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Classes of graphs that can be partitioned to satisfy all their vertices

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Abstract

In a given graph, we want to partition the set of its vertices into two subsets, such that each vertex is satisfied in that it has at least as many neighbours in its own subset as in the other. We exhibit some sufficient conditions for the existence of a partition where all vertices are satisfied. In particular, we characterize for which graphs of girth at least 5 and which line-graphs of triangle-free graphs there exists a solution to the problem.

1 Introduction

We consider an undirected, finite, simple graph $G = (V, E)$. The neighbourhood of a vertex v is denoted by $N(v)$. Let $d_G(v)$ be the degree of vertex v, and $\delta(G)$ (resp. $\Delta(G)$) the minimum (resp. maximum) degree in G. In the following we shall use additional graph theory terms whose definitions can be found in [2].

We want to partition the set of vertices of a given graph into two non-empty subsets such that each vertex has at least as many neighbours in its own subset as in the other. A vertex is said to be *satisfied* if it has at least as many neighbours in its

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subset as in the other. Our problem, the *Satisfactory Graph Partitioning* problem, consists in determining if a graph can be partitioned into two such that each vertex is satisfied. More precisely, we define the Satisfactory Graph Partitioning SGP problem as follows.

INSTANCE SGP: A graph $G = (V, E)$.

QUESTION: Is there a partition of V in two non-empty subsets such that each vertex is satisfied?

For an interpretation of this problem we consider a group of people, among which certain persons are friends. We want to partition the group into two such that every person is satisfied, i.e., everyone has at least as many friends in his group as in the other, and therefore does not want to change group. This problem can be modelled by an instance of SGP on a graph where the people are the vertices and two vertices are linked if the corresponding persons are friends.

For a graph $G = (V, E)$ and a partition $(V', V \backslash V')$ of V, we define, for each vertex v, $IN(v)$ as the number of neighbours of v that are in the same subset of the partition as v, and $OUT(v)$ as the number of neighbours of v that are in the other subset than v (and therefore $d_G(v) = IN(v) + OUT(v)$). The SGP problem then consists of deciding the existence of a partition such that $IN(v) \geq OUT(v)$ for each vertex v.

The Different Than Majority Labelling (DTML) problem [8] is a graph partition problem closely related to SGP: partition the vertices of an undirected graph into two sets such that for every vertex v, $OUT(v) \geq IN(v)$. This can be done in polynomial time. This problem is also referred to as the unfriendly graph partition problem by [1] and [11]. While there exists an unfriendly partition for any graph, this is not the case for satisfactory partitions (take a clique for example). The complexity of determining the existence of a satisfactory partition is open (see Section 5).

While an algorithmic approach to solving SGP has been presented in [4], in this paper we will provide some theoretical results on SGP and provide directions for future research. More precisely, we start in Section 2 with some initial observations. We then show in Section 3 exactly which graphs of girth at least 5 have a satisfactory partition. The *girth* of a graph is the length of its shortest chordless induced cycle. In Section 4, we characterize the line-graphs of triangle-free graphs that have a satisfactory partition. Finally, we conclude in Section 5 with some final remarks and open questions.

2 Some initial observations

To simplify the reading, we say a graph is *partitionable*, or can be *partitioned*, when the corresponding problem SGP has a solution.

There exist both graphs that can be partitioned, and graphs that cannot be partitioned. For example, the complete bipartite graph $K_{p,q}$ is partitionable if and only if p and q are even (in particular, stars cannot be partitioned). The complete graphs are also examples of graphs which cannot be partitioned.

It is easy to see that every disconnected graph can be partitioned. Some other graphs can also be partitioned in a simple way. For example, in the case of a cycle on $n \geq 4$ vertices, we put $1 < i < n-1$ consecutive vertices in one subset and the remaining $n-i$ vertices in the other subset. If the graph is a tree, but not a star, the partition of the vertices is given by the connected components obtained by removing a non-pendant edge of the tree (see Proposition 2 below). Also, any graph G can be made partitionable by for example adding $d_G(v)$ leaves to any vertex v of G.

Since SGP is trivial for disconnected graphs, we will only consider connected graphs in the remainder of this paper.

We say that a cut is *non-trivial* if it partitions the set of vertices into two subsets of size greater than 1. As the unfriendly partition problem mentioned in the introduction has a close link to maximum cuts (every maximum cut provides an unfriendly partition), SGP has a link to minimum cuts. Indeed, any non-trivial minimum cut provides a satisfactory partition: if a vertex was not satisfied, then a smaller cut could be obtained by flipping that vertex to the other subset. Moreover, all minimum cuts in a graph can be generated in polynomial time (see for example [9]). Hence, we have:

Proposition 1 *If a graph G has a non-trivial minimum cut, then G is partitionable. Moreover, the solution can be found in polynomial time.*

Therefore, all non-partitionable graphs G have $\delta(G)$ equal to the size of a minimum cut, and all minimum cuts are trivial.

Another simple result concerns graphs that are "almost disconnected". In a graph G, a *pendant edge* is an edge with (at least) one endpoint of degree 1 in G.

Proposition 2 *If a graph G has a*

- *disconnecting edge that is a non-pendant edge, or*
- *disconnecting vertex that is not an endpoint of a pendant edge,*

then G is partitionable.

Proof. The first condition is a corollary of Proposition 1.

Consider the connected components $V_1, V_2, \ldots, V_k$ ($k \geq 2$) obtained by removing the disconnecting vertex v. For any positive integer $k' < k$, we define $c_1 = |N(v) \cap V_1 \cap V_2 \cap \ldots \cap V_{k'}|$ and $c_2 = |N(v) \cap V_{k'+1} \cap V_{k'+2} \cap \ldots \cap V_k|$. If $c_1 \geq c_2$, then we set $V' = V_1 \cup V_2 \cup \ldots \cup V_{k'} \cup \{v\}$, otherwise we set $V' = V_1 \cup V_2 \cup \ldots \cup V_{k'}$. We now have for every vertex $w \notin N(v) \cup \{v\}$

$$IN(w) \geq 0 = OUT(w),$$

for every vertex $w \in N(v)$

$$IN(w) \geq 1 \geq OUT(w),$$

(since v is not an endpoint of a pendant edge), and for vertex v

$$IN(v) = \max(c_1, c_2) \geq \min(c_1, c_2) = OUT(v). \square$$

3 Graphs of girth at least 5

If $G = (V, E)$, and $A \subseteq V$, $G[A]$ is the subgraph of G induced by A. For an induced subgraph H and a vertex x of H, $d_H(x)$ represents the number of neighbours of x in H. To simplify the notation, $d_{G[A]}(x)$ will be written as $d_A(x)$.

Studying a somewhat related partitioning problem, Stiebitz [12] proved the following result.

Theorem 1 [12] *Let $G = (V, E)$ be a graph and $a, b : V \to I\!N$ two functions. Assume that $d_G(x) \geq a(x) + b(x) + 1$ for every vertex $x \in V$. Then there is a partition $(V', V \backslash V')$ of V such that:*

- *$IN(x) \geq a(x)$ for every vertex $x \in V'$, and*
- *$IN(x) \geq b(x)$ for every vertex $x \in V \backslash V'$.*

We would like to use $a(x) = b(x) = \left\lceil \frac{d_G(x)}{2} \right\rceil$ in order to have a satisfactory partition, but this does not satisfy the condition $d_G(x) \geq a(x) + b(x) + 1$. Theorem 1 only shows that for any graph $G = (V, E)$, there is a partition $(V', V \backslash V')$ such that $IN(x) \geq OUT(x) - 2$ for every vertex x (by choosing $a(x) = b(x) = \left\lfloor \frac{d_G(x)-1}{2} \right\rfloor$). We therefore need to improve the lower bound $a(x) + b(x) + 1$.

As a corollary of Theorem 1, the vertex set of a graph with minimum degree at least $s + t + 1$ can be partitioned into two parts inducing subgraphs with minimum degree at least s and t, respectively. In [7], Kaneko showed that this result holds for triangle-free graphs with minimum degree $s + t$. Then Diwan showed that the bound can be improved to $s + t - 1$ for graphs with girth at least 5 when $s, t \geq 2$ [3]. Unfortunately, both papers [7, 3] only consider minimum degrees, and no longer the more general situation of Theorem 1 where different values can be assigned to different vertices. On the other hand, the bound used in [3] suits our need.

Hence, to obtain further results for SGP, we first generalize Diwan's theorem [3] as follows.

Theorem 2 *Let $G = (V, E)$ be a graph of girth at least 5, and $a, b : V \to I\!N$ two functions. Assume that $d_G(x) \geq a(x) + b(x) - 1$, $a(x) \geq 2$ and $b(x) \geq 2$ for every vertex $x \in V$. Then there is a partition $(V', V \backslash V')$ of V such that:*

- *$d_{V'}(x) = IN(x) \geq a(x)$ for every vertex $x \in V'$, and*
- *$d_{V \backslash V'}(x) = IN(x) \geq b(x)$ for every vertex $x \in V \backslash V'$.*

Although the proof of this generalization will closely follow Diwan's proof in [3], we nevertheless present it here to allow us to obtain further results (such as Theorem 3 and its corollary below) that require some changes to the proof. Before giving the proof, let us introduce further helpful notions.

A pair (A, B) is said to be *feasible* if A and B are disjoint non-empty subsets of V such that

- $d_A(x) \geq a(x)$ for every vertex $x \in A$, and
- $d_B(x) \geq b(x)$ for every vertex $x \in B$.

We have to show the existence of a feasible partition of V. In fact, as the following property shows, it is enough to show the existence of a feasible pair (A, B).

Proposition 3 [12] *If there exists a feasible pair, then there exists a feasible partition of* V.

Proof. Let (A, B) a feasible pair such that $A \cup B$ is maximal. Assume $A \cup B \neq V$. Therefore, $C = V \backslash (A \cup B)$ is not empty. By maximality of $A \cup B$, $(A, B \cup C)$ is not feasible, and there is a vertex $x \in C$ such that $d_{B \cup C}(x) \leq b(x) - 1$. Since $d_G(x) \geq a(x) + b(x) - 1$, we have $d_A(x) \geq a(x)$. But then $(A \cup \{x\}, B)$ is a feasible pair, contradicting the maximality of (A, B). □

An induced subgraph H of $G = (V, E)$ is said to be $(a-1)$-*degenerate* (resp. $(b-1)$-*degenerate*) if for every induced subgraph H' of H there is a vertex x such that $d_{H'}(x) \leq a(x) - 1$ (resp. $d_{H'}(x) \leq b(x) - 1$).

By an (a, b)-*partition* of V, we mean a partition (A, B) of V such that $G[A]$ is $(a-1)$-degenerate and $G[B]$ is $(b-1)$-degenerate (both A and B not empty). The *weight* $w(A, B)$ of an (a, b)-partition is the number of edges that have both endpoints in a same subset, plus $\sum_{x \in A} b(x) + \sum_{x \in B} a(x)$.

Finally, an a-*good* (resp. b-*good*) set $V' \subset V$ is a non-empty subset such for every vertex $x \in V'$, we have $d_{V'}(x) \geq a(x)$ (resp. $d_{V'}(x) \geq b(x)$).

Proof of Theorem 2. The three claims along the proof will point out where we use the fact that $a(x) \geq 2$ and $b(x) \geq 2$ for every vertex $x \in V$.

Suppose there is a graph $G = (V, E)$, with functions a and b, for which there is no feasible partition (hence also no feasible pair by Proposition 3), and choose such a graph with minimum number of edges. We can therefore assume that, for every edge $(u, v) \in E$, we have $d_G(u) = a(u) + b(u) - 1$ or $d_G(v) = a(v) + b(v) - 1$ (otherwise, remove the edges that do not satisfy this condition, and the feasible partition found on that subgraph is also feasible for G).

We first show that there exists an (a, b)-partition. Consider an (inclusionwise) minimal a-good subset $S \subset V$.

Claim 1 *S is a proper subset of V.*

Proof. Consider a vertex $v \in V$. For any vertex $x \neq v$,

$$d_{V \backslash \{v\}}(x) \geq d_G(x) - 1 \geq a(x) + b(x) - 2 \geq a(x)$$

since $b(x) \geq 2$. Hence, $V \backslash \{v\}$ is an a-good set. □

Let $T = V\backslash S$. Since there is no feasible pair in G (by Proposition 3), $G[T]$ must be $(b-1)$-degenerate. By minimality of S, there is a vertex u in S with $d_S(u) = a(u)$. Moreover, for any proper non-empty subset S' of S, there is a vertex $v \in S'$ with $d_{S'}(v) \leq a(v) - 1$. By setting $S' = S\backslash\{u\}$, we find a vertex $v \in S$, adjacent to u, with $d_S(v) = a(v)$. We have $d_G(u) = a(u) + b(u) - 1$ or $d_G(v) = a(v) + b(v) - 1$; let us assume $d_G(u) = a(u) + b(u) - 1$. Then $(S\backslash\{u\}, T \cup \{u\})$ is an (a, b)-partition, since $d_T(u) = b(u) - 1$ and $S\backslash\{u\}$ is not empty (contains v).

Claim 2 *For any (a, b)-partition (S, T), we have $|S| \geq 2$ and $|T| \geq 2$.*

Proof. Indeed, by definition, there is a vertex $u \in S$ with $d_S(u) \leq a(u) - 1$ and a vertex $v \in T$ with $d_T(v) \leq b(v) - 1$. By the condition on functions a and b, this implies $|T| \geq d_T(u) \geq a(u) + b(u) - 1 - d_S(u) \geq b(u) \geq 2$ and $|S| \geq d_S(v) \geq a(v) + b(v) - 1 - d_T(v) \geq a(v) \geq 2$. □

Among all (a, b)-partitions (we now know there is at least one), consider an (a, b)-partition (S, T) that has maximum weight. Let $C = \{u \in S \mid d_S(u) \leq a(u) - 1\}$ and $D = \{v \in T \mid d_T(v) \leq b(v) - 1\}$. By definition of an (a, b)-partition, neither C nor D is empty.

We now show that every vertex in C is adjacent to every vertex in D. Let $u \in C$ and $v \in D$ be two vertices, and assume they are not adjacent. The partition $(S\backslash\{u\}, T \cup \{u\})$ cannot be an (a, b)-partition, since

$$w(S\backslash\{u\}, T \cup \{u\}) - w(S, T) \geq -(a(u) - 1) + b(u) - b(u) + a(u) = 1 \ .$$

The only possibility is that $G[T \cup \{u\}]$ is not $(b - 1)$-degenerate. Therefore, there exists a b-good subset $B \subseteq T \cup \{u\}$. Since $d_T(v) \leq b(v) - 1$ and v is not adjacent to u, vertex v does not belong to B. Similarly, there exists an a-good subset $A \subseteq S \cup \{v\}$, that does not contain u. But then, (A, B) forms a feasible pair, a contradiction.

Since G is triangle-free, and every vertex in C is adjacent to every vertex in D, both C and D must be independent sets.

Claim 3 *We have $C \neq S$ and $D \neq T$.*

Proof. As mentioned above, for any vertex $v \in D$, there exists an a-good set $A \subseteq S \cup \{v\}$. For a vertex $x \in A$, we have $d_A(x) \geq a(x) \geq 2$. If $C = S$, C being an independent set, we cannot have $d_A(x) \geq 2$. Hence, $C \subset S$. Similarly, $D \subset T$. □

Since G has girth at least 5, a vertex in $S\backslash C$ has at most one neighbour in C (otherwise, consider a vertex in $S\backslash C$, two of its neighbours in C and a vertex in D). S being $(a - 1)$-degenerate, there exists a vertex u_1 in $S\backslash C$ such that $d_{S\backslash C}(u_1) \leq a(u_1) - 1$. But since $u_1 \notin C$, we also have $d_S(u_1) \geq a(u_1)$. Therefore $d_S(u_1) = a(u_1)$, and u_1 has exactly one neighbour u in C. Similarly, there is a vertex v_1 in $T\backslash D$ with $d_T(v_1) = b(v_1)$ and adjacent to exactly one vertex v in D.

We already know there exists a b-good subset $B \subseteq T \cup \{u\}$. If such a set does not contain v, then every a-good subset $A \subseteq S \cup \{v\}$ must contain u; otherwise, A and B

are disjoint and we have a feasible pair. So, either every a-good subset $A \subseteq S \cup \{v\}$ contains u or every b-good subset $B \subseteq T \cup \{u\}$ contains v. Without loss of generality, we assume the latter. Since $d_T(v) \leq b(v) - 1$, B also contains all neighbours of v in T, in particular v_1. Therefore, both $G[(T \cup \{u\}) \backslash \{v\}]$ and $G[(T \cup \{u\}) \backslash \{v_1\}]$ are $(b-1)$-degenerate. Depending on whether $d_G(v) = a(v) + b(v) - 1$ or $d_G(v_1) = a(v_1) + b(v_1) - 1$ (at least one holds), we consider two cases.

Assume $d_G(v) = a(v) + b(v) - 1$. Since v belongs to every b-good subset of $T \cup \{u\}$, $d_T(v) = b(v) - 1$. Hence, $d_S(v) = a(v)$ and $d_{S \backslash \{u\}}(v) = a(v) - 1$. Therefore, $G[(S \backslash \{u\}) \cup \{v\}]$ is $(a-1)$-degenerate, and $((S \backslash \{u\}) \cup \{v\}, (T \cup \{u\}) \backslash \{v\})$ forms an (a, b)-partition. This (a, b)-partition also has maximum weight, since

$$w((S \backslash \{u\}) \cup \{v\}, (T \cup \{u\}) \backslash \{v\}) - w(S, T) \geq a(v) + b(u) - b(u) - a(v) = 0 \ .$$

G being triangle-free, v is not adjacent to u_1 and u is not adjacent to v_1. Therefore, u_1 has degree $a(u_1) - 1$ in $G[(S \backslash \{u\}) \cup \{v\}]$ and v_1 has degree $b(v_1) - 1$ in $G[(T \cup \{u\}) \backslash \{v\}]$. But we showed earlier that in an $(a, b)-$partition (S, T) of maximum weight, every vertex $x \in S$ with $d_S(x) \leq a(x) - 1$ has to be adjacent to every vertex $x' \in T$ with $d_T(x') \leq b(x') - 1$. Therefore, u_1 and v_1 must be adjacent, giving a cycle (u, v, v_1, u_1) of length 4 in G, a contradiction.

Assume now $d_G(v_1) = a(v_1) + b(v_1) - 1$. Similarly as above, $((S \backslash \{u\}) \cup \{v_1\}, (T \cup \{u\}) \backslash \{v_1\})$ is an (a, b)-partition and has maximum weight. To avoid the cycle (u, v, v_1, u_1) of length 4, u_1 and v_1 are not adjacent. Therefore, $d_{(S \backslash \{u\}) \cup \{v_1\}}(u_1) = a(u_1) - 1$ and $d_{(T \cup \{u\}) \backslash \{v_1\}}(v) = d_T(v) \leq b(v) - 1$. As previously, this implies that u_1 and v are adjacent, yielding a triangle in G. This concludes the proof of Theorem 2. □

As mentioned above, Theorem 2 generalizes the result in [3]. But, more important for us in the study of SGP, Theorem 2 has the following corollary.

Corollary 1 *Let G be a graph of girth at least 5 and minimum degree at least 3. Then G is partitionable.*

Proof. For every vertex x, set $a(x) = b(x) = \left\lceil \frac{d_G(x)}{2} \right\rceil$. Since the minimum degree in G is at least 3, we have $a(x) = b(x) \geq 2$ for every vertex x, and we can apply Theorem 2. □

Theorem 2 cannot be improved by having a bound on the girth of less than 5; indeed, complete bipartite graphs $K_{p,q}$, with p odd, are not partitionable (in the SGP sense), and have girth 4. Similarly, Theorem 2 also cannot be improved by having a lower bound on the minimum value of functions a and b, as illustrated by the cycle on $n \geq 5$ vertices, a the constant function 1 and b the constant function 2.

But there is room for improving the result stated in Corollary 1. Indeed, when we consider $a(x) = b(x) = \left\lceil \frac{d_G(x)}{2} \right\rceil$, we have the additional information that $a(x) = b(x)$ for everty vertex x. This property is useful to prove the following result.

Theorem 3 *Let G be a graph of girth at least 5, and minimum degree at least 2. Then G is partitionable.*

Proof. In the proof of Theorem 2, the fact that $a(x) \geq 2$ and $b(x) \geq 2$ for every vertex x has only been used to prove the three claims. We now show how to prove these claims in the specific case where $a(x) = b(x) = \left\lceil \frac{d_G(x)}{2} \right\rceil$ and every vertex has degree at least 2.

Let us first consider Claim 1. Let v be a vertex of degree 2. If no such vertex exists, or if every neighbour of v has degree at least 3, the same argument as in the proof of Theorem 2 applies. Otherwise, there is a neighbour u of v that has degree 2. But then, $(\{u, v\}, V\backslash\{u, v\})$ is a feasible partition:

$$IN(u) = IN(v) = 1 = OUT(u) = OUT(v)$$

and, since G is triangle-free,

$$IN(x) \geq d_G(x) - 1 \geq 1 \geq OUT(x)$$

for every vertex x in $V\backslash\{u, v\}$.

Let us now consider Claim 2, and assume $|S| = 1$. Since $G[T]$ is $(b-1)$-degenerate, there is a vertex $v \in T$ with $d_T(v) \leq b(v) - 1$. Hence, we have $1 \geq d_S(v) \geq a(v)$, implying $a(v) = 1$. But this in turn implies that $d_T(v) \leq b(v) - 1 = 0$, contradicting the fact that $d_G(v) = d_S(v) + d_T(v) \geq 2$. Therefore $|S| \geq 2$, and similarly $|T| \geq 2$.

Finally, let us consider Claim 3, and assume $C = S$. Let v be a vertex in D. As shown before Claim 3 in the proof of Theorem 2, there exists an a-good set $A \subseteq S \cup \{v\}$. We also know at this point that C, and hence S, is an independent set. Since $a(x) \geq 1$ for any vertex x, A must contain at least two vertices, including v. We will show that $(A, V\backslash A)$ is a feasible pair, a contradiction. By definition, A is a-good. It remains to show that $V\backslash A$ is b-good.

Let $A' = A \cap S$. For any vertex $x \in A'$, we have $1 = d_A(x) \geq a(x)$. Therefore, every vertex x in A' satisfies $a(x) = b(x) = 1$ and has degree 2. For such a vertex, let $v(x)$ be the neighbour of x different from v (every vertex $x \in A' \subseteq C$ is adjacent to $v \in D$). If x and x' are two distinct vertices of A', $v(x)$ and $v(x')$ are two distinct vertices (of T); indeed, if that was not the case, we would have a cycle $(x, v(x) = v(x'), x', v)$ of length four. Moreover, since G is triangle-free, $v(x)$ cannot be adjacent to v. Hence, $d_{V\backslash A}(v(x)) = d_G(v(x)) - 1$. A vertex $w \in V\backslash A$ that is not a vertex $v(x)$ for any $x \in A'$ can only have v as neighbour in A (the vertices in A' have degree 2), and therefore $d_{V\backslash A}(w) \geq d_G(w) - 1$. Hence, $d_{V\backslash A}(w) \geq d_G(w) - 1$ for any vertex $w \in V\backslash A$. If $d_G(w)$ is even, $d_G(w) = a(w) + b(w)$, and thus $d_{V\backslash A}(w) \geq d_G(w) - 1 = a(w) + b(w) - 1 \geq b(w)$. If $d_G(w)$ is odd, $d_G(w) \geq 3$, hence $a(w) = b(w) \geq 2$, and thus $d_{V\backslash A}(w) \geq d_G(w) - 1 = a(w) + b(w) - 2 \geq b(w)$. □

Corollary 2 *Let G be a graph of girth at least 5. Either G is a star $K_{1,m}$ (and is therefore not partitionable), or G is partitionable.*

Proof. If $G = (V, E)$ is not connected, G is trivially partitionable; so let us assume that G is connected. Let G' be the graph obtained from G by removing all leaves (vertices of degree 1) and their incident edge. If G' has no edge (zero or one vertex), G is a star $K_{1,m}$ with $m \geq 1$.

If G' has minimum degree 1, let u be a vertex of G' of degree 1, and v its neighbour in G'. Then G can be partitioned as follows: $V' \subset V$ contains u and all neighbours of u of degree 1 in G (that is, $V' = (\{u\} \cup N_G(u)) \backslash \{v\}$). It is easy to check that this is indeed a satisfactory partition.

If G' has minimum degree at least 2, G' has a satisfactory partition by Theorem 3. This partition can be extended to a satisfactory partition of G by putting every leaf in the same subset as its unique neighbour. □

As mentioned earlier, complete bipartite graphs $K_{p,q}$, with p odd, are not partitionable and have girth 4. It is therefore tempting to generalize the previous corollary to graphs of girth at least 4 and excluding those complete bipartite graphs. Unfortunately, there are other graphs of girth 4 that are not partitionable. Consider a graph G constructed as follows. We start with $K_{p,q}$, $p \geq 3$ odd and $q \geq 2$. We then add another $q' < q$ vertices, each linked to at least one of the p vertices. Such a bipartite graph G is not partitionable (notice that all q vertices have to be in the same subset of any satisfactory partition, since p is odd). Moreover, there exist also non-bipartite graphs that are not partitionable. Consider a graph $G = (V, E)$ defined by:

$$V = V_0 \cup V_1 \cup V_2 \cup V_3 \cup V_4 \quad \text{with} \quad |V_i| = i + 1 \quad 0 \leq i \leq 4 \quad .$$

The edge set E consists of all possible edges between V_i and $V_{(i+1 \mod 5)}$, $0 \leq i \leq 4$. This graph is not bipartite and is not partitionable (all vertices of V_4 have to be in the same subset of a satisfactory partition, since they have an odd number of neighbours). This construction can easily be generalized to create more non-bipartite non-partitionable graphs of girth 4.

Notice that in all these non-partitionable graphs of girth 4, there are vertices of odd degree. In fact, we can extend the result of Kaneko [7] in the same way as we did for Diwan's result; this would allow us to prove that every triangle-free graph with all vertices of even degree and minimum degree at least 4 is partitionable. Here again, the condition on the minimum degree can be dropped with a little more work (while in a counter-example graph $|S| \geq 2$ ($|T| \geq 2$) no longer holds for every (a, b)-partition (S, T) (now defined slightly differently than above), it still holds for (a, b)-partitions of maximum weight, which is enough for our needs). For space reasons, we do not include this proof here.

We now know exactly which graphs of girth at least 5 are partitionable. Can we also determine whether their line-graphs are partitionable? The main theorem of the next section answers this question and goes even further: it indicates which line-graphs of graphs of girth at least 4 are partitionable.

4 Partitionable line-graphs

Given a graph $G = (V, E)$, its line-graph $L(G) = (E, F)$ has the edge-set of G as vertex-set, and there is an edge (e_1, e_2) in $L(G)$ if and only if in G the two edges e_1

and e_2 are adjacent.

Let $T_{1,m,m'}$, with $m' \leq m$, be the graph defined by the vertex-set

$$V = \{v_0, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_{m'}\}$$

and edge-set

$$E = \{(v_0, v_i)\ i = 1, 2, \ldots, m\} \cup \{(v_i, w_i)\ i = 1, 2, \ldots, m'\}$$

Notice that $T_{1,m,0}$ is the star $K_{1,m}$.

Theorem 4 *Let G be a triangle-free graph. Then its line-graph $L(G)$ is partitionable if and only if G is not $T_{1,m,m'}$ with $m' < m$ or m odd.*

Proof. Let us first consider the case where G is not $T_{1,m,m'}$ for any $m' \leq m$, and let v be a vertex of maximum degree in $G = (V, E)$. We partition the vertices of $L(G)$ by E' and $E \backslash E'$, with E' the set of edges $e = (x, y)$ of G such that:

- either e is incident to v, or
- $d_G(x) = 2$, $d_G(y) = 1$, and x is adjacent to v.

In this situation, E' is a proper subset of E. We now show that it is a satisfactory partition for $L(G)$. For $e = (v, y)$, we have:

$$IN(e) \geq d_G(v) - 1 \geq d_G(y) - 1 \geq OUT(e)\ .$$

For $e = (x, y) \in E'$ not incident to v, we have $IN(e) = 1$ and $OUT(e) = 0$. For $e = (x, y) \in E \backslash E'$, we consider two cases. If x is adjacent to v, then $IN(e) \geq d_G(x) + d_G(y) - 3 \geq 1 = OUT(e)$. Otherwise, by symmetry, neither x nor y is adjacent to v, and then $IN(e) \geq 0 = OUT(e)$.

We now consider the case where $G = (V, E)$ is $T_{1,m,m}$ with $m > 0$ even, that is,

$$V = \{v_0, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m\}$$

and

$$E = \{(v_0, v_i)\ i = 1, 2, \ldots, m\} \cup \{(v_i, w_i)\ i = 1, 2, \ldots, m\}\ .$$

We then set

$$E' = \left\{(v_0, v_i)\ i = 1, 2, \ldots, \frac{m}{2}\right\} \cup \left\{(v_i, w_i)\ i = 1, 2, \ldots, \frac{m}{2}\right\}\ .$$

For any edge $e = (v_0, v_i)$, we have $IN(e) = \frac{m}{2} = OUT(e)$. For an edge $e = (v_i, w_i)$, we have $IN(e) = 1$ and $OUT(e) = 0$. Therefore, E' and $E \backslash E'$ form a satisfactory partition of $L(G)$.

Finally, let us consider the case where G is $T_{1,m,m'}$ with $0 \leq m' < m$ or m odd. $L(G)$ is then a clique on vertices $e_1, e_2, \ldots, e_m$ (corresponding to the edges (v_0, v_i) of G), with m' additional vertices e'_i, each linked to e_i. Assume there is a satisfactory partition $(E', E\backslash E')$ of E, with $e_m \in E'$. Consider first the situation where $m' < m$. Then,

$$|E'| \geq IN(e_m) + 1 \geq \left\lceil \frac{m-1}{2} \right\rceil + 1 \ .$$

But for a vertex e_i in $E\backslash E'$ we now have

$$OUT(e_i) \geq |E'| \geq \left\lceil \frac{m-1}{2} \right\rceil + 1 > \frac{m}{2} \geq \frac{d_{L(G)}(e_i)}{2} \ ,$$

a contradiction. If $m' = m$ and m' is odd, we have

$$|E'| \geq IN(e_m) + 1 \geq \left\lceil \frac{m}{2} \right\rceil + 1 \ ,$$

and for a vertex e_i in $E\backslash E'$:

$$OUT(e_i) \geq |E'| - 1 \geq \frac{m+1}{2} > \frac{m-1}{2} = \left\lfloor \frac{d_{L(G)}(e_i)}{2} \right\rfloor \ ,$$

again a contradiction. □

The first part of this proof can be used to show the following result.

Theorem 5 *Let G be a graph without vertices of degree 2. Then its line-graph $L(G)$ is partitionable if and only if G is not a star $K_{1,m}$.*

Proof. Let v be a vertex of maximum degree in $G = (V, E)$. We partition the vertices of $L(G)$ into E' and $E\backslash E'$ with E' the set of edges $e = (v, y)$ of G. If G is not a star $K_{1,m}$, E' is a proper subset of E. We show that it is a satisfactory partition for $L(G)$. For $e = (v, y)$, we have:

$$IN(e) = d_G(v) - 1 \geq d_G(y) - 1 = OUT(e) \ .$$

For $e = (x, y) \in E\backslash E'$, we consider two cases. If both x and y are adjacent to v, their degree is at least 3, and we have

$$IN(e) = d_G(x) + d_G(y) - 4 \geq 2 = OUT(e) \ .$$

If x and y are not both adjacent to v, we have

$$IN(e) = d_G(x) + d_G(y) - 3 \geq 1 \geq OUT(e)$$

(G being connected x and y cannot both have degree 1). □

While it is possible to extend this result a little bit further (with conditions about the vertices of degree 2), it would be more interesting to determine exactly which line-graphs are partitionable and which are not.

5 Final remarks and open questions

The goal of this paper is to somewhat serve as an introduction to the Satisfactory Graph Partitioning (SGP) problem, by providing some initial theoretical results and provide interesting research directions. We showed that the only graphs of girth at least 5 that do not admit a partition satisfying all their vertices are the stars $K_{1,m}$. Similarly, we characterized the line-graphs of triangle-free graphs that admit a satisfactory partition. Unfortunately, the complexity of the general SGP problem remains open.

Open question 1 *Is the SGP problem NP-complete in general?*

As shown in [5], more general versions of the SGP problem with weight functions on the edges and/or the vertices of the graph are NP-complete in the strong sense. More precisely, the Satisfactory Graph Partitioning SGP(s, w) problem is defined as follows.

INSTANCE SGP(s, w): A graph $G = (V, E)$ and two weight functions $s : V \to \mathbb{Z}_+^*$ and $w : E \to \mathbb{Z}_+^*$.

QUESTION: Is there a partition of V in two non-empty subsets V' and $V \backslash V'$ such that each vertex is satisfied, that is, for each vertex $v \in V'$ ($v \in V \backslash V'$ respectively) we have $f_{V'}(v) \geq f_{V \backslash V'}(v)$ ($f_{V \backslash V'}(v) \geq f_{V'}(v)$ respectively) with $f_A(v) = \sum_{v' \in N(v) \cap A} w(v, v')s(v')$?

Both special cases where either s or w is a constant function, denoted by SGP$(1,w)$ and SGP$(s, 1)$ are NP-complete in the strong sense [5]. It would therefore be interesting to determine classes of graphs on which these more general problems become polynomially solvable. Such classes are already known for the unweighted problem SGP, as we have seen with the classes of graphs of girth at least 5 and the line-graphs of triangle-free graphs in this paper. The polynomiality of SGP on classes such as the cographs and distance hereditary graphs are consequences of the general results in [6].

While we exhibited various types of graphs that do not admit a satisfactory partition, it seems that the more vertices a non-partitionable graph has, the higher its maximum degree is. We could therefore set n_Δ to be the smallest value such that every graph with maximum degree Δ and at least n_Δ vertices is partitionable ($n_\Delta = \infty$ if non-partitionable graphs with maximum degree Δ can be arbitrarily large).

Open question 2 *Is n_Δ finite for every Δ?*

It is a simple exercise to see that $n_2 = 4$. Determining n_3 is also not very difficult. Indeed, consider a graph G with $\Delta(G) = 3$. If G has girth at least 5 and G is not $K_{1,3}$, G is partitionable by Corollary 2. If $G = (V, E)$ has girth 3, G has a triangle (v, w, x). Starting with $V' = \{v, w, x\}$, all vertices in V' are satisfied, and there are at most $c(V', V\backslash V') = 3$ edges with one endvertex in V' and the other endvertex in $V\backslash V'$. If there is an unsatisfied vertex y in $V\backslash V'$, we can switch it to V'. We still have that all the vertices in V' are satisfied, and $c(V', V\backslash V')$ has been reduced by at least 1 (since y was not satisfied). If there is still an unsatisfied vertex z in $V\backslash V'$, we proceed as for y and switch z over to V'. Again, V' contains only satisfied vertices, and $c(V', V\backslash V') \leq 1$. At this stage, if $V\backslash V'$ contains at least 2 vertices, then all the vertices in $V\backslash V'$ must be satisfied. If that was not the case, we could switch an unsatisfied vertex over to V' and we would have $c(V', V\backslash V') = 0$ which is not possible in a connected graph. In summary, if $G = (V, E)$ has girth 3 and has at least 7 vertices, G is partitionable. If G has girth 4, a similar analysis can be done. Starting with V' containing the vertices of a cycle of length 4 in G, and noticing that $c(V', V\backslash V') \leq 4$, we can prove that if G has at least 9 vertices, then G is partitionable. We therefore have $n_3 \leq \max(5, 7, 9) = 9$. Following the above procedure, it is an easy exercise to show that $n_4 \leq \max(6, 10, 13) = 13$.

Unfortunately, the same procedure doesn't work for larger values of Δ, since a cycle does no longer consist in a set of satisfied vertices. But if a graph G has a clique on $\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$ vertices, the above technique can be applied by starting with the vertices of such a clique. One can then prove that if a graph G has a clique of size $\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$ and has at least $\left\lceil \frac{\Delta^2+4\Delta+7}{4} \right\rceil$ vertices, then G is partitionable.

Open question 3 *What other sufficient conditions for a graph to be partitionable can be found, in particular for graphs with low minimum degree and small cycles?*

Independently of the complexity of the SGP problem, it would be interesting to be able to characterize which graphs in certain classes are or are not partitionable. A case study on the length of the largest induced cycle in an outerplanar graph provides a characterization of the partitionable outerplanar graphs. Indeed, the only outerplanar graph without leaves that is not partitionable consists of a path on $m \geq 2$ vertices and an additional vertex adjacent to all the vertices of the path.

Open question 4 *Is there a good characterization of partitionable planar graphs?*

Bipartite graphs are also of interest, since their structure often simplify the study of combinatorial problems on this class of graphs.

Open question 5 *Is there a good characterization of partitionable bipartite graphs?*

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References

[1] R. Aharoni, E.C. Milner and K. Prikry, Unfriendly partitions of a graph, J. Combinatorial Theory Series B **50** (1990), 1–10.

[2] C. Berge, Graphs and hypergraphs, North-Holland, Amsterdam, 1973.

[3] A.A. Diwan, Decomposing graphs with girth at least 5 under degree constraints, J. Graph Theory **33** (2000), 237–239.

[4] M.U. Gerber and D. Kobler, Algorithmic approach to the satisfactory graph partitioning problem, European J. Operational Research **125** (2000), 283–291.

[5] M.U. Gerber and D. Kobler, Partitioning a weighted graph to satisfy all vertices, Technical Report ORWP 98/03, Swiss Federal Institute of Technology, 1998.

[6] M.U. Gerber and D. Kobler, Algorithms for vertex partitioning problems on graphs with fixed clique-width, Theoretical Computer Science **299** (2003), 719–734.

[7] A. Kaneko, On decomposition of triangle-free graphs under degree constraints, J. Graph Theory **27** (1998), 7–9.

[8] M. Luby, A Simple parallel algorithm for the maximal independent set problem, SIAM J. Computing **15** (1986), 1036–1053.

[9] H. Nagamochi, K. Nishimura and T. Ibaraki, Computing all small cuts in an undirected network, SIAM J. Discrete Math. **10** (1997), 469–481.

[10] A.A. Schäffer and M. Yannakakis, Simple local search problems that are hard to solve, SIAM J. Computing **20** (1991), 56–87.

[11] S. Shelah and E.C. Milner, Graphs with no unfriendly partitions, in A. Baker, B. Bollobás and A. Hajnal (eds.), A Tribute to Paul Erdös, Cambridge University Press, 1990, 373–384.

[12] M. Stiebitz, Decomposing graphs under degree constraints, J. Graph Theory **23** (1996), 321–324.

Note added in proof

In a paper titled "Complexity of the satisfactory partition problem" written after this paper has been accepted for publication, C. Bazgan, Z. Tuza and D. Vanderpooten have been able to provide a positive answer to open question 1 (SGP is NP-complete).

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