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Magic labellings of graphs over finite abelian groups

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Abstract

A *total labelling* of a graph with v vertices and e edges is a one-to-one map taking the vertices and edges onto the set $\{1, 2, 3, \ldots, v + e\}$. A labelling can be used to define a weight for each vertex and edge. For a vertex the weight is the sum of the label of the vertex and the labels of the incident edges. For an edge $\{x, y\}$ the weight is the sum of the label of the edge and the labels of the end vertices x and y. A labelling is *vertex-magic* if all the vertices have the same weight. A labelling is *edge-magic* if all the edges have the same weight. A labelling is *totally-magic* if it is both vertex-magic and edge-magic. In this paper we generalize these concepts to $\mathbb{A}$-labellings of a graph, that is labellings with the elements of an abelian group $\mathbb{A}$ of order $v + e$. We consider in detail $\mathbb{A}$-labellings of star graphs.

1 Introduction

There are many types of graph labellings, and a detailed description and survey of many of them can be found in the dynamic survey of graph labellings by Gallian [7]. A *labelling* of a graph is an assignment of labels to the vertices (a *vertex labelling*), or an assignment of the labels to the edges (an *edge labelling*), or an assignment of the labels to the combined set of vertices and edges of the graph (a *total labelling*). A *magic labelling* of a graph with v vertices and e edges is a total labelling of the

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graph by the integers $1, 2, 3, \ldots, v + e$ with constant edge or vertex weights. It can be viewed as a generalisation of the concept of a *magic square*. Magic labellings have recently been considered by various authors, for example vertex-magic labellings by MacDougall et al [10], edge-magic labellings by Baskaro et al [1] and labellings which are both vertex-magic and edge-magic by Exoo et al [5] and by Wood [14]. There are many more references in the book by Wallis [13].

Clearly a labelling over consecutive integers can be viewed as a labelling over a cyclic group. There are many natural generalisations of this idea. In this paper we introduce the idea of $\mathbb{A}$-magic labellings, where the labels are the elements of an arbitrary abelian group, $\mathbb{A}$, of order $v + e$.

We illustrate the idea of magic labellings over groups by considering star graphs. Labellings (not group labellings) of star graphs were considered by Exoo et al [5], who show that, although there are various edge-magic labellings, there are no vertex-magic labellings of any star with more than 2 rays. The situation is different with labellings of stars over groups. In our main result we prove that for all choices of abelian group $\mathbb{A}$ of appropriate order, all stars have various edge-magic $\mathbb{A}$-labellings, and further any star with more than than 4 rays has a plentiful supply of vertex-magic $\mathbb{A}$-labellings.

Previously some authors have considered labelling graph elements by groups, but none were considering total labellings or labellings with "magic" properties. Fukuchi [6] and Egawa [4] considered labelling the vertices of a graph by an elementary abelian group such that the connected components of the graph (rather than vertices, or edges) have constant weight. Gimbel [8] and Edelman and Saks [3] considered labellings of vertices and labellings of edges over abelian groups, and the relationship between vertex and edge labellings.

2 $\mathbb{A}$-labellings of graphs

2.1 Definitions

In this paper, a graph G is a finite graph with no loops and no multiple edges. The graph need not be connected. The vertex set is $V = V(G)$, with $v = |V| > 0$. The edge set $E = E(G)$ is a set of unordered pairs of vertices, with $e = |E| \geq 0$. The set $V \cup E$ is the set of graph elements. If x and y are vertices, then $x \sim y$ means there is an edge between x and y, and the edge is denoted by xy (or yx). The complete graph with v vertices, and $e = \binom{v}{2}$, is denoted by K_v. The circuit with v vertices (and $e = v$) is denoted C_v. The path with v vertices and $v - 1$ edges is denoted P_v. The star with e edges (and $v = e + 1$) is denoted T_e. There are many general references for graph theory concepts, for example [2] or [9]. We take $v + e$ as a measure of the size of G.

In this paper the group $\mathbb{A}$ is always a finite abelian group. (It would be nice to generalise labellings to non-abelian groups as well, but it is not clear to the authors how to do this.) Note that in the original definition of a labelling over the integers $\{1, 2, \ldots\}$, the weights were defined by addition, but we find it convenient to consider our groups to be multiplicative. The cyclic group of order n is denoted $\mathbb{C}_n$, and in any example where it will not cause confusion, we will consider $\mathbb{C}_n$ to be generated by a.

Let G be a graph, and $\mathbb{A}$ a group with order $|\mathbb{A}| = v + e$. Then a *total* $\mathbb{A}$*-labelling* of G, or a *total labelling of* G *over* $\mathbb{A}$, is a bijection from $V \cup E$ to $\mathbb{A}$. Our $\mathbb{A}$-labellings will always be total, that is they will always be labellings of the union $V \cup E$ and not just vertex labellings or edge labellings, so, without risk of confusion, we will refer to them as $\mathbb{A}$*-labellings* or *labellings of* G *over* $\mathbb{A}$. Clearly the total labellings considered in [1], [5] and [10] all yield labellings over the appropriate cyclic group. See Example 7 for an illustration of this when we consider labellings of the path P_3 over $\mathbb{C}_5$.

Let λ be a labelling of G over a group $\mathbb{A}$. We define the λ*-weight*, $\omega = \omega_\lambda$, of the graph elements as follows:
(i) for $x \in V$, the weight is the product of the label of x and the labels of the edges incident with x, that is

$$\omega(x) = \lambda(x) \times \prod_{y \in V:\ x \sim y} \lambda(xy);$$

(ii) for $xy \in E$, the weight is the product of the label of xy and the labels of x and y, that is

$$\omega(xy) = \lambda(x) \times \lambda(xy) \times \lambda(y).$$

The labelling λ is a *vertex-magic* $\mathbb{A}$*-labelling* of G if there is an element $h = h(\lambda)$ of $\mathbb{A}$ such that for every $x \in V$, $\omega(x) = h$. The element h is the *vertex constant.*

The labelling λ is an *edge-magic* $\mathbb{A}$*-labelling* of G if there is an element $k = k(\lambda)$ of $\mathbb{A}$ such that for every $xy \in E$, $\omega(xy) = k$. The element k is the *edge constant.*

The labelling λ is a *totally-magic* $\mathbb{A}$*-labelling* of G if it is both a vertex-magic $\mathbb{A}$-labelling and an edge-magic $\mathbb{A}$-labelling.

A graph G is said to be *vertex* $\mathbb{A}$*-magic* if there exists a vertex-magic labelling over $\mathbb{A}$. Similarly G is *edge* $\mathbb{A}$*-magic* if there exists an edge-magic labelling over $\mathbb{A}$; and G is *totally* $\mathbb{A}$*-magic* if there exists a totally magic labelling over $\mathbb{A}$.

2.2 Examples

Example 1. *The case $v + e = 1$. The graph K_1, with $v = 1$, $e = 0$, is (trivially) vertex-magic, edge-magic, and totally-magic over the trivial group.*

Example 2. *The case $v + e = 2$. The only graph in this category has two vertices and no edges. The graph is trivially edge-magic and not vertex magic over $\mathbb{C}_2$.*

Example 3. *The case $v + e = 3$. The only graph here with $e \neq 0$ is P_2. The only group is $\mathbb{C}_3$. Any labelling over $\mathbb{C}_3$ is trivially edge-magic. There are no vertex-magic $\mathbb{C}_3$-labellings of P_2. No disconnected graph with P_2 as a connected component has a vertex magic labelling over any group.*

Example 4. *The case $v + e = 4$. The only graph here with $e \neq 0$ is $P_2 \cup K_1$, and has $v = 3$ and $e = 1$.*

This does not have any vertex-magic labelling. There are two groups of order 4, these are $\mathbb{C}_4$ and $\mathbb{C}_2 \times \mathbb{C}_2$. (This is the smallest order for which there is a non-cyclic abelian group, however we do not here get any interesting labellings.) Since $e = 1$, the graph is trivially edge-magic for any labelling over either group.

2.3 Equivalence of labellings

A graph automorphism of a graph G can be viewed as a bijection $f : V \cup E \longrightarrow V \cup E$ which preserves the graph structure.

Two $\mathbb{A}$-labellings λ_1, λ_2 of G are *graph equivalent*, denoted $\lambda_1 \sim_G \lambda_2$ if there exists a graph automorphism f such that $\forall z \in V \cup E$, $\lambda_1(z) = \lambda_2(f(z))$. The labelling λ_1 is the *composite of* λ_2 *and* f, and is denoted $\lambda_1 = \lambda_2 \circ f$. Clearly graph equivalence is an equivalence relation on labellings.

Two $\mathbb{A}$-labellings λ_1, λ_2 of G are *group equivalent*, denoted $\lambda_1 \sim_{\mathbb{A}} \lambda_2$, if there exists a group automorphism ρ such that $\forall z \in V \cup E$, $\lambda_1(z) = (\lambda_2(z))^\rho$. The labelling λ_1 is the *product of* λ_2 *and* ρ, and denoted $\lambda_1 = (\lambda_2)^\rho$. Clearly group equivalence is an equivalence relation on labellings.

If λ is a vertex-magic $\mathbb{A}$-labelling of G then every labelling which is graph equivalent or group equivalent to λ, is also vertex-magic. Similarly, if λ is an edge-magic $\mathbb{A}$-labelling of G then every labelling which is graph equivalent or group equivalent to λ, is also edge-magic.

The actions of composition and product are independent right and left actions on the set of $\mathbb{A}$-labellings of G. That is:

Lemma 5. *If λ_1 and λ_2 are $\mathbb{A}$-labellings of G, and there exists an $\mathbb{A}$-labelling λ such that $\lambda_1 \sim_G \lambda$ and $\lambda \sim_{\mathbb{A}} \lambda_2$, then there exists an $\mathbb{A}$-labelling μ such that $\lambda_1 \sim_{\mathbb{A}} \mu$ and $\mu \sim_G \lambda_2$.*

Proof. Since $\lambda_1 \sim_G \lambda$, there exists a graph automorphism f such that $\lambda_1 = \lambda \circ f$. Since $\lambda \sim_{\mathbb{A}} \lambda_2$, there exists a group automorphism ρ such that $\lambda = \lambda_2^\rho$. Therefore, for all $z \in V \cup E$, $\lambda_1(z) = \lambda(f(z)) = (\lambda_2(f(z)))^\rho = (\lambda_2 \circ f(z))^\rho$. Hence, taking $\mu = \lambda_2 \circ f$, $\lambda_1 \sim_{\mathbb{A}} \mu$ and $\mu \sim_G \lambda_2$. □

Two $\mathbb{A}$-labellings λ_1, λ_2 of G are *equivalent*, denoted $\lambda_1 \sim \lambda_2$ if there exists an $\mathbb{A}$-labelling λ such that $\lambda_1 \sim_G \lambda$ and $\lambda \sim_{\mathbb{A}} \lambda_2$. By the previous lemma, this is an equivalence relation.

Two $\mathbb{A}$-labellings λ_1, λ_2 of G are *translation equivalent*, denoted $\lambda_1 \sim_\tau \lambda_2$ if there exists $a \in \mathbb{A}$, such that $\forall z \in V \cup E$, $\lambda_1(z) = a\lambda_2(z)$. The labelling λ_1 is the *translation of* λ_2 *by* a, and written $\lambda_1 = a\lambda_2$. Clearly translation equivalence is an equivalence relation on labellings.

The following lemma is immediate:

Lemma 6. *If λ is edge-magic with edge-constant k, then any translation of λ is also edge-magic. Furthermore, if $a \in \mathbb{A}$, then $a\lambda$ has edge-constant a^3k.*

If λ is vertex-magic, then translations of λ are not necessarily vertex-magic unless the graph itself is regular.

A similar argument to the above shows that we can combine all three concepts of equivalence to get an equivalence relation where two $\mathbb{A}$-labellings λ_1, λ_2 of G are equivalent if there exist labellings λ and μ such that $\lambda_1 \sim_G \lambda$, $\lambda \sim_{\mathbb{A}} \mu$ and $\mu \sim_\tau \lambda_2$.

2.4 Further Examples

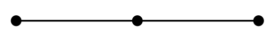
Example 7. *The case $v + e = 5$.*

The only group is $\mathbb{C}_5$, with elements $\{1, a, a^2, a^3, a^4\}$ and $a^5 = 1$. There are two graphs with $e \neq 0$:

1. The union of P_2 and two isolated vertices.

This is trivially edge-magic and not vertex-magic.

2. $P_3 = T_2$.



In [5] it is shown that P_3 has two totally-magic labellings. Viewed as labellings over $\mathbb{C}_5$ these are group equivalent labels. We explain this in some detail:

A labelling in [5] is a labelling from the set $\{1, 2, 3, 4, 5, 6\}$. Writing the labels in the sequence vertex-edge-vertex-edge-vertex the two labellings given are:

labels 4, 2, 3, 1, 5, *vertex-magic constant* 6, *edge-magic constant* 9,
labels 3, 4, 1, 2, 5, *vertex-magic constant* 7, *edge-magic constant* 8.

Each of these labellings could be written in the reverse order to give a magic-labelling with the same magic numbers (magic constants). By its construction the labelling is graph equivalent to its "reverse" labelling. Dual (or inverse) labellings are obtained by replacing any label, i, by $6 - i$. The dual labellings are edge-magic, with magic numbers 9, and 10 respectively, but are not vertex-magic.

Corresponding to the two labellings above, the totally-magic $\mathbb{C}_5$-labellings of P_3 are

$a^4, a^2, a^3, a, 1$ vertex constant is a, edge constant is a^4,
$a^3, a^4, a, a^2, 1$ vertex constant is a^2, edge constant is a^3.

The dual (inverse) $\mathbb{C}_5$-labellings are both totally $\mathbb{C}_5$-magic:

$a, a^3, a^2, a^4, 1$ vertex constant is a^4, edge constant is a,
$a^2, a, a^4, a^3, 1$ vertex constant is a^3, edge constant is a^2.

All of these are group equivalent.

Viewed as a star graph rather than a path, $P_3 = T_2$. This is the largest star which has a vertex-magic labelling, as shown in [5]. In this paper we show that almost every star T_n has magic labellings, edge-magic labellings, and totally-magic labellings over every (abelian) group of the appropriate order.

2.5 Abelian groups and involutions

Recall that any group element of order 2 is called an *involution*, and that a non-trivial finite group has elements of order 2 if and only if the order of the group is even.

The structure of finite abelian groups is well known, see for example van der Waerden [11], [12]. Let $\mathbb{A}$ be a non-trivial abelian group, then $\mathbb{A}$ can be expressed as a direct product of cyclic groups of prime power orders. This product is unique up to the order of the direct product. If k be the number of these cyclic components whose order is a power of 2, then $\mathbb{A}$ has $2^k - 1$ involutions, and the product of all the group elements is equal to the product of the involutions and the identity element $1_{\mathbb{A}}$. This product we denote by $i_{\mathbb{A}}$.

$$i_{\mathbb{A}} = \prod_{g \in \mathbb{A}} g = \prod_{g \in \mathbb{A}, g^2 = 1} g.$$

The following lemma follows immediately by induction on k:

Lemma 8. *Let $\mathbb{A}$ be an abelian group.*
(i) If $\mathbb{A}$ has exactly one involution i, say, then $i_{\mathbb{A}} = i$.
(ii) If $\mathbb{A}$ has no involutions, or more than one involution, then $i_{\mathbb{A}} = 1_{\mathbb{A}}$.

2.6 Preliminary lemmas

Lemma 9. *Let G be a graph with a vertex-magic $\mathbb{A}$-labelling with vertex constant h. Then the product of the edge labels is equal to $h^v \times i_{\mathbb{A}}$.*

Proof. Suppose λ is a vertex-magic $\mathbb{A}$-labelling of G with vertex constant h. The product of the weights of all the vertices consists of the product of the labels of the vertices and the squares of the labels of the edges, since each edge enters the product twice. That is:

$$h^v = \prod_{y \in V} \omega(y) = \prod_{y \in V} \lambda(y) (\prod_{zy \in E} \lambda(zy))^2 = \prod_{f \in \mathbb{A}} f \times \prod_{zy \in E} \lambda(zy) = i_{\mathbb{A}} \times \prod_{zy \in E} \lambda(zy)$$

Hence

$$\prod_{zy \in E} \lambda(zy) = i_{\mathbb{A}} \times h^v = h^v \times i_{\mathbb{A}}.$$

□

Corollary 10. *Let G be a graph with a vertex-magic $\mathbb{A}$-labelling with vertex constant h, for some group $\mathbb{A}$ of odd order. Then the product of the edge labels is h^v.*

Analogously, for edge-magic $\mathbb{A}$-labellings we have:

Lemma 11. *Let G be a graph, and λ an edge-magic $\mathbb{A}$-labelling of G with edge constant k. For each vertex x denote the degree of x by d_x. Then*

$$k^e = i_{\mathbb{A}} \times \prod_{w \in V} \lambda(w)^{d_w - 1}.$$

Proof. The product of the weights of all the edges consists of the product of all the labels of edges and the labels of the their end-vertices and is:

$$\begin{aligned} k^e &= \prod_{xy\in E} \omega(xy) \\ &= \prod_{xy\in E} \lambda(xy)\lambda(x)\lambda(y) \\ &= \prod_{xy\in E} \lambda(xy) \prod_{z\in V} \lambda(z)^{d_z} \\ &= \prod_{xy\in E} \lambda(xy) \prod_{z\in V} \lambda(z) \prod_{w\in V} \lambda(w)^{d_w-1} \\ &= i_{\mathbb{A}} \times \prod_{w\in V} \lambda(w)^{d_w-1}. \end{aligned}$$

□

Corollary 12. *Let $\mathbb{A}$ be a group of odd order, and let λ be an edge magic $\mathbb{A}$-labelling of a graph G. Suppose that the vertex constant is k, and for each vertex x denote the degree of x by d_x. Then*

$$k^e = \prod_{w\in V} \lambda(w)^{d_w-1}.$$

Finally, putting these together for $\mathbb{A}$-labellings which are both vertex and edge-magic, we have:

Lemma 13. *Let λ be a totally-magic $\mathbb{A}$-labelling of a graph, with vertex constant h and edge constant k. For each vertex x denote the degree of x by d_x. Then*

$$k^{-e} \prod_{w\in V} \lambda(w)^{d_w-1} = h^{-v} \prod_{zy\in E} \lambda(zy) = i_{\mathbb{A}}.$$

3 The star T_n

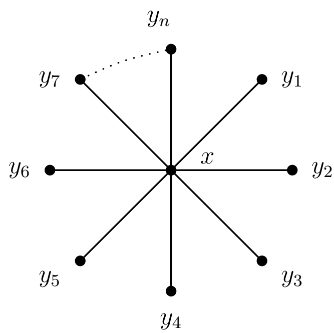
Theorem 14. *Let T_n be the star with $v = n+1$ and $e = n$, and let $\mathbb{A}$ be a group of order $2n+1$. Then*

(i) T_n has an edge-magic $\mathbb{A}$-labelling with edge constant k if and only if $k = a^3$ for some $a \in \mathbb{A}$;

(ii) T_n has a vertex-magic $\mathbb{A}$-labelling with vertex constant $h = 1_{\mathbb{A}}$ except when $\mathbb{A}$ is $\mathbb{C}_3$, $\mathbb{C}_5$ or $\mathbb{C}_3 \times \mathbb{C}_3$; and

(iii) for $h \in \mathbb{A}$, $h \neq 1_{\mathbb{A}}$, T_n has a vertex-magic $\mathbb{A}$-labelling with vertex constant h except when $\mathbb{A}$ is $\mathbb{C}_3$ or $\mathbb{C}_3 \times \mathbb{C}_3$.

Proof. For convenience we denote the central vertex of the star by x and the other vertices by $y_1, y_2, \ldots, y_n$. Then the edges are $xy_1, xy_2, \ldots, xy_n$.



The graph has $e = n$, $v = n + 1$. The size $v + e = 2n + 1$ is odd. Let $\mathbb{A}$ be a group of order $2n + 1$, and λ be a $\mathbb{A}$-labelling of T_n. The product of the elements of $\mathbb{A}$ is the identity, that is $i_{\mathbb{A}} = 1_{\mathbb{A}}$.

The weight of an edge xy_i is $w(xy_i) = \lambda(x)\lambda(xy_i)\lambda(y_i) = \lambda(x)w(y_i)$, and so if the vertices have constant weight h, then the edges have constant weight $k = h\lambda(x)$. Therefore, if λ is a vertex-magic $\mathbb{A}$-labelling of T_n then it is also edge-magic.

Suppose λ is edge-magic with edge constant k. Then, by Corollary 12,

$$k^n = (\lambda(x))^{n-1}.$$

Squaring this gives

$$k^{-1} = k^{2n+1-1} = k^{2n} = (\lambda(x))^{2n-2} = (\lambda(x))^{2n+1-3} = (\lambda(x))^{-3}.$$

Therefore $k = (\lambda(x))^3$, and hence $k = a^3$ for some $a = \lambda(x) \in \mathbb{A}$.

Now the identity $1_{\mathbb{A}}$ must label a vertex or an edge.

Suppose firstly that $\lambda(x) = 1_{\mathbb{A}}$. Then, by Corollary 12, $k^n = (\lambda(x))^{n-1} = 1_{\mathbb{A}}$. Therefore, since n and $2n + 1$ are coprime, $k = 1_{\mathbb{A}}$. Now, for each $i = 1, 2, \ldots, n$,

$$1_{\mathbb{A}} = \omega(xy_i) = 1_{\mathbb{A}} \times \lambda(xy_i) \times \lambda(y_i) = \lambda(xy_i)\lambda(y_i).$$

Since $\mathbb{A}$ has odd order, there are many ways of partitioning the non-identity elements of $\mathbb{A}$ into two sequences $a_1, a_2, \ldots, a_n$ and $a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}$. For any such partition, setting $\lambda(y_i) = a_i$ and $\lambda(xy_i) = a_i^{-1}$ for each $i = 1, 2, \ldots, n$ gives a $\mathbb{A}$-labelling of T_n which is edge-magic with $k = 1_{\mathbb{A}}$.

Will any, or all, of these be vertex-magic? If such a labelling were vertex-magic with vertex-magic constant h, then this would require $h = 1_{\mathbb{A}}$ and $a_1a_2 \ldots a_n = 1_{\mathbb{A}}$. Now for an odd order abelian group $\mathbb{A}$ there are many partitions as described above, and for some groups there is a suitable partition and for others there is not. For example, in the case of T_4, there is a vertex-magic labelling over $\mathbb{C}_9$ with $\lambda(x) = 1_{\mathbb{A}}$ but there is no such vertex-magic labelling over $\mathbb{C}_3 \times \mathbb{C}_3$. In Theorem 15 in the

Appendix to this paper we prove that such a partition will exist for any odd order abelian group $\mathbb{A}$ except when $\mathbb{A} = \mathbb{C}_3$, $\mathbb{C}_5$ or $\mathbb{C}_3 \times \mathbb{C}_3$. This proves (ii).

Next suppose that $\lambda(x) = a \neq 1_{\mathbb{A}}$. Choosing any partition as above, and setting $\lambda(y_i) = aa_i$ and $\lambda(xy_i) = aa_i^{-1}$ for each $i = 1, 2, \ldots, n$ gives a $\mathbb{A}$-labelling of T_n which is edge-magic with $k = a^3$. Therefore there is an edge-magic $\mathbb{A}$-labelling of T_n with edge constant $k = a^3$ for each element $a \in \mathbb{A}$. This completes the proof of (i).

Let λ be an edge-magic $\mathbb{A}$-labelling of T_n with $\lambda(x) = a$, then the translation $a^{-1}\lambda$ is edge-magic with $a^{-1}\lambda(x) = 1_{\mathbb{A}}$. So every edge-magic $\mathbb{A}$-labelling of T_n with edge constant $k = a^3$ is of the form that has just been described.

Finally, will any such labelling be vertex-magic? If it is, the vertex constant h is determined, $h = aa_iaa_i^{-1} = a^2$. For the labelling to be vertex-magic requires in addition that $a^2 = \omega(x) = a^{n+1}a_1^{-1}a_2^{-1}\ldots a_n^{-1}$, and hence a partition in which $a_1a_2\ldots a_n = a^{n-1}$.

In Theorem 15 in the appendix to this paper we prove that if $\mathbb{A}$ is not $\mathbb{C}_3$, $\mathbb{C}_5$ or $\mathbb{C}_3 \times \mathbb{C}_3$, then for each $f \in \mathbb{A}$ there is a partition with $a_1a_2\ldots a_n = f$.

If $\mathbb{A} = \mathbb{C}_5$ then the star is T_2, $n = 2$ and $a^{n-1} = a \neq 1_{\mathbb{A}}$. In Theorem 15 in the Appendix we show that if $a \in \mathbb{C}_5$, $a \neq 1_{\mathbb{C}_5}$, there is a partition with $a_1a_2 = a$.

Since $\mathbb{A}$ has odd order, for any $h \in \mathbb{A}$, $h = a^2$ for some $a \in \mathbb{A}$. Therefore, if $\mathbb{A}$ is not $\mathbb{C}_3$ or $\mathbb{C}_3 \times \mathbb{C}_3$ we have shown that there is a vertex-magic labelling with constant h for any $h \in \mathbb{A}$.

If $\mathbb{A} = \mathbb{C}_3$, then the star is $T_1 = P_2$. There are no vertex-magic labellings of P_2 since any vertex-magic labelling would have to have the same label on both end vertices.

If $\mathbb{A} = \mathbb{C}_3 \times \mathbb{C}_3$, then the star is T_4, and $a^{n-1} = a^3 = 1$. By Theorem 15 there is no partition of $\mathbb{C}_3 \times \mathbb{C}_3 = \{1_{\mathbb{C}_3\times\mathbb{C}_3}\} \cup \{a_1, a_2, a_3, a_4\} \cup \{a_1^{-1}, a_2^{-1}, a_3^{-1}, a_4^{-1}\}$ for which $a_1a_2a_3a_4 = 1_{\mathbb{A}}$. Therefore T_4 has no vertex-magic labellings over $\mathbb{C}_3 \times \mathbb{C}_3$.

This completes the proof of (iii).

□

4 Appendix: Partitions of inverse pairs in Abelian groups of odd order

Let $\mathbb{A}$ be an abelian group of odd order greater than 1. A set $\{g, g^{-1}\}$ with $g \neq 1_{\mathbb{A}}$, will be called an *inverse pair*. The set of non-identity elements in $\mathbb{A} \setminus \{1_{\mathbb{A}}\}$ is a disjoint union of inverse pairs. We are interested to know which elements of $\mathbb{A}$ can be written as a product of a set of representatives of such pairs. We call an element $g \in \mathbb{A}$ *representable* if it can be represented as a product of a set of representatives of inverse pairs.

In this section we prove the following theorem:

Theorem 15. *Let $\mathbb{A}$ be an abelian group of odd order greater than 1.*

(i) If $\mathbb{A}$ is $\mathbb{C}_3$, $\mathbb{C}_5$, or $\mathbb{C}_3 \times \mathbb{C}_3$, then every element except the identity is representable.

(ii) If $\mathbb{A}$ is not $\mathbb{C}_3$, nor $\mathbb{C}_5$, nor $\mathbb{C}_3 \times \mathbb{C}_3$, every group element is representable.

Let $\mathrm{Aut}(\mathbb{A})$ be the automorphism group of $\mathbb{A}$. Inverse pairs are preserved under automorphisms of $\mathbb{A}$. Hence the set of representable elements is preserved by automorphisms of $\mathbb{A}$. Further, the number of distinct sets of representatives which multiply to give an element $g \in \mathbb{A}$ depends only on the $\mathrm{Aut}(\mathbb{A})$ orbit of g.

Lemma 16. *Let $\mathbb{C}_n$ be a cyclic group of odd order $n = 2r + 1 > 1$.*

(i) If $n \geq 7$, then every element of $\mathbb{C}_n$ is representable.

(ii) If $n = 3$ or 5, then every element is $\mathbb{C}_n$ is representable except the identity.

Proof. Assume that $\mathbb{C}_n$ is cyclic of odd order $2r + 1$, $r \geq 1$, with generator a. Then

$$\{a, a^{-1}\} \cup \{a^2, a^{-2}\} \cup \cdots \cup \{a^r, a^{-r}\}$$

partitions $\mathbb{C}_n \setminus \{1_{\mathbb{C}_n}\}$ as a disjoint union of inverse pairs.

To determine which elements of $\mathbb{C}_n$ are representable, i.e. to find those g with

$$g = \prod_{i=1}^{r} g_i, \quad g_i \in \{a^i, a^{-i}\},$$

reduces to determining which congruence classes modulo $2r + 1$ can be represented as a sum

$$\epsilon_1 + \cdots + \epsilon_r, \quad \epsilon_i = \pm i.$$

Taking successively $r = 1, 2, 3$, etc we can write down a list of these values.

$r = 1$: $-1, 1$

Subtracting 2 and adding 2 to each sum above:

$r = 2$: $-3, -1, 1, 3$

Subtracting 3 and adding 3 to each sum above:

$r = 2$: $-6, -4, -2, 0, 2, 4, 6$

Subtracting 4 and adding 4 to each sum above:

$r = 3$: $-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10$

Continuing, we have inductively that for each r the sums $\sum_{i=1}^{r} \epsilon_i$, $\epsilon_i = \pm i$, form an arithmetic progression with first term $-\frac{1}{2}r(r+1)$, last term $\frac{1}{2}r(r+1)$ and common difference 2. As there are $\frac{1}{2}r(r+1) + 1$ terms in the sequence, and 2 is prime to $2r + 1$ we get all residue classes as soon as $\frac{1}{2}r(r+1) + 1 \geq 2r + 1$, i.e. for $r \geq 3$. In the case $r = 1$ or 2 we get all classes but the zero class. □

Lemma 17. *Let $\mathbb{H}$ and $\mathbb{K}$ be non-trivial abelian groups of odd order. Suppose h is representable in $\mathbb{H}$ and k is representable in $\mathbb{K}$. Then (h, k), $(h, 1_{\mathbb{K}})$ and $(1_{\mathbb{H}}, k)$ are representable in $\mathbb{H} \times \mathbb{K}$.*

Proof. The inverse pairs of $\mathbb{H} \times \mathbb{K}$ are of three types,

$$\{(f_1, f_2), (f_1^{-1}, f_2^{-1})\}, \qquad \{(f_1, 1_{\mathbb{K}}), (f_1^{-1}, 1_{\mathbb{K}})\}, \quad \{(1_{\mathbb{H}}, f_2), (1_{\mathbb{H}}, f_2^{-1})\},$$

where $\quad 1_{\mathbb{H}} \neq f_1 \in \mathbb{H}, \quad 1_{\mathbb{K}} \neq f_2 \in \mathbb{K}$.

We can make up a set of inverse pair representatives as follows. We have $|\mathbb{H}| = 2s+1$ for some $s \geq 1$. Let $X = \{h_1, \dots h_s\}$ be a set of representatives of inverse pairs in $\mathbb{H}$. For each $i = 0, 1, \dots, s$, let Y_i be a set of representatives of inverse pairs in $\mathbb{K}$. Let Z be the subset of $\mathbb{H} \times \mathbb{K}$ made up of all elements (h_i, y) and (h_i^{-1}, y) with $h_i \in X$ and $y \in Y_i$, for $i = 1, 2, \dots, s$, together with all $(h_i, 1_{\mathbb{K}})$ with $h_i \in X$, and all $(1_{\mathbb{H}}, y)$ with $y \in Y_0$. Then Z is a set of representatives of inverse pairs of $\mathbb{H} \times \mathbb{K}$.

Suppose h is representable in $\mathbb{H}$. Let $k_0, k_1, \dots, k_s$ be a sequence of s representable elements in $\mathbb{K}$. Since squaring is an automorphism of any abelian group of odd order we have for each $i \geq 1$, $k_i = l_i^2$ for some (unique) representable l_i in $\mathbb{K}$. We can choose X such the product of its elements is h, Y_0 such that the product of its elements is k_0, and for each $i \geq 1$, Y_i such the product of its elements is l_i. Then the product of the corresponding set Z of inverse pair representatives in $\mathbb{H} \times \mathbb{K}$ is

$$(h, k_0 l_1^2 \dots l_s^2) = (h, k_0 k_1 \dots k_s),$$

which is therefore representable in $\mathbb{H} \times \mathbb{K}$.

Now suppose k is representable in $\mathbb{K}$. Then again, as squaring is an automorphism, $k = l^2$ for some (unique) representable l in $\mathbb{K}$. We now consider two sequences of representable elements of $\mathbb{K}$. The first is the sequence with

$$k_0 = k,\ k_1 = k^{-1},\ k_2 = k,\ k_3 = k^{-1},\ \dots$$

The second is the sequence with

$$k_0 = l,\ k_1 = l\ , k_2 = k^{-1},\ k_3 = k,\ k_4 = k^{-1},\ \dots$$

The first sequence has product $k_0 k_1 \dots k_s$ equal to k when s is even, and to the identity $1_{\mathbb{K}}$ when s is odd. The second sequence has product $k_0 k_1 \dots k_s$ equal to the identity $1_{\mathbb{K}}$ when s is even, and to k when s is odd. Thus the first sequence gives a set of inverse pair representatives Z with product (h, k) if s is even, and product $(h, 1_{\mathbb{K}})$ if s is odd. The second gives a set of inverse pair representatives Z with product $(h, 1_{\mathbb{K}})$ if s is even, and product (h, k) if s is odd. Consequently, if h is representable in $\mathbb{H}$ and k is representable in $\mathbb{K}$, then both $(h, 1_{\mathbb{K}})$ and (h, k) are representable in $\mathbb{H} \times \mathbb{K}$. By symmetry we deduce also that $(1_{\mathbb{H}}, k)$ is representable in $\mathbb{H} \times \mathbb{K}$. □

We are now ready to prove Theorem 15.

Proof. Recall that any non-trivial finite abelian group is, by the theory of elementary divisors, a product

$$\mathbb{A} = \mathbb{C}_{d_1} \times \mathbb{C}_{d_2} \times \cdots \times \mathbb{C}_{d_n}$$

of non-trivial cyclic groups $\mathbb{C}_{d_i}$, with $d_1 | d_2 | \cdots | d_n$. For $\mathbb{A}$ of odd order all the $\mathbb{C}_{d_i}$ are cyclic of odd order greater than 1.

From Lemma 17 we deduce that if two non-trivial abelian groups of odd order each has every element, except possibly its identity, representable, then every element in their product, except possibly for its identity, is representable. Further if every element in one of the factors is representable, and every element, except possibly the identity, of the other factor is representable, then so is every element of their product.

By Lemma 16, and induction on the number of elementary divisors, we deduce that any non-trivial abelian group $\mathbb{A}$ of odd order has all its elements representable, except possibly its identity. Further if every element in any one of the cyclic factors is representable then so is every element of $\mathbb{A}$. Again by Lemma 16 this last condition holds except when all the d_i are either 3 or 5.

In the possible exceptional cases $\mathbb{A}$ is an elementary abelian p-group $(\mathbb{C}_p)^n$ with p either 3 or 5. For these groups every element, except possibly the identity, is representable. Further, if some $(\mathbb{C}_p)^m$ has every element representable, then the same is true for all $(\mathbb{C}_p)^n$, with $n > m$, since then $(\mathbb{C}_p)^m$ is a factor of $\mathbb{A}$. To show every element in a particular such group is representable it remains only to show that its identity element is representable.

We know that $\mathrm{Aut}(\mathbb{A})$ acts transitively on the non-identity elements in an elementary p-group. There are 2^r, $r = \frac{1}{2}(p^n - 1)$ choices of a set of representatives of inverse pairs. Let A be the number of representations of the identity, and B those of any non-identity element. Then

$$A + (p^n - 1)B = 2^r.$$

Thus A cannot be zero unless $p^n - 1$ is a power of 2. Since $5^2 - 1 = 24$ is not a power of 2, the identity is representable in $\mathbb{C}_5 \times \mathbb{C}_5$, and thus in all higher order elementary abelian 5-groups. Since $3^3 - 1 = 26$ is not a power of 2, the identity is representable in $\mathbb{C}_3 \times \mathbb{C}_3 \times \mathbb{C}_3$, and thus in all higher order elementary abelian 3-groups.

To complete the proof of Theorem 15 it remains to show that the identity is not representable in $\mathbb{C}_3 \times \mathbb{C}_3$, i.e. that for this group $A = 0$. Suppose a generates $\mathbb{C}_3$. Then

$$(a, 1)(1, a^{-1})(a^{-1}, a)(a, a) = (a^{-1}, 1)(1, a)(a, a^{-1})(a, a) = (a, a),$$

exhibits two distinct representations of (a, a). For this group with $A + 8B = 16$ we have therefore $B \geq 2$. This is only possible if $B = 2$ and $A = 0$. □

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