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Domination and irredundance in tournaments

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Abstract

We study the concepts of minimal dominating and maximal irredundant sets of vertices in tournaments.

1 Introduction

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. Domination in graphs is a well-studied branch of graph theory, and is the subject of two books by Haynes, Hedetniemi and Slater [8, 9]. However, about 90% of the papers on domination have considered only undirected graphs. Thus, relatively little is known about domination and related concepts in directed graphs, and much of what is known is related to the study of kernels in digraphs. For an excellent survey of most of this literature the reader is referred to a chapter on this topic by Ghoshal, Laskar and Pillone [6]. The focus of this paper is the application of the concepts of domination and irredundance in undirected graphs to the study of tournaments. These terms are defined in the next section.

2 Definitions and terminology

Let $D = (V, A)$ be a directed graph with a set of *vertices* V and a set $A \subseteq V \times V$ of directed edges, called *arcs*. If $(u, v) \in A$, we write $u \to v$ and say u *dominates* v

or u *beats* v. Define the *outset* of a vertex $v \in V$ as $O(v) = \{w \in V | v \to w \in A\}$ and the *inset* of v as $I(v) = \{u \in V | u \to v \in A\}$. We also define $O[u] = O(u) \cup \{u\}$ and $I[u] = I(u) \cup \{u\}$. The *outdegree* of a vertex u is defined as $od(u) = |O(u)|$. Similarly, the *indegree* of u is $id(u) = |I(u)|$. In the obvious way, we can define $O(S)$ for any subset $S \subseteq V$ by: $O(S) = \bigcup_{v \in S} O(v)$. The definitions of $I(S)$, $I[S]$ and $O[S]$ are similar. Also, let $\Delta^+(D) = \max\{od(u) | u \in V\}$.

A digraph T is a *tournament* if for every pair u, v of distinct vertices, either $u \to v$ or $v \to u$, but not both. Furthermore, if vertex u beats every vertex in a set S we use the notation $u \Rightarrow S$. Note that if $T = (V, A)$ is a tournament, then the subgraph $T[S] = (S, A \cap S \times S)$ *induced* by any subset $S \subseteq V$ is also a tournament.

The subtournament induced by a set $S \subseteq V$ is *transitive* if its vertices can be (uniquely) ordered $u_1, u_2, \ldots, u_k$, such that $u_i \to u_j \in A$ if and only if $i < j$. In part of this paper we are interested in transitive subtournaments of tournaments. Let us define $tr(T)$ and $TR(T)$ to equal the minimum and maximum orders, respectively, of a maximal transitive subtournament of a tournament T.

A set $S \subseteq V$ is a *dominating set* in a directed graph $D = (V, A)$ if for every vertex $v \in V - S$ there exists a vertex $u \in S$ for which $u \to v \in A$; equivalently, S is a dominating set if $O[S] = V(D)$. The *domination number* of a digraph D, denoted $\gamma(D)$, equals the minimum cardinality of a dominating set in D. The *upper domination number* of D, denoted $\Gamma(D)$, equals the maximum cardinality of a minimal dominating set in D.

In a directed graph $D = (V, A)$, if $S \subseteq V$, the *private neighbor set* of a vertex $u \in S$ with respect to S is the set $pn(u, S) = O[u] - O[S - \{u\}]$. So $pn(u, S) = \{x \in O[u] | x \Rightarrow S - \{u\}\}$. A set $S \subseteq V$ is *irredundant* if for every vertex $u \in S$, $pn(u, S) \neq \emptyset$. If $pn(u, S) \neq \emptyset$ then every vertex in $pn(u, S)$ is called a *private neighbor* of u (with respect to S). Note that if $u \in pn(u, S)$, then no vertex in $S - \{u\}$ dominates u, from which it follows that if S is an irredundant set in a tournament T, then at most one vertex $u \in S$ can satisfy $u \in pn(u, S)$. Let $ir(D)$ and $IR(D)$ denote, respectively, the minimum and maximum cardinalities of a maximal irredundant set of vertices in D; these invariants are called the *irredundance number* and the *upper irredundance number* of D, respectively.

In a directed graph $D = (V, A)$, a set $S \subseteq V$ is called *independent* if no two vertices in S are joined by an arc. The *independent domination number*, $i(D)$, and the *independence number*, $\beta(D)$, equal the minimum and maximum cardinalities, respectively, of a maximal independent set in D.

Analogously, the independence, domination and irredundance parameters can be defined for undirected graphs G, and are related by the following well-known inequality chain [1]:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G). \tag{1}$$

However, for arbitrary directed graphs this sequence of inequalities does not hold. The first and fifth of these inequalities follow from the following simple result.

Proposition 1 *Every minimal dominating set in a digraph D is a maximal irredundant set.*

Corollary 1 *For any digraph D,*

$$ir(D) \leq \gamma(D) \leq \Gamma(D) \leq IR(D).$$

For undirected graphs, it is easy to see that every maximal independent set is a minimal dominating set, and therefore

$$\gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G).$$

However, these inequalities are not always true for directed graphs. Consider the directed 3-cycle D_1 with three vertices, u, v and w, and with $u \to v$ and $v \to w$ and $w \to u$. For this graph,

$$\gamma(D_1) = 2 > i(D_1) = 1.$$

Also, for the simple digraph D_2 consisting of three vertices x, y and z, with $x \to y$ and $x \to z$, one can see that

$$\beta(D_2) = 2 > \Gamma(D_2) = 1.$$

The problem is that maximal independent sets in digraphs are not necessarily dominating sets, and as we can see from the digraph D_1, not every digraph has an independent dominating set. Furthermore, no directed cycle of odd length has an independent dominating set.

This is well known to those who study kernels in digraphs, which are defined as follows. A set $S \subseteq V$ in a digraph $D = (V, A)$ is called *absorbant* if for every vertex $v \in V - S$ there is a vertex $u \in S$ such that $v \to u$. That is, S is a dominating set in the *directional dual* $D^* = (V, A^*)$, where $A^* = \{u \to v | v \to u \in A\}$. A set S which is both independent and absorbant is called a *kernel*.

So a set S is an independent dominating set in a digraph D if and only if it is a kernel in the dual digraph D^*.

Thus, while in general it is not true that

$$\gamma(D) \leq i(D) \leq \beta(D) \leq \Gamma(D),$$

it is true for all digraphs D which have at least one independent dominating set. It is worth noting, however, that it is an NP-complete problem to decide, given an arbitrary digraph D, whether D has an independent dominating set [5].

Some Gallai-type results involving pairs of $\gamma(D)$, max degree in D, $i(D)$ and $ir(D)$ have been obtained by Merz and Stewart [13].

The next result establishes a relationship between dominating sets and transitive subtournaments in a digraph.

Proposition 2 *The vertex set of every maximal transitive subtournament of a tournament T is a dominating set, but not necessarily a minimal dominating set of T.*

Proof: Let $S = \{u_1, u_2, \ldots, u_k\}$ be the unique ordering of the vertices of a maximal transitive subtournament of a tournament T. Assume that S is not a dominating set. Then there exists a vertex $w \in V(T) - S$ which is not dominated by any vertex in S, i.e. $w \rightarrow u_i$ for every vertex $u_i \in S$. But this means that $S' = \{w, u_1, u_2, \ldots, u_k\}$ is a transitive subtournament of T, which contradicts the maximality of S. □

Corollary 2 *For every tournament T,*

(i) $ir(T) \leq \gamma(T) \leq tr(T) \leq TR(T) \leq \Delta^+(T)$;
(ii) $ir(T) \leq \gamma(T) \leq \Gamma(T) \leq IR(T)$;
(iii) $\gamma(T) \leq n - \Delta^+(T)$;
(iv) $\gamma(T) \leq \delta^-(T) + 1$.

We note that in general, no inequalities hold between either of $\{tr(T), TR(T)\}$ and either of $\{\Gamma(T), IR(T)\}$. For example, for the transitive tournament TT_n on n vertices, $\Gamma(TT_n) = IR(TTn) = 1$, while $tr(TT_n) = TR(TT_n) = n$.

Proposition 3 *If T is a tournament that is not strongly connected, with strong components $T_1, T_2, \ldots, T_m$, where every vertex in T_i dominates every vertex in T_j whenever $1 \leq i < j \leq m$, then*

(i) $\gamma(T) = \gamma(T_1)$;
(ii) $\Gamma(T) = \Gamma(T_1)$;
(iii) $ir(T) = min\{ir(T_i) | 1 \leq i \leq m\}$;
(iv) $IR(T) = max\{IR(T_i) | 1 \leq i \leq m\}$.

3 Domination in tournaments

We quote freely from [8] in order to review some pertinent background. In 1962, K. Schütte [3] indirectly raised the question of whether there exist tournaments with arbitrarily large domination numbers. In fact, he raised a slightly different question: given any positive integer $k > 0$, does there exist a tournament $T_{n(k)}$ in which for any set S of k vertices, there is a vertex u which dominates all vertices in S. Such a tournament is said to have property S_k.

Notice the following. If a tournament does not have property S_k, then there exists a set S' of k vertices such that for every vertex $w \notin S'$, there is a vertex $v \in S'$ for which $v \rightarrow w$. That is, S' is a dominating set in T of order k. Thus, a tournament T has property S_k if and only if $\gamma(T) > k$, or $\gamma(T) = k$ if and only if T has property S_{k-1} but does not have property S_k.

In [3] Erdös showed, by probabilistic arguments, that such a tournament $T_{n(k)}$ does exist, for every positive integer k.

Proposition 4 *If $T_{n(k)}$ has property S_k, then there is a tournament W of order $n(k) + 1$ with property S_k.*

Proof: Fix vertex $u \in V(T_{n(k)})$. Form W from $T_{n(k)}$ by adjoining a new vertex u' such that $O(u') = O[u]$ and $I(u') = I(u)$. Let $S \subseteq V(W)$, $|S| = k$. If $u' \notin S$,

$S \subseteq V(T_{n(k)})$, so there is a vertex $v \in V(T_{n(k)})$ so that $v \Rightarrow S$, since $T_{n(k)}$ has property S_k. If $u' \in S$, either $u \notin S$ or $u \in S$.

If $u \notin S$, then $|(S - \{u'\}) \cup \{u\}| = k$, and $(S - \{u'\}) \cup \{u\} \subseteq V(T_{n(k)})$. As $T_{n(k)}$ has S_k, there is a vertex $v' \in V(T_{n(k)})$ with $v' \Rightarrow (S - \{u'\}) \cup \{u\}$. In particular, $v' \in I(u)$. Thus, $v' \to u'$ in W, so $v' \Rightarrow S$ in W.

If $u \in S$, let z be any vertex of $V(T_{n(k)})$, $z \notin S$. Then $|(S - \{u'\}) \cup \{z\}| = k$ and $(S - \{u'\}) \cup \{z\} \subseteq V(T_{n(k)})$, so there is a vertex $v" \in V(T_{n(k)})$ with $v" \Rightarrow ((S - \{u'\}) \cup \{z\}$. Since $u \in (S - \{u'\}) \cup \{z\}$, $v" \in I(u)$. So $v" \to u'$ in W, and $v" \Rightarrow S$ in W. In any case, there is a vertex $w \in W$ so that $w \Rightarrow S$. So, W has S_k.
□

Corollary 3 *If $T_{n(k)}$ has property S_k, then for every $n \geq n(k)$, there is a tournament of order n with property S_k.*

If we let $f(k)$ be the minimum value of $n(k)$ for which a $T_{n(k)}$ exists, then Erdös showed that

$$f(k) \leq k^2 2^k (\log 2 + \epsilon),$$

for any $\epsilon > 0$, provided k is sufficiently large.

We can restate this theorem as follows.

Theorem 1 *(Erdös) For every $\epsilon > 0$, there is an integer K such that for every $k \geq K$, there exists a tournament T_k with no more than $k^2 2^k(\log 2 + \epsilon)$ vertices, for which $\gamma(T_k) > k$.*

Proof: Let T be a random tournament on n vertices, where for every pair of vertices u and v, either the $u \to v$ arc or the $v \to u$ arc is chosen with equal probability, and independently of the other arcs of T. The probability, therefore, that vertex u dominates vertex v is 1/2. For every set S of k vertices and every vertex $u \notin S$, the probability that u dominates every vertex in S is 2^{-k}. The probability that S is a dominating set is therefore $(1 - 2^{-k})^{n-k}$. The expected number of dominating sets of cardinality k is

$$\binom{n}{k}(1 - 2^{-k})^{n-k}.$$

If n is sufficiently large, the value of this expression will be less than 1, and therefore there exists a tournament T on n vertices with $\gamma(T) > k$. In fact, if $n > k^2 2^k(\log 2 + \epsilon)$ then

$$\binom{n}{k}(1 - 2^{-k})^{n-k} < 1.$$

□

The fact that there are tournaments with $\gamma(T) > k$, for arbitrary positive integers k, is also discussed in Moon's 1968 monograph on tournaments [14] (cf. Exercise 5, p. 32). Let $\gamma(n)$ be the maximum of $\gamma(T)$ over all tournaments T with n vertices, so that for each n there is some tournament with n vertices for which $\gamma(T) = \gamma(n)$. Moon attributes the following result to Leo Moser (without any reference):

$$\log n - 2\log(\log n) \leq \gamma(n) \leq \log(n+1),$$

where $n \geq 2$ and log is to the base 2. Thus, there is a tournament T for which $\gamma(T) \geq \log n - 2\log(\log n)$, i.e. for every positive integer k there is a tournament T for which $\gamma(T) > k$.

Szekeres and Szekeres [17] later established a lower bound for $f(k)$:

$$(k+2)2^{k-1} - 1 \leq f(k). \tag{2}$$

Still later, Graham and Spencer [7] gave an explicit construction of a tournament $T_{n(k)}$ which has property S_k, although their construction takes $n(k)$ to be larger than $k^2 2^{2k-2}$. Their construction is as follows:

Select the smallest prime number $p > k^2 2^{2k-2}$, where $p \equiv 3 \pmod 4$. The vertices of $T_{n(k)}$ correspond to $\{0, 1, \ldots, p-1\}$. For two distinct vertices u and v, $u \to v$ if $u - v \equiv a^2 \pmod p$, for some $a \in \{0, 1, \ldots, p-1\}$. This is the quadratic residue tournament, denoted QRT_p.

They pointed out, however, the following [7]:

> The value $k^2 2^{2k-2}$ is nearly the square of the nonconstructive upper bound of Erdös. Specific constructions show that much smaller values p suffice to endow T_p with property S_k. For example, QRT_7 has property S_2 and QRT_{19} has property S_3. In [17] it is shown that $f(2) = 7$ and $f(3) = 19$, so that these tournaments are minimal. Also, it is true that QRT_{67} has property S_4. Since (2) gives $f(4) \geq 47$, it is possible that QRT_{67} is also minimal.

There is another method for constructing tournaments with arbitrarily large domination number, but the order of the tournaments becomes quite large. Recall that a tournament T has property S_k if and only if $\gamma(T) > k$.

Theorem 2 *(Tyszkiewicz [18]) Let T be a tournament with n-set V as vertex set. Suppose T has property S_k. Form a new tournament W with n^3-set $V \times V \times V$ as vertex set in which (a_1, b_1, c_1) beats (a_2, b_2, c_2) if at least two of the pairs (a_1, a_2), (b_1, b_2), (c_1, c_2) describe arcs in T (and all* other *arcs in W are abritrary). Then W has property $S_{\lfloor 3k/2 \rfloor}$.*

Starting with QRT_7 (which has property S_2) and using Theorem 2 repeatedly yields a tournament with 7^{3^m} vertices that has property S_k, with k slightly smaller than $2(3/2)^m$. Although the order of this tournament is much larger than required by Erdös'proof and by the construction of Graham and Spencer, the construction is simple and elementary.

As pointed out by Duncan and Jacobson [2], whenever there is a tournament T with $\gamma(T) > k$, by deleting vertices one can obtain a tournament T' with $\gamma(T') = k$. Duncan and Jacobson also give a construction of a tournament with exactly m minimum dominating sets of order k. It starts with a tournament T of order n with $\gamma(T) > k$ and requires $(k + m - 1) + kn$ vertices. We describe the construction for $m = 1$.

Theorem 3 *(Duncan and Jacobson [2]) Let T be a tournament with $\gamma(T) > k$. Form a new tournament W as follows: use k copies of T, $T_1, T_2, \ldots, T_k$, where the copy of vertex a of T in T_i is denoted (a, i), $1 \leq i \leq k$, and add k new vertices $x_1, x_2, \ldots, x_k$. In W, vertex x_i beats each vertex in $V(T_j)$ if and only if $i = j$ $(1 \leq i \leq k;\ 1 \leq j \leq k)$, each of $x_2, x_3, \ldots, x_k$ beats x_1, for $a \neq b$ in $V(T)$, (a, i) beats (b, j) if and only if a beats b in T, and for $a = b$ in $V(T)$, (a, i) beats (b, j) if and only if $i < j$. Arcs of W between vertices in $\{x_2, x_3, \ldots, x_k\}$ are arbitrary. Then $\{x_1, x_2, \ldots, x_k\}$ is the unique minimum dominating set in W, and $\gamma(W) = k$.*

In 1988 Megiddo and Vishkin [12] revisited this old problem, but from a computational point of view.

TOURNAMENT DOMINATING SET
INSTANCE: A tournament $T = (V, A)$ and a positive integer k.
QUESTION: Does T have a dominating set of cardinality at most k?

The following theorem is attributed by Moon to Erdös (cf. [14], p. 28).

Theorem 4 *(Erdös) If T is a tournament with $n \geq 2$ vertices, then $\gamma(T) \leq \lceil \log_2 n \rceil$.*

Proof: If $od(u)$ equals the number of vertices dominated by u, then clearly $\sum_{u \in V} od(u) = n(n-1)/2$. It follows that there must be at least one vertex which dominates at least $\lceil \frac{(n-1)}{2} \rceil$ vertices. Select a vertex u_1 which dominates at least $\lceil \frac{(n-1)}{2} \rceil$ vertices. We remove this vertex and all of the vertices it dominates. We repeat this process on the remaining tournament which has at most $\lceil \frac{(n-1)}{2} \rceil$ vertices, by selecting a second vertex u_2 which dominates at least half of the remaining vertices, and then deleting u_2 and the vertices it dominates. By continuing this process we can find a dominating set with no more than $\lceil \log_2 n \rceil$ vertices. □

Corollary 4 *(Megiddo, Vishkin) A minimum dominating set in a tournament can be found in $n^{O(\log n)}$ time.*

Proof: The proof of Theorem 4 implies that a minimum dominating set can be found by examining all subsets of V of cardinality no greater than $\lceil \log_2 n \rceil$. There are $\Sigma_{i=1}^{\lceil \log_2 n \rceil} \left(\begin{array}{c} n \\ i \end{array} \right)$ such subsets. □

In effect, what Megiddo and Vishkin are saying is that there is an algorithm for computing the domination number of a tournament which runs in subexponential, yet superpolynomial time. It remains an open problem whether it is possible to compute the domination number of a tournament in polynomial time.

Figure 1 provides examples of tournaments with domination numbers 2 and 3. The tournament QRT_3 in Figure 1 is called the cyclic triple and requires two vertices to dominate it; in fact it is the unique smallest tournament with $\gamma(T) = 2$. The tournament $C_3[C_3]$, which is the composition of the cyclic triple with itself, consists of three cyclic triples; all of the vertices in one cyclic triple beat all of the vertices in the next cyclic triple, in cyclic order. It is a nice exercise to show that $\gamma(C_3[C_3]) = 3$. The tournament labeled QRT_7 is the smallest tournament with $\gamma(T) = 3$.

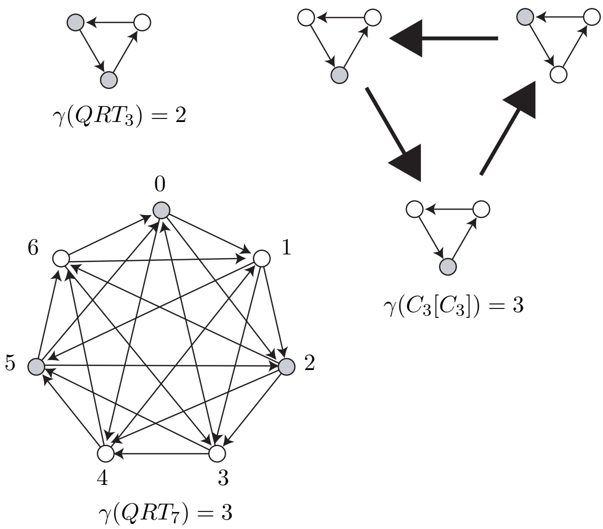

Figure 1: Tournaments with small domination numbers.

Proposition 5 *For every tournament T with less than seven vertices, $\gamma(T) \leq 2$.*

Proof: Let T be a tournament with six vertices. Since T must have at least one vertex which dominates at least $\lceil (n-1)/2 \rceil = 3$ vertices, select a vertex of maximum outdegree, say u, for a dominating set of T. If $od(u) = 5$, then $\{u\}$ is a dominating set for T and $\gamma(T) = 1$. If $od(u) = 4$, then $\gamma(T) > 1$ and exactly one vertex v dominates u, so $\{u, v\}$ is a dominating set for T and $\gamma(T) = 2$. So, assume that $od(u) = 3$. At most two vertices, say x and y are left undominated. Either $x \to y$ or $y \to x$. Thus, either $\{u, x\}$ or $\{u, y\}$ is a dominating set for T. Notice that if every tournament with six vertices has $\gamma(T) \leq 2$, then every tournament with fewer than six fewer vertices also satisfies $\gamma(T) \leq 2$. □

In order to prove a result about tournaments with $\gamma(T) \leq 3$, we will need a few preliminary results. Proposition 3 yields the following:

Observation 1 *If T is a tournament on n vertices and $\gamma(T) = k$, then for every $m > n$ there exists a tournament T_m on m vertices with $\gamma(T_m) = k$.*

Proposition 6 *If T is a tournament for which $\gamma(T) = k$, for some $k > 1$, then T contains a subtournament W for which $\gamma(W) = k - 1$.*

Proof: Let T be a tournament for which $\gamma(T) = k$ and let $S = \{x_1, x_2, \ldots, x_k\}$ be a dominating set in T. By Proposition 1, S is irredundant. Let $pn(x_i, S) = \{x \in V(T) \cap O[x_i] | x \Rightarrow S - \{x_i\}\}$ be the private neighbor set of x_i. Note that every vertex in $S' = V(T) - (\bigcup_{i=1}^{k} pn(x_i, S) \cup S)$ is dominated by at least two vertices in S.

Let $T[W]$ be the subtournament of T induced by the set $W = V(T) - (\{x_i\} \cup pn(x_i, S))$. Now, $S - \{x_i\}$ is a dominating set of $T[W]$ of cardinality $k-1$. For, if $z \in W - S$, and $z \Rightarrow S - \{x_i\}$, then $x_i \rightarrow z$, as S is a dominating set of T. But then $z \in pn(x_i, S)$, contrary to the assumption that $z \in W$. So, $S - \{x_i\}$ is a dominating set of $T[W]$.

Any dominating set S' in $T[W]$ of cardinality less than or equal to $k-2$ yields a dominating set $S' \cup \{x_i\}$ in T with cardinality at most $k-1$, which contradicts the fact that $\gamma(T) = k$. Thus, $\gamma(T[W]) = k-1$. □

Corollary 5 *If T is a tournament for which $\gamma(T) = k$, then there exist nested subtournaments $T[W_1] \subset T[W_2] \subset \ldots \subset T[W_{k-1}] \subset T$ such that for $1 \leq i \leq k-1$, $\gamma(T[W_{k-i}]) = k-i$.*

Proposition 7 *Let T be a smallest tournament such that $\gamma(T) = k \geq 2$. Then for every vertex $v \in V(T)$, $\gamma(T[I(v)]) = k-1$.*

Proof: Let $v \in V(T)$. Since $\gamma(T) = k \geq 2$, we know that $|I(v)| \geq 1$. If $T[I(v)]$ has a dominating set S' with at most $k-2$ vertices, then $S' \cup \{v\}$ is a dominating set of T with at most $k-1$ vertices, which contradicts the assumption that $\gamma(T) = k$. On the other hand, if $\gamma(T[I(v)]) \geq k+r$ $(r \geq 0)$, then by Corollary 5, $T[I(v)]$ contains a subtournament $T[W]$ with $\gamma(T[W]) = k$. But W has fewer vertices than T, which contradicts the minimality of T. □

Proposition 8 *Let T be a smallest tournament with $\gamma(T) = k$, and let T have n vertices. Then a smallest tournament W with $\gamma(W) = k+1$ must have at least $2n+1$ vertices.*

Proof: Let W be a smallest tournament with $\gamma(W) = k+1$, and let W have m vertices. Let v be any vertex in W with $od(v) \geq \lceil (m-1)/2 \rceil$. Then the tournament $T'[I(v)]$ has at most $m - \lceil (m-1)/2 \rceil - 1$ vertices, and has, by Proposition 7, $\gamma(T'[I(v)]) = k$. But since a smallest tournament T with $\gamma(T) = k$ has n vertices, we have $n \leq m - \lceil (m-1)/2 \rceil - 1$ vertices. This implies that $m \geq 2n+1$. □

Proposition 9 *Let T be a smallest tournament with $\gamma(T) = k \geq 2$. Then for every vertex $v \in V(T)$, if S is a minimum dominating set for $T[I(v)]$ (and so, by Proposition 7, $|S| = k-1$), then*

- *a. at least one vertex $w \in O(v)$ dominates all vertices in S (otherwise, S dominates all of T);*
- *b. no set $S' \subseteq I(v)$ with $|S'| < k-2$ dominates all vertices in $V(T) - O(v) - S$ (otherwise, $\{v\} \cup \{w\} \cup S'$ dominates T).*

Theorem 5 *For every tournament with less than 19 vertices, $\gamma(T) \leq 3$.*

Proof: Let T be a smallest tournament with $\gamma(T) = 4$, and let v be an arbitrary vertex in $V(T)$. Then $\gamma(T[I(v)]) = 3$ (by Proposition 7) and $|V(T[I(v)])| \geq 7$ (by the comment preceding Proposition 5). Let u be a vertex having maximum outdegree in $T[I(v)]$. Suppose that $id(v) \leq 8$. Then there will be three vertices in $I(v) - O[u]$ (if fewer, then $\gamma(T[I(v)]) < 3$). If $S = I(v) - O[u]$ is a (minimum) dominating set for $T[I(v)]$, then, if w is as in condition a in Proposition 9, $\{u, v, w\}$ is a dominating set for T, contradicting the assumption that $\gamma(T) = 4$. If $I(v) - O[u]$ is not a dominating set for $T[I(v)]$, then there exists some vertex $w \in O(u) \cap I(v)$ that dominates every vertex in $I(v) - O[u]$. But then $\{u, w\}$ forms a dominating set for $T[I(v)]$, contradicting the assumption that $\gamma(T[I(v)]) = 3$.

Therefore, $id(v) \geq 9$, and since vertex v was chosen arbitrarily, all vertices in T must have indegree at least 9. Therefore, T must must have at least 19 vertices. □

We next present a quadratic residue tournament QRT_{19} on 19 vertices whose domination number equals four. The vertices of QRT_{19} are labeled $\{0, 1, \ldots, 18\}$. For $0 \leq j \leq 18$, vertex j dominates vertex $(j+k)$ mod 19, for all $k \in \{1, 4, 5, 6, 7, 9, 11, 16, 17\}$.

By constructing the adjacency matrix of QRT_{19}, one can verify by hand that no set of three vertices dominates QRT_{19}, while, for example, $\{0, 1, 5, 8\}$ is a minimum cardinality dominating set. So, $\gamma(QRT_{19}) = 4$.

Thus, from Theorem 5, we know that a smallest tournament T with $\gamma(T) = 4$ has 19 vertices. It is interesting to note that since $k^2 2^{2k-2} = (16)2^6 = 1024$, the Graham and Spencer construction requires 1031 vertices in order to construct a tournament with $\gamma(T) = 4$.

We report some computational results due to Fisher [4]. A *rotational tournament* of order $n = 2m+1$ has as its vertex set $\{0, 1, 2, \ldots, 2m\}$ and vertex i beats vertex j whenever $j - i \in S$, where S is an m-subset of $\{1, 2, \ldots, 2m\}$ such that $s_1 + s_2 \neq 0$ for all $s_1, s_2 \in S$, where arithmetic is modulo $n = 2m + 1$. For example, QRT_p, defined earlier, is a rotational tournament where S is the set of quadratic residues modulo p. Fisher verified via computer that

- the smallest rotational tournament with domination number 5 is QRT_{67},
- the smallest QRT_p with domination number k has $p = 331$ if $k = 6$, and has $p = 1163$ if $k = 7$,
- the smallest QRT_p with domination number 8 has $p \geq 3079$.

Lu, Wang and Wong [11] studied bounded domination numbers of tournaments and proved that the minimum number of stars of degree at most k needed to cover the vertex set is $\lceil n/(k+1) \rceil$ for a tournament of order $n \geq 14k \log k$.

An upper bound on another domination parameter, the α-domination of a tournament, has been studied by Langley, Merz, Stewart and Ward [10].

4 Irredundance in Tournaments

In this section we will examine some of the basic properties of irredundant sets in tournaments and provide results relating the irredundance and domination numbers of tournaments.

A *transmitter* in a digraph is a vertex v with $id(v) = 0$. A *receive*r is a vertex with $od(v) = 0$. So, a transmitter in a tournament is a vertex which beats every other vertex, while a receiver is beaten by every other vertex. Clearly, for any tournament T, $\gamma(T) = 1$ if and only if T has a transmitter.

Proposition 10 *For any tournament T, $\gamma(T) = 2$ if and only if T has no transmitter and there exist distinct vertices u and v such that $I(u) \subseteq O(v)$.*

Proof: Assume that T has no transmitter (so $\gamma(T) > 1$) and contains two distinct vertices u and v such that $I(u) \subseteq O(v)$. Consider any other vertex w. Either u dominates w or not. If u does not dominate w then $w \in I(u)$. But this implies that $w \in O(v)$, i.e. v dominates w. Thus, the set $\{u, v\}$ is a dominating set and $\gamma(T) = 2$.

Conversely, assume that $\gamma(T) = 2$ and let $\{u, v\}$ be a minimum dominating set. Clearly T can have no transmitter, else $\gamma(T) = 1$. Consider the set $I(u)$. Vertex u does not dominate any vertex in $I(u)$. But since $\{u, v\}$ is a dominating set, it must be the case that $v \Rightarrow I(u)$, and therefore $I(u) \subseteq O(v)$, as required. $\square$

Lemma 1 *Let u and v be two distinct vertices in a tournament T. Then u and v are contained in some (directed) 3-cycle if and only if $\{u, v\}$ is an irredundant set.*

Proof: Without loss of generality, assume that $u \to v$. Let $S = \{u, v\}$. Suppose u and v are contained in some 3-cycle with vertex w. Then $v \to w$ and $w \to u$. Note that $u \Rightarrow S - \{u\}$, so $u \in pn(u, S)$. Also, $w \in O[v]$ and $w \Rightarrow S - \{v\}$, so $w \in pn(v, S)$. As every vertex in S has a private neighbor, S is irredundant.

Conversely, assume that S is irredundant. So v has a private neighbor, say z. Then $z \in O[v]$ and $z \Rightarrow S - \{v\}$, that is, $z \in O(v) \cup \{v\}$ and $z \to u$. Now $z \neq v$, since $u \to v$, so $z \in O(v)$ and $T[\{u, v, z\}]$ is a 3-cycle in T. $\square$

Proposition 11 *For any tournament T, $ir(T) = 1$ if and only if T contains a strong component which consists of a single vertex.*

Proof: Recall that every singleton set is irredundant. It follows that $ir(T) = 1$ if and only if there exists a vertex u which has the property that for every vertex $v \neq u$, the set $\{u, v\}$ is not irredundant. By Lemma 1, this is equivalent to saying that u and v are in no 3-cycle, for all $v \in V(T)$. And this is equivalent to saying that $\{u\}$ is a strong component of T. $\square$

Corollary 6 *For any tournament T, $ir(T) = 1$ if and only if T contains a vertex u such that for every vertex $v \neq u$, either (i) $v \Rightarrow O[u]$ or (ii) $u \Rightarrow O[v]$.*

The regular tournament of order 5 in Figure 2 is an example of a tournament with $ir(T) = 2$. Since any two vertices are in a 3-cycle, any two vertices form an irredundant set. Furthermore, $\gamma(T) > 1$ by Proposition 11.

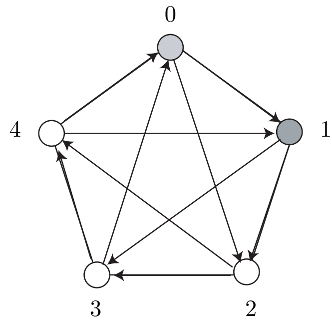

Figure 2: A tournament with $ir(T) = 2$.

The quadratic residue rotational tournament QRT_7, shown in Figure 1, is an example of a tournament with $ir(QRT_7) = 3$. Every pair of vertices is in a 3-cycle, so all 2-sets of vertices are irredundant, by Lemma 1. Let $\{i, j\}$ be any 2-set of vertices. We show that there is a $k \neq i, j$ so that $\{i, j, k\}$ is irredundant.

Without loss of generality, assume that $i \to j$. So, $j - i \in \{1, 2, 4\}$. If $j - i = 1$, then $\{i, i+1, i+2\}$ is irredundant, since $i \Rightarrow S - \{i\}$, $i+1 \to i+5$, $i+5 \Rightarrow S - \{i+1\}$, $i + 2 \to i + 6$, and $i + 6 \Rightarrow S - \{i + 2\}$.

If $j - i = 2$, then $\{i, i + 1, i + 2\}$ is irredundant as above.

If $j - i = 4$, then $\{i, i+3, i+4\}$ is irredundant, since $i+3 \Rightarrow S - \{i+3\}$, $i \to i+2$, $i + 2 \Rightarrow S - \{i\}$, $i + 4 \to i + 6$, and $i + 6 \Rightarrow S - \{i + 4\}$.

That is, no irredundant set of two vertices is maximal. Thus, $ir(QRT_7) \geq 3$.

Finally, by Corollary 2(i) and the comments preceding Proposition 5, $ir(QRT_7) \leq \gamma(QRT_7) = 3$. Thus, $ir(QRT_7) = 3$.

Note also that the rotational tournament QRT_{19} given earlier is an example of a tournament for which $ir(T) = 4$. In QRT_{19}, $S = \{0, 1, 5, 8\}$ is an irredundant set, since $0 \to 4$ and $4 \Rightarrow S - \{0\}$, $1 \to 18$ and $18 \Rightarrow S - \{1\}$, $5 \to 3$ and $3 \Rightarrow S - \{5\}$, $8 \to 13$ and $13 \Rightarrow S - \{8\}$. Furthermore, since there is no irredundant set with five vertices, $\{0, 1, 5, 8\}$ is a maximal irredundant set. The fact that $ir(T) = 4$ follows from the observation that QRT_{19} has no maximal irredundant set of size less than four.

Proposition 12 *For every positive integer k, there is a tournament T for which*

$$ir(T) = 1 < \gamma(T) = k.$$

Proof: Let T_2 be the unique tournament on two vertices, $u \to v$. Let T_k be any tournament with $\gamma(T_k) = k$. We know that such a tournament exists by Erdös' Theorem 1. Now construct a tournament T from T_2 and T_k by adding the arcs in $V(T_k) \Rightarrow V(T_2)$. Since v is a receiver, $ir(T) = 1$, yet $\gamma(T) = \gamma(T_k) = k$. □

The following is an example of a rotational tournament T_{13} with 13 vertices for which $\gamma(T) < \Gamma(T)$. The vertices of T_{13} are the integers modulo 13, where a vertex

i beats vertices $(i+d) \bmod 13$ for every integer $d \in \{1,2,3,5,6,9\}$; thus T_{13} is a 6-regular tournament. A minimal dominating set of size four is the set $D = \{0,1,2,3\}$. Let $u \mapsto v$ denote the fact that v is a private neighbor of u, i.e. $v \in pn(u,D)$. It is easy to see that the set D is a dominating set of T_{13}. The fact that D is a minimal dominating set follows from the observation that every vertex in D has a private neighbor, i.e. $0 \mapsto 0$, $1 \mapsto 10$, $2 \mapsto 11$ and $3 \mapsto 12$. Thus, $\Gamma(T_{13}) \geq 4$. On the other hand, $D' = \{0,1,6\}$ is also a minimal dominating set, which implies that $\gamma(T_{13}) \leq 3$. In fact, it can be seen that $\gamma(T_{13}) = 3$. Therefore, $\gamma(T_{13}) < \Gamma(T_{13})$.

The tournament T_{13} consists of the three fourth powers of elements of Z_{13} and their doubles. It is the extremal tournament for the disproof of a conjecture of Erdös and Moser that there is a tournament of order $2^k - 1$ which contains no transitive subtournament with $k+1$ vertices [16]. The result in question is that every tournament with $n \geq 14$ vertices contains a transitive subtournament of order five. Furthermore, every tournament with 13 vertices, save one, contains a transitive subtournament of order five. The lone exceptional 13-tournament that does not contain a transitive subtournament of order five is the one described above. It does, however, contain a transitive subtournament of order four, as does every tournament with $n \geq 8$ vertices.

Reversing the 2-path $4 \to 0 \to 1$ results in the regular tournament of order 5 that is shown in Figure 3 results in an example of a smallest tournament T with $\gamma(T) < \Gamma(T)$. One can see that $\{1,3\}$ is a minimum cardinality dominating set in T, while $\{0,3,4\}$ is a minimal dominating set in T.

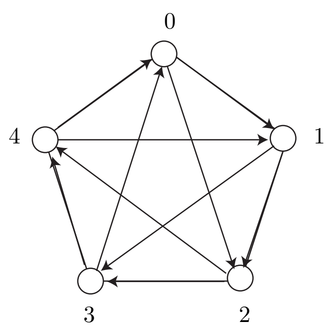

Figure 3: $\gamma(T) = 2 < \Gamma(T) = 3$.

The example in Figure 3 easily generalizes. Consider two transitive tournaments of order n, T_1 with $V(T_1) = \{v_1, v_2, \ldots, v_n\}$ and T_2 with $V(T_2) = \{u_1, u_2, \ldots, u_n\}$. Form the tournament of order $2n$, denoted T_{2n} from T_1 and T_2 by adjoining arcs $v_i \to u_i$, for $1 \leq i \leq n$, and $u_i \to v_j$ if $i \neq j$. For this class of tournaments, $\gamma(T_{2n}) = 2$ while $\Gamma(T_{2n}) = n$.

The family of tournaments T_{2n} can also be used to show that the difference $IR(T) - \Gamma(T)$ can be arbitrarily large. To each tournament T_{2n}, add a new vertex w for which $w \Rightarrow V(T_{2n})$; let T_{2n}^{+} be the resulting tournament. It is easy to see that $\gamma(T_{2n}^{+}) = \Gamma(T_{2n}^{+}) = 1$, while $IR(T_{2n}^{+}) = n$, since the set V_1 is a maximal irredundant set. In fact, from this example the following is clear.

Proposition 13 *For any tournament T of order n,*

$$\Gamma(T) \leq IR(T) \leq \lceil n/2 \rceil.$$

Proof: Let T be a tournament of order n. Let S be a maximal irredundant set of maximum cardinality in T. If there is a vertex $s \in S$ so that $s \Rightarrow S - \{s\}$, then consider the function $pn : S - \{s\} \rightarrow V(T) - S$, given by $pn(x)$ is the private neighbor of x (with respect to S). Then pn is one-to-one, so $|S - \{s\}| \leq n - |S|$ or $n \geq 2|S| - 1$. Thus, $IR(T) = |S| \leq (n+1)/2$. If there is no such vertex $s \in S$, then $pn : S \rightarrow V(T) - S$, given as above is one-to-one, so $|S| \leq n - |S|$, or $IR(T) = |S| \leq n/2$. In any case, $IR(T) \leq \lceil n/2 \rceil$. □

5 Open Problems

We conclude this paper by presenting a collection of open problems suggested by this research.

1. Can the value of $\gamma(T)$ be computed in polynomial time for an arbitrary tournament T?

2. Can you settle the NP-completeness questions related to $ir(T)$, $tr(T)$, $TR(T)$, $\Gamma(T)$ and $IR(T)$?

3. Can you characterize tournaments T for which $ir(T) = 2$?

4. What are the smallest orders of tournaments with $ir(T) = 4$ and $ir(T) = 5$? It is easy to see that the cyclic triple is the smallest tournament with $ir(T) = 2$, and one can verify, using Proposition 1, that the tournament QRT_7 in Figure 1 is a smallest tournament with $ir(T) = 3$. One can show that no two (or one) element set of vertices in QRT_7 is a maximal irredundant set.

5. What is the smallest order of a tournament with $\gamma(T) = 5$?

   Note that from Proposition 8 and the fact (near the end of Section 3) that $\gamma(QRT_{19}) = 4$, we only know that $|V(T)| \geq 39$.

   Proposition 14 *If $\gamma(T) \geq 5$, then $|V(T)| \geq 47$.*

   Proof: Suppose $\gamma(T) \geq 5$. Let $a \in V(T)$. Pick $b \in I(a)$ and $c \in I(a) \cap I(b)$. Let $S = I(a) \cap I(b) \cap I(c)$. If $I(a) = \emptyset$ then $\{a\}$ is a dominating set. If $I(a) \cap I(b) = \emptyset$, then $\{a, b\}$ is a dominating set. If $S = \emptyset$, then $\{a, b, c\}$ is a dominating set. But $\gamma(T) \geq 5$, so $I(a)$, $I(a) \cap I(b)$ and S are all non-empty.

If there is a vertex x so that $x \Rightarrow S$, then $\{a, b, c, x\}$ form a dominating set in T, contrary to the fact that $\gamma(T) \geq 5$. Since S is a dominating set for T, $|S| \geq \gamma(T) \geq 5$. As c was arbitrary in $I(a) \cap I(b)$, every vertex in $I(a) \cap I(b)$ has indegree at least five in $T[I(a) \cap I(b)]$. Thus, $|I(a) \cap I(b)| \geq 11$. As b was arbitrary in $I(a)$, every vertex in $I(a)$ has indegree at least 11 in $T[I(a)]$. Thus, $|I(a)| = id(a) \geq 23$. As a was arbitrary in T, every vertex in T has indegree at least 23. Thus, $|V(T)| \geq 47$. □

Corollary 7 *If* $\gamma(T) \geq 5$, *then* $|V(T)| \geq 48$.

Proof: From the proof of Proposition 14, $|V(T)| \geq 47$. If $|V(T)| = 47$, then equality holds throughout the proof above and T is a triply regular $(5, 11, 23)$-tournament. But there are no non-trivial triply regular tournaments (Reid and Brown, 1972 [15]), so we have a contradiction and $|V(T)| \geq 48$. □

6. Can $tr(T) - \gamma(T)$ be arbitrarily large?

7. Can $TR(T) - tr(T)$ be arbitrarily large?

8. Is there a tournament with $TR(T) < \Gamma(T)$?

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