessary and sufficient conditions for v and λ. We are interested in solving this problem for all groups of sufficiently small order and for families of groups. Much work has been done in the case when the group is abelian, and our particular interest is in the case where the group is non-abelian.

The existence of generalized Bhaskar Rao designs of block size 3 over the abelian group $C_2 \times C_2 \times C_3$ has been settled by Seberry [16]. For the non-abelian groups

* Supported in part under The University of New South Wales Research Support Program
† Supported in part under The University of New South Wales Research Support Program
‡ Part of this work was conducted while the author was visiting UNSW

D_6 and A_4 the existence has been settled by Palmer and Seberry [13] and Combe, Palmer, and Unger [3] respectively.

In this paper we consider the existence of generalized Bhaskar Rao designs of block size 3 over the remaining groups of order 12: the cyclic group, C_{12}, and the dicyclic group, Q_6. In particular, we show that, if $\mathbb{G}$ is one of C_{12} or Q_6, then a GBRD$(v, 3, \lambda = 12t; \mathbb{G})$ exists for $v \geq 3$ when t is even and for $v \geq 4$ when t is odd.

We consider the *cyclic group* C_{12} to be be generated by c with $c^{12} = 1$. The *dicyclic group*, Q_6, can be generated by a and b subject to the defining relations below. Note that the conditions $a^6 = 1$ and $a^3 = b^2$ also imply $b^4 = 1$.

$$a^6 = 1, a^3 = b^2, ab = ba^{-1}.$$

2 Bhaskar Rao designs

Definition 1. Let $\mathbb{G}$ be a finite group of order g, written multiplicatively, and let $0 \notin \mathbb{G}$ be a zero symbol. Let $v \geq k$. We define a generalized Bhaskar Rao design GBRD$(v, b, r, k, \lambda; \mathbb{G})$ to be a $v \times b$ matrix each entry of which is either 0 or an element of $\mathbb{G}$ such that:

1. each row has exactly r group element entries and exactly $b - r$ zeroes;
2. each column has exactly k group element entries and exactly $v - k$ zeroes;
3. for each pair of distinct rows $(x_1, x_2, \ldots, x_b)$ and $(y_1, y_2, \ldots, y_b)$ consider the columns in which both rows have non-zero entries. The list

$$x_i y_i^{-1} : i = 1, 2, \ldots, b, \quad x_i \neq 0, \quad y_i \neq 0,$$

contains each element of the group λ/g times.

We note immediately that $\lambda \equiv 0 \pmod{g}$ and $\lambda \leq r \leq b$.

If $v > k$, replacing the group element entries in a GBRD$(v, b, r, k, \lambda; \mathbb{G})$ by 1 and the other entries by 0, produces a $(0, 1)$-matrix which is an incidence matrix for a balanced incomplete block design, BIBD(v, b, r, k, λ). One method of constructing GBRDs over a group is to take the incidence matrix of a BIBD or a matrix of all 1s and to *sign* it over the group, that is to replace the the non-zero entries by elements of the group so that the above conditions hold. It is well-known, see, for example, Mathon and Rosa [10], that the five numbers v, b, r, k and λ for a balanced incomplete block design, are not independent: they satisfy the relations $bk = vr$ and $\lambda(v - 1) = r(k - 1)$. For a generalized Bhaskar Rao design in which $v = k$, there are no zero entries and $b = r = \lambda$, so the relations: $bk = vr$ and $\lambda(v - 1) = r(k - 1)$ still hold. Thus it is usual to denote the GBRD$(v, b, r, k, \lambda; \mathbb{G})$ by the shorter notation GBRD$(v, k, \lambda; \mathbb{G})$.

Note that if $v = k$, then a GBRD$(v, k, \lambda; \mathbb{G})$ is a $(g, k, \lambda/g)$ *difference matrix* over $\mathbb{G}$ and gives a *transversal design*, $TD_{\lambda/g}(k, g)$. More generally, from a GBRD$(v, k, \lambda; \mathbb{G})$ we can construct a *group divisible design* with v groups of size g and index λ/g. Small group divisible designs are used in experimental design and larger ones in recursive constructions of other designs such as balanced incomplete block designs and pairwise balanced designs. For further details about difference matrices, transversal designs, group divisible designs, and other types of designs the reader is referred to Colbourn and Dinitz [2].

Example 2. A GBRD$(3, 24, 24, 3, 24; C_{12})$, (a GBRD$(3, 3, 24; C_{12})$) is given below:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & c^4 & c^8 & c & c^5 & c^9 & 1 & c^4 & c^8 & c & c^5 & c^9 & c^2 & c^6 & c^{10} & c^3 & c^7 & c^{11} & c^2 & c^6 & c^{10} & c^3 & c^7 & c^{11} \\ c & c^9 & c^5 & 1 & c^8 & c^4 & c^3 & c^{11} & c^7 & c^2 & c^{10} & c^6 & 1 & c^8 & c^4 & c & c^9 & c^5 & c^2 & c^{10} & c^6 & c^3 & c^{11} & c^7 \end{bmatrix}$$

In this example the number of columns, $b = 24$. There are no zero entries, and each entry of the array is an element of C_{12}. For the second and third rows, the list: $c.1^{-1} = c$, $c^9.(c^4)^{-1} = c^5$, $\ldots$, $c^7.(c^{11})^{-1} = c^8$, contains each of the group elements $1, c, \ldots, c^{11}$ exactly twice.

Example 2 was constructed by applying Theorem 2.1 of [11] to a GBRD$(3, 3, 3; C_3)$ and a GBRD$(3, 3, 8; C_4)$ which appears in [4].

Some generalized Bhaskar Rao designs, for example a GBRD$(11, 4, 6; C_6)$ shown in Example 3, can be developed from a set of *GBRD initial blocks*. See, for example, Seberry [14], and also Combe, Palmer and Unger [3].

Example 3. [6, Theorem 7.1.2] A set of GBRD initial blocks (mod 11, C_6) for a GBRD$(11, 4, 6; C_6)$ is given below:

$$(0_1, 1_1, 2_c, 3_1), \quad (0_1, 1_{c^2}, 5_{c^4}, 7_{c^3}), \quad (0_1, 1_{c^3}, 4_{c^4}, 5_{c^2}),$$
$$(0_1, 2_{c^2}, 4_1, 7_{c^5}), \quad (0_1, 2_{c^3}, 5_1, 8_{c^2}).$$

3 Construction Theorems

Let v and λ be positive integers and let K be a set of positive integers. A *pairwise balanced design*, denoted by PBD$(v; K; \lambda)$, is an arrangement of the v elements of a set X into a collection of (not necessarily distinct) subsets (called *blocks*) of X, for which:

1. each pair of distinct elements of X appears together in exactly λ blocks;

2. if a block contains exactly k elements of X then k belongs to K.

A pairwise balanced design PBD$(v; \{k\}; \lambda)$, where $K = \{k\}$ consists of exactly one integer, is a BIBD(v, k, λ). It is well-known, see, for example, Street and Wallis [17], that a PBD$(v - 1; \{k, k - 1\}; \lambda)$ can be obtained from a BIBD(v, b, r, k, λ).

From Bennett et al. [1, p. 206] we have:

Lemma 4. *A PBD$(v; K; 1)$ exists when:*

1. $v \geq 3$ and $K = \{3, 4, 5, 6, 8\}$;

2. $v \geq 4$ *and* $K = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 23\}$.

We use extensively in this paper the following two corollaries to the construction theorem in Combe, Palmer and Unger [3] and Palmer [12]. Note that Corollary 5, based upon pairwise balanced designs, was first proved in de Launey and Seberry [5, 6], originally for generalized Bhaskar Rao designs over the group Z_2 and then for generalized Bhaskar Rao designs over any finite group. Corollary 6 was first proved in Palmer [11].

Corollary 5. *Given a PBD*$(v; K; \lambda)$*, and for each* $k \in K$*, a GBRD*$(k, j, \mu; \mathbb{G})$*, one can construct a GBRD*$(v, j, \lambda\mu; \mathbb{G})$*.*

Corollary 6. *Suppose that* $\mathbb{N}$ *is a normal subgroup of a finite group* $\mathbb{G}$*. Then, given a GBRD*$(v, k, \lambda; \mathbb{G}/\mathbb{N})$ *and a GBRD*$(k, j, \mu; \mathbb{N})$ *one can construct a GBRD*$(v, j, \lambda\mu; \mathbb{G})$*.*

4 Existence result

In the following section let $\mathbb{G}$ be either the dicyclic group Q_6 or the cyclic group C_{12}. Then the Sylow 2-subgroup of $\mathbb{G}$ is cyclic. Therefore, by Drake's Theorem [7], whenever t is odd, there is no $(12, 3, t)$ difference matrix over $\mathbb{G}$. Since the existence of a GBRD$(3, 3, \lambda = 12t; \mathbb{G})$ is equivalent to the existence of a $(12, 3, t)$ difference matrix over $\mathbb{G}$ we have the following:

Lemma 7. *For* $\mathbb{G} = C_{12}$ *or* Q_6*, there is no GBRD*$(3, 3, \lambda = 12t; \mathbb{G})$ *if* t *is odd.*

For a GBRD$(v, 3, \lambda; \mathbb{G})$ to exist there must exist a BIBD$(v, 3, \lambda)$.
From Hanani [9], a BIBD$(v, 3, \lambda)$ exists if and only if:

$$\begin{aligned} v &\geq 3 \\ \lambda(v-1) &\equiv 0 \pmod 2 \\ \lambda v(v-1) &\equiv 0 \pmod 6. \end{aligned}$$

These conditions together with the condition $\lambda \equiv 0 \pmod{|\mathbb{G}|}$ are described as "BIBD conditions" as they are determined by the underlying block design and the order of $\mathbb{G}$. For $\lambda = 12t$ the "BIBD conditions" impose no constraints.

The subgroup C_6 is normal in $\mathbb{G}$ and the factor group $\mathbb{G}/C_6$ is cyclic of order 2. Thus, by Gibbons and Mathon [8, Theorem 2], if a GBRD$(v, 3, \lambda; \mathbb{G})$ exists then a GBRD$(v, 3, \lambda; C_2)$ exists.

By [15] a GBRD$(v, 3, \lambda; C_2)$ exists if and only if

$$\begin{aligned} \lambda &\equiv 0 \pmod 2 \\ v &\geq 3 \\ \lambda v(v-1) &\equiv 0 \pmod{24}. \end{aligned}$$

These conditions are described as "C_2 conditions". For $\lambda = 12t$, the "C_2 conditions" impose no constraints.

Combining the "C_2 conditions", the "cyclic Sylow 2-subgroup condition" and "BIBD conditions" we have:

Lemma 8. *Let $\mathbb{G} = C_{12}$ or Q_6. If a GBRD$(v, 3, \lambda = 12t; \mathbb{G})$ exists then either: $v \geq 4$; or $v = 3$ and t is even.*

For $u \in \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 23\}$, we now construct the designs GBRD$(u, 3, 12; \mathbb{G})$. Wherever there exist suitable GBRDs over normal subgroups of $\mathbb{G}$, we apply Corollary 6 to construct a GBRD$(u, 3, 12; \mathbb{G})$. For example, applying Corollary 6, with a GBRD$(11, 4, 6; C_6)$, whose initial blocks are shown in Example 3, and a GBRD$(4, 3, 2; C_2)$ (Seberry [15]), we can construct a GBRD$(11, 3, 12; \mathbb{G})$. When, for a given u, suitable GBRDs over normal subgroups of $\mathbb{G}$ cannot be found, we find a GBRD$(u, 3, 12; \mathbb{G})$ directly by hand-calculation or by computer search. The GBRDs whose initial blocks are exhibited in Examples 13, 14 and 15, were found by hand-calculation. Examples 9, 10, 11 and 12, were found via computer searches using Fortran.

Whence by Lemma 4 and Corollary 5 we construct the designs GBRD$(v, 3, 12; \mathbb{G})$ for all $v \geq 4$.

Example 9. A set of initial blocks (mod 5, Q_6) for a GBRD$(5, 3, 12; Q_6)$ is given below:

$$\begin{array}{cccc}
(0_1, 1_1, 2_{ba^3}), & (0_b, 1_a, 2_{a^2}), & (0_1, 1_{a^4}, 2_{ba^2}), & (0_1, 1_{ba^5}, 2_b), \\
(0_1, 2_1, 4_{ba^4}), & (0_b, 2_{ba}, 4_{ba^3}), & (0_1, 2_{ba^5}, 4_{ba^3}), & (0_1, 2_a, 4_{a^4}).
\end{array}$$

Example 10. A set of initial blocks (mod 7, Q_6) for a GBRD$(8, 3, 12; Q_6)$ is given below:

$$\begin{array}{cccc}
(0_1, 1_1, 2_{ba^2}), & (0_1, 1_{a^2}, 2_{a^5}), & (0_1, 2_1, 4_{ba^3}), & (0_1, 2_{a^2}, 4_1), \\
(0_b, 4_a, 1_1), & (0_1, 4_{a^3}, 1_{ba^4}), & (0_b, 1_{a^2}, 5_1), & (0_1, 1_{a^4}, 5_{ba^2}), \\
(0_1, 1_a, 3_{ba^2}), & (0_1, 1_{ba^5}, 3_{a^4}), & (0_1, 1_b, \infty_1), & (0_1, 1_{a^5}, \infty_b), \\
(0_1, 2_a, \infty_{a^2}), & (0_1, 2_{a^3}, \infty_{ba^2}), & (0_1, 4_a, \infty_{a^4}), & (0_1, 4_b, \infty_{a^5}).
\end{array}$$

Example 11. A set of initial blocks (mod 11, Q_6) for a GBRD$(11, 3, 12; Q_6)$ is given below:

$$\begin{array}{cccc}
(0_1, 1_b, 4_{ba}), & (0_1, 1_1, 2_{a^3}), & (0_1, 3_1, 1_{ba^5}), & (0_1, 3_{a^2}, 6_{ba}), \\
(0_{ba^2}, 9_b, 3_a), & (0_b, 9_1, 7_{a^4}), & (0_{ba^2}, 5_{a^2}, 9_{ba^2}), & (0_b, 5_{a^3}, 10_{a^4}), \\
(0_1, 4_{a^4}, 5_{a^5}), & (0_1, 4_1, 8_{ba^3}), & (0_1, 1_{ba^3}, 2_{ba}), & (0_1, 1_{a^5}, 4_{a^3}), \\
(0_1, 3_{a^3}, 6_{a^4}), & (0_1, 3_{ba^2}, 1_{ba}), & (0_1, 9_{ba^2}, 7_{a^5}), & (0_1, 9_{ba}, 3_{ba}), \\
(0_1, 5_{ba^5}, 10_{ba}), & (0_1, 5_{ba}, 9_{a^5}), & (0_1, 4_{a^2}, 8_b), & (0_1, 4_{a^5}, 5_{a^3}).
\end{array}$$

Example 12. A set of initial blocks (mod 13, Q_6) for a GBRD$(14, 3, 12; Q_6)$ is given below:

$$\begin{array}{cccc}
(1_1, 3_{ba^5}, 9_{ba^4}), & (1_{ba^4}, 3_{ba^5}, 9_b), & (2_{a^2}, 6_{a^4}, 5_{ba^5}), & (2_1, 6_{ba}, 5_{ba^2}), \\
(0_1, 1_{ba^3}, 5_1), & (0_1, 1_{a^2}, 2_{ba^3}), & (0_{a^3}, 3_1, 2_1), & (0_1, 3_{ba}, 6_{a^4}), \\
(0_{ba^4}, 9_1, 6_{a^5}), & (0_b, 9_{a^4}, 5_{a^3}), & (0_{ba^2}, 5_{a^2}, 6_{ba^2}), & (0_1, 5_{ba^2}, 10_1), \\
(0_1, 2_a, 5_{a^3}), & (0_1, 2_{a^2}, 4_1), & (0_1, 6_{ba}, 2_{ba^4}), & (0_1, 6_{ba^4}, 12_a), \\
(0_1, 3_{a^5}, 6_{a^3}), & (0_1, 1_{ba^2}, \infty_1), & (0_{ba^3}, 9_{ba}, 5_{a^2}), & (0_1, 3_{ba^4}, \infty_{a^4}), \\
(0_1, 1_{ba^5}, 2_{ba}), & (0_1, 9_a, \infty_{a^3}), & (0_1, 5_a, \infty_{ba^2}), & (0_1, 2_1, 6_{ba^3}), \\
(0_1, 2_{ba^2}, \infty_{a^5}), & (0_1, 6_{a^2}, 5_{a^5}), & (0_1, 6_{ba^5}, \infty_{ba^4}), & (0_1, 5_{a^4}, 2_b).
\end{array}$$

Example 13. A set of initial blocks (mod 23, Q_6) for the GBRD$(23, 3, 12; Q_6)$ is given below, where $t = 1, \ldots, 5$:

$$\begin{array}{lll}
(0_1, (2t+1)_1, (22-2t)_b), & (0_1, 1_{ba^3}, 2_{ba^3}), & (0_1, 1_{ba^5}, 2_{ba}), \\
(0_1, (2t+1)_{a^2}, (22-2t)_{ba^4}), & (0_1, 1_{ba}, 2_{ba^5}), & (0_1, 5_{a^3}, 7_{a^3}), \\
(0_1, (2t+1)_{a^4}, (22-2t)_{ba^2}), & (0_1, 5_{a^5}, 7_a), & (0_1, 5_a, 7_{a^5}), \\
(0_1, (2t)_{a^3}, (23-2t)_{ba^3}), & (0_1, 1_{a^3}, 11_{a^5}), & (0_1, 1_{a^5}, 11_{a^3}), \\
(0_1, (2t)_{a^5}, (23-2t)_{ba}), & (0_1, 3_{a^3}, 9_a), & (0_1, 3_a, 9_{a^3}), \\
(0_1, (2t)_a, (23-2t)_{ba^5}), & (0_1, 1_a, 9_{a^5}), & (0_1, 3_{a^5}, 11_a), \\
(0_1, 4_{a^4}, 8_1), & (0_1, 4_1, 10_1). &
\end{array}$$

Example 14. A set of initial blocks (mod 7, C_{12}) for a GBRD$(7, 3, 12; C_{12})$ is given below:

$$\begin{array}{ccc}
(0_1, 1_1, 6_c), & (0_1, 1_{c^4}, 6_{c^9}), & (0_1, 1_{c^8}, 6_{c^5}), \\
(0_1, 2_{c^2}, 5_{c^3}), & (0_1, 2_{c^6}, 5_{c^{11}}), & (0_1, 2_{c^{10}}, 5_{c^7}), \\
(0_1, 3_1, 4_c), & (0_1, 3_{c^4}, 4_{c^9}), & (0_1, 3_{c^8}, 4_{c^5}), \\
(0_1, 1_{c^2}, 3_{c^2}), & (0_1, 1_{c^6}, 3_{c^{10}}), & (0_1, 1_{c^{10}}, 3_{c^6}).
\end{array}$$

Example 15. A set of initial blocks (mod 23, C_{12}) for a GBRD$(23, 3, 12; C_{12})$ is given below, where $t = 1, \ldots, 5$:

$$\begin{array}{lll}
(0_1, (2t+1)_1, (22-2t)_c), & (0_1, 1_{c^3}, 2_{c^3}), & (0_1, 1_{c^7}, 2_{c^{11}}), \\
(0_1, (2t+1)_{c^4}, (22-2t)_{c^9}), & (0_1, 1_{c^{11}}, 2_{c^7}), & (0_1, 5_{c^2}, 7_{c^2}), \\
(0_1, (2t+1)_{c^8}, (22-2t)_{c^5}), & (0_1, 5_{c^6}, 7_{c^{10}}), & (0_1, 5_{c^{10}}, 7_{c^6}), \\
(0_1, (2t)_{c^2}, (23-2t)_{c^3}), & (0_1, 1_{c^2}, 11_{c^6}), & (0_1, 1_{c^6}, 11_{c^2}), \\
(0_1, (2t)_{c^6}, (23-2t)_{c^{11}}), & (0_1, 3_{c^2}, 9_{c^{10}}), & (0_1, 3_{c^{10}}, 9_{c^2}), \\
(0_1, (2t)_{c^{10}}, (23-2t)_{c^7}), & (0_1, 1_{c^{10}}, 9_{c^6}), & (0_1, 3_{c^6}, 11_{c^{10}}), \\
(0_1, 4_{c^8}, 8_1), & (0_1, 4_1, 10_1). &
\end{array}$$

Lemma 16. *Let $\mathbb{G} = C_{12}$ or Q_6. If $u \in \{4, 6, 7, 9, 10, 12, 15, 18, 19\}$ then a GBRD$(u, 3, 12; \mathbb{G})$ exists.*

Proof. A GBRD$(3, 3, 3; C_3)$ exists. For $u \in \{4, 6, 7, 9, 10, 12, 15, 18, 19\}$ there exists a GBRD$(u, 3, 4; C_4)$ (de Launey and Seberry [4]). In $\mathbb{G}$ there is a normal subgroup of order 3. Thus, for all $u \in \{4, 6, 7, 9, 10, 12, 15, 18, 19\}$, using a GBRD$(v, 3, 4; C_4)$ and a GBRD$(3, 3, 3; C_3)$, we can construct a GBRD$(v, 3, 12; \mathbb{G})$ by applying Corollary 6. □

Lemma 17. *If $u \in \{5, 8, 11, 14, 23\}$ then a GBRD$(u, 3, 12; C_{12})$ exists.*

Proof. For $u \in \{5, 8\}$, a GBRD$(u, 4, 3; C_3)$ exists from de Launey and Seberry [6, Lemma 5.1.4 and Theorem 5.1.5]. A GBRD$(4, 3, 4; C_4)$ exists by de Launey et al. [4]. The cyclic group C_{12} contains a normal subgroup isomorphic to C_4. Thus, for $u \in \{5, 8\}$, we can apply Corollary 6 to construct a GBRD$(u, 3, 12; C_{12})$.

A GBRD$(11, 4, 6; C_6)$ exists and a GBRD$(14, 4, 6; C_6)$ exists (de Launey and Seberry [6, Theorem 7.1.2]). In particular, a GBRD$(11, 4, 6; C_6)$ is exhibited in Example 3. A GBRD$(4, 3, 2; C_2)$ exists (Seberry [15]). The subgroup $C_6 \lhd C_{12}$ so on applying Corollary 6 we construct GBRD$(11, 3, 12; C_{12})$ and GBRD$(14, 3, 12; C_{12})$.

A GBRD$(23, 3, 12; C_{12})$ is exhibited in Example 15. □

Lemma 18. *If* $u \in \{5, 8, 11, 14, 23\}$ *then a GBRD*$(u, 3, 12; Q_6)$ *exists.*

Proof. The required designs are shown in Examples 9, 10, 11, 12 and 13. □

Note that the general subgroup constructions, via Corollary 6, used extensively in proving Lemma 17 are not available in proving Lemma 18 as, for example, Q_6 does not have a normal C_4 subgroup. So, in this case, we are forced to treat each $u \in \{5, 8, 11, 14, 23\}$ on an ad hoc basis.

Theorem 19. *Let* $\mathbb{G} = C_{12}$ *or* Q_6. *When* $v \geq 4$ *a GBRD*$(v, 3, 12; \mathbb{G})$ *exists.*

Proof. For all $v \geq 4$, we can construct a GBRD$(v, 3, 12; C_{12})$ by applying Lemmas 4, 16 and 17, together with Corollary 5. Likewise we can construct a GBRD$(v, 3, 12; Q_6)$. □

Example 20. A GBRD$(3, 3, 24; Q_6)$ is given below:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & a^2 & a^3 & a^4 & a^5 & b & ba^5 & ba^4 & ba^3 & ba^2 & ba & 1 & a & a^2 & a^3 & a^4 & a^5 & b & ba^5 & ba^4 & ba^3 & ba^2 & ba \\ ba^3 & ba^2 & ba & b & ba^5 & ba^4 & ba^3 & ba^4 & ba^5 & b & ba & ba^2 & 1 & a^5 & a^4 & a^3 & a^2 & a & a & a^2 & a^3 & a^4 & a^5 & 1 \end{bmatrix}$$

We are now able to prove our main theorem:

Theorem 21. *Let* $\mathbb{G} = C_{12}$ *or* Q_6. *A GBRD*$(v, 3, 12t; \mathbb{G})$ *exists for all* $v \geq 3$ *when* t *is even; and for all* $v \geq 4$ *when* t *is odd.*

Proof. For $v \geq 4$, a GBRD$(v, 3, 12; \mathbb{G})$ exists by Theorem 19; if $t > 1$, take t copies of this design. For $v = 3$, a GBRD$(3, 3, 24; C_{12})$ and a GBRD$(3, 3, 24; Q_6)$ are exhibited in Examples 2 and 20. If $t > 2$ is even, a GBRD$(3, 3, 12t; \mathbb{G})$ can be obtained by taking $t/2$ copies of one of these two designs. For $v = 3$ and t odd, no GBRD$(v, 3, 12t; \mathbb{G})$ exists by the 'cyclic Sylow 2-subgroup condition' in Lemma 7. □

References

[1] F.E. Bennett, H-D.O.F. Gronau, A.C.H. Ling and R.C. Mullin, *PBD-closure*, in: The CRC Handbook of Combinatorial Designs (eds C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton, FL, (1996), 203–213.

[2] C.J. Colbourn and J.H. Dinitz, Editors, *The CRC Handbook of Combinatorial Designs*, CRC Press Series on Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 1996.

[3] D. Combe, W.D. Palmer and W.R. Unger, *Bhaskar Rao designs and the alternating group* A_4, Australas. J. Combin. **24** (2001), 275–283.

[4] W. de Launey, D.G. Sarvate and J. Seberry, *Generalized Bhaskar Rao designs of block size* 3 *over* Z_4, Ars Combin. **19A** (1985), 273–285.

[5] W. de Launey and J. Seberry, *On Bhaskar Rao designs of block size four*, Combinatorics and Applications (Calcutta, 1982), Indian Statist. Inst., Calcutta (1984), 311–316.

[6] W. de Launey and J. Seberry, *On generalized Bhaskar Rao designs of block size four*, Congr. Numer. **41** (1984), 229–294.

[7] D.A. Drake, *Partial λ-geometries and generalized Hadamard matrices*, Canad. J. Math. **31** (1979), 139–149.

[8] P.B. Gibbons and R. Mathon, *Construction methods for Bhaskar Rao and related designs*, J. Austral. Math. Soc. Ser. A **42** (1987), 5–30; *ibid.* **43**, 420.

[9] H. Hanani, *Balanced incomplete block designs and related designs*, Discrete Math. **11** (1975), 255–369.

[10] R. Mathon and A. Rosa, 2-(v, k, λ) *Designs of small order*, in: The CRC Handbook of Combinatorial Designs (eds C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton, FL, (1996), 3–41.

[11] W.D. Palmer, *A composition theorem for generalized Bhaskar Rao designs*, Australas. J. Combin. **6** (1992), 221–228.

[12] W.D. Palmer, *Partial generalized Bhaskar Rao designs*, Ph.D. Thesis, University of Sydney, 1993.

[13] W.D. Palmer and J. Seberry, *Bhaskar Rao designs over small groups*, Ars Combin. **26A** (1988), 125–148.

[14] J. Seberry, *Some families of partially balanced incomplete designs*, in: Combinatorial Mathematics IX, Lecture Notes in Mathematics **952**, Springer-Verlag, Berlin (1982), 378–386.

[15] J. Seberry, *Regular group divisible designs and Bhaskar Rao designs with block size three*, J. Statist. Plann. Inference **10** (1984), 69–82.

[16] J. Seberry, *Generalized Bhaskar Rao designs of block size three*, J. Statist. Plann. Inference **11** (1985), 373–379.

[17] A.P. Street and W.D. Wallis, *Combinatorial Theory: An Introduction*, Charles Babbage Research Centre, Winnipeg, 1977.

(Received 20 Dec 2002)