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The algorithmic complexity of certain functional variations of total domination in graphs

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Abstract

A two-valued function f defined on the vertices of a graph $G = (V, E)$, $f : V \to \{-1, 1\}$, is a signed total dominating function if the sum of its function values over any open neighborhood is at least one. That is, for every $v \in V, f(N(v)) \geq 1$, where $N(v)$ consists of every vertex adjacent to v. The weight of a total signed dominating function is $f(V) = \sum f(v)$, over all vertices $v \in V$. The total signed domination number of a graph G, denoted $\gamma_t^s(G)$, equals the minimum weight of a total signed dominating function of G. If, instead of the range $\{-1, 1\}$, we allow the range $\{-1, 0, 1\}$, then we get the concept of a total minus dominating function. Its associated parameter, called the total minus domination number of a graph G, is denoted $\gamma_t^-(G)$. In this paper, we show that the decision problem corresponding to the computation of the total minus domination number of a graph is NP-complete, even when restricted to bipartite graphs or chordal graphs. For a fixed k, we show that the decision problem corresponding to determining whether a graph has a total minus dominating function of weight at most k may be NP-complete, even when restricted to bipartite or chordal graphs. Linear time algorithms for computing $\gamma_t^-(T)$ and $\gamma_t^s(T)$ for an arbitrary tree T are also presented.

1 Introduction

Generally, we will use the notation of [1]. Let $G = (V, E)$ be a graph, and let v be a vertex in V. The open neighborhood of v is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. A *total dominating set* (TDS) of a graph G without isolated vertices is a subset $S \subseteq V(G)$ such that every vertex in $V(G)$ is adjacent to a vertex in S (other than itself). A total dominating set of minimum cardinality is called the *total domination number* of G, denoted $\gamma_t(G)$. A total dominating set of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2] and is now well studied in graph theory (see, for example, [14] and [22]). The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [18, 19].

For a real-valued function $f: V \to R$ the *weight* of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. Let $f: V \to \{0, 1\}$ be a function which assigns to each vertex of a graph an element in the set $\{0, 1\}$. We say f is a *total dominating function* (TDF) if for every $v \in V$, $f(N(v)) \geq 1$. To ensure existence of a TDF, we henceforth restrict our attention to graphs without isolated vertices. A TDF f is minimal if no $g < f$ is also a TDF.

Several authors have suggested changing the allowable weights. Well-known is fractional total domination where the weights are allowed to be in the range $[0, 1]$. For a graph $G = (V, E)$, a function $f: V \to [0, 1]$ is called a *fractional total dominating function* (FTDF) of G if $f(N(v)) \geq 1$ for each $v \in V$. The *fractional total domination number* of G is the minimum weight among all FTDFs of G, and so $\gamma_t(G) = \{w(f) \mid f$ is a TDF of $G\}$. The integer-valued TDFs are precisely the characteristic functions of total dominating sets. This fractional version of total domination has been studied in [4, 5, 6, 7, 15, 30, 33, 34] and elsewhere.

Let $f: V \to \{-1, 1\}$ be a function which assigns to each vertex of G an element of the set $\{-1, 1\}$. The function f is defined in [12] to be *signed dominating function* of G if $\sum_{u \in N[v]} f(u) \geq 1$ for every $v \in V$. The *signed domination number*, denoted $\gamma_s(G)$, of G is the minimum weight of a signed dominating function on G. Signed domination has been studied in [3, 12, 13, 16, 17, 20, 21, 27, 32, 35, 36] and elsewhere.

Let $f: V \to \{-1, 0, 1\}$ be a function which assigns to each vertex of G an element of the set $\{-1, 0, 1\}$. The function f is defined in [11] to be *minus dominating function* of G if $\sum_{u \in N[v]} f(u) \geq 1$ for every $v \in V$. The *minus domination number*, denoted $\gamma^-(G)$, of G is the minimum weight of a minus dominating function on G. Minus domination has been studied in [8, 9, 10, 11, 24, 25, 26, 28, 31] and elsewhere.

Recently, Henning [23] introduced the concept of total signed domination that arises when one changes "closed" neighborhood in the definition of signed domination to "open" neighborhood. Let $f: V \to \{-1, 1\}$ be a function which assigns to each vertex of a graph $G = (V, E)$ an element of the set $\{-1, 1\}$. We define the function f to be *total signed dominating function* (TSDF) of G if $f(N(v)) \geq 1$ for every $v \in V$. The *total signed domination number*, denoted $\gamma_t^s(G)$, of G is the minimum weight of a TSDF on G. A TSDF f is *minimal* if no $g < f$ is also a TSDF. A (minimal) TSDF of weight $\gamma_t^s(G)$ will be called a $\gamma_t^s(G)$-function. The concept of total minus domination may be defined similarly. Specifically, let $f\colon V \to \{-1,0,1\}$ be a function which assigns to each vertex of a graph $G=(V,E)$ an element of the set $\{-1,0,1\}$. We define the function f to be *total minus dominating function* (TMDF) of G if $f(N(v)) \geq 1$ for every $v \in V$. The *total minus domination number*, denoted $\gamma_t^-(G)$, of G is the minimum weight of a TMDF on G. A (minimal) TSDF of weight $\gamma_t^-(G)$ will be called a $\gamma_t^-(G)$-function.

In this paper, we show that the decision problem for the total minus domination number of a graph is NP-complete, even when restricted to bipartite graphs or chordal graphs. For a fixed k, we show that the decision problem corresponding to determining whether a graph has a TMDF of weight at most k may be NP-complete, even when restricted to bipartite or chordal graphs. Linear time algorithms for computing $\gamma_t^-(T)$ and $\gamma_t^s(T)$ for an arbitrary tree T are also presented.

The motivation for studying this variation of the total domination number is rich and varied from a modelling perspective. For example, by assigning the values -1, 0 or $+1$ to the vertices of a graph we can model networks of people or organizations in which global decisions must be made (e.g. negative, neutral or positive responses or preferences). We assume that each individual has one vote and that each individual has an initial opinion. We assign $+1$ to vertices (individuals) which have a positive opinion, 0 to vertices which have no opinion and -1 to vertices which have a negative opinion. We also assume, however, that an individual's vote is affected by the opinions of neighboring individuals. In particular, each individual gives equal weight to the opinions of neighboring individuals (thus individuals of high degree have greater "influence"). A voter votes 'aye' if there are more vertices in its (open) neighborhood with positive opinion than with negative opinion, otherwise the vote is 'nay'. We seek an assignment of opinions that guarantee an unanimous decision; that is, for which every vertex votes aye. We call such an assignment of opinions a uniformly positive assignment. Among all uniformly positive assignments of opinions, we are interested primarily in the minimum number of vertices (individuals) who have a positive or neutral opinion. The total minus domination number is the minimum possible sum of all opinions, -1 for a negative opinion, 0 for a neutral opinion and $+1$ for a positive opinion, in a uniformly positive assignment of opinions. The total minus domination number represents, therefore, the minimum number of individuals which can have positive or neutral opinions and in doing so force every individual to vote aye.

2 Complexity Issues

In this section we discuss complexity issues regarding the computation of $\gamma_t^-(G)$ and $\gamma_t^s(G)$ for a graph G.

The following decision problem corresponding to the computation of the total domination number is known to be NP-complete, even when restricted to bipartite graphs or chordal graphs [29].

Total Domination (TD)

Instance: A graph $G=(V,E)$ and a positive integer $k \leq |V|$.

Question: Does G have a total dominating set of cardinality k or less?

We will demonstrate a polynomial time reduction from this problem to the following decision problem:

Total Minus Domination (TMD)

Instance: A graph $H = (V, E)$ and a positive integer $\ell \leq |V|$.

Question: Does H have a TMDF of weight ℓ or less?

Theorem 1 **TMD** *is NP-complete, even when restricted to bipartite or chordal graphs.*

Proof. It is obvious that **TMD** is a member of NP since we can, in polynomial time, guess a function $f : V \to \{-1, 0, 1\}$ and verify that f has weight at most ℓ and is a TMDF. We next show how a polynomial time algorithm for **TMD** could be used to solve **TD**. Given a graph $G = (V, E)$ and a positive integer k, construct the graph H by adding to each vertex v_i of G a path of length four, consisting of the consecutive vertices v_i, w_i, x_i, y_i and z_i. It is easy to see that the graph H can be constructed in polynomial time, and that if G is a bipartite or chordal graph, then so too is H.

Lemma 2 $\gamma_t^-(H) = \gamma_t(H) = \gamma_t(G) + 2|V(G)|$.

Proof. Let $v_i \in V(G)$ and let f be a $\gamma_t^-(H)$-function. Since $N(z_i) = \{y_i\}$ and $f(N(z_i)) \geq 1$, we have $f(y_i) = 1$. Also, $1 \leq f(N(y_i)) = f(z_i) + f(x_i)$, so that $f(z_i) \geq 0$ and $f(x_i) \geq 0$. Similarly, using the facts that $1 \leq f(N(x_i))$ and $1 \leq f(N(w_i))$, we have $f(w_i) \geq 0$ and $f(v_i) \geq 0$.

Thus, $\mathrm{Im}(f) \subseteq \{0, 1\}$, and so f is a TDF of H. Consequently, $\gamma_t(H) \leq f(V(H)) = \gamma_t^-(H)$. On the other hand, if S is a $\gamma_t(H)$-set, then the characteristic function h of S is a TMDF of H, so $\gamma_t^-(H) \leq h(V(H)) = \gamma_t(H)$. Consequently, $\gamma_t^-(H) = \gamma_t(H)$.

Let $n = |V(G)|$ and let S be a $\gamma_t(G)$-set. Then $S \cup \bigcup_{i=1}^n \{x_i, y_i\}$ is a TDS of H. Thus, $\gamma_t(H) \leq \gamma_t(G) + 2n$.

To see that the reverse inequality holds, let S be a $\gamma_t(H)$-set for which $|S \cap (\bigcup_{i=1}^n \{w_i, x_i, y_i, z_i\})|$ is minimized.

We may assume, without loss of generality, $z_i \notin S$ and $\{x_i, y_i\} \subseteq S$. For suppose $z_i \in S$. It follows $y_i \in S$. If $x_i \in S$, then $S - \{z_i\}$ is a TDS, contradicting the minimality of S. Thus, $x_i \notin S$, and $S' = S - \{z_i\} \cup \{x_i\}$ is a $\gamma_t(H)$-set such that $z_i \notin S'$ and $\{x_i, y_i\} \subseteq S'$.

We next show that $w_i \notin S$ for all $1 \leq i \leq n$. For suppose, to the contrary, $w_i \in S$ for some $1 \leq i \leq n$. Since $S - \{w_i\}$ is not a TDS, v_i is uniquely (open) dominated by w_i. Let v_j be any vertex adjacent to v_i. Then $v_j \notin S$. If $v_i \in S$, then $S' = S - \{w_i\} \cup \{v_j\}$ is a $\gamma_t(H)$-set with $|S' \cap (\bigcup_{i=1}^n \{w_i, x_i, y_i, z_i\})| < |S \cap (\bigcup_{i=1}^n \{w_i, x_i, y_i, z_i\})|$, which is a contradiction. We may, therefore, assume $v_i \notin S$. If v_j is dominated by some vertex $v_\ell \in S$, then $S - \{w_i\} \cup \{v_j\}$ is a $\gamma_t(H)$-set, contradicting our choice of S, as before. Thus, v_j must be uniquely dominated by w_j. But then $S - \{w_i, w_j\} \cup \{v_i, v_j\}$ is a $\gamma_t(H)$-set, again contradicting our choice of S.

Since $w_i \notin S$ for all $1 \leq i \leq n$, $S - \cup_{i=1}^n \{x_i, y_i\}$ is a TDS of G, so $\gamma_t(G) \leq |S| - 2n = \gamma_t(H) - 2n$. It now follows that $\gamma_t(H) = \gamma_t(G) + 2|V(G)|$. $\Box$

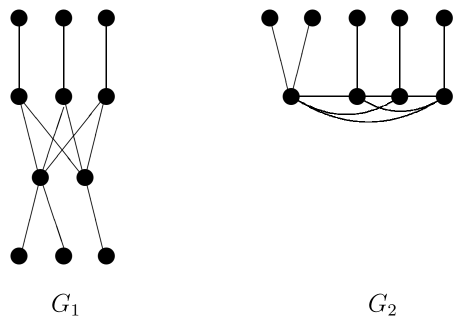

Figure 1:

Lemma 2 implies that if we let $\ell = k + 2|V(G)|$, then $\gamma_t(G) \leq k$ if and only if $\gamma_t^-(H) \leq \ell$, and our proof is complete. □

Problem **TDS** is polynomial for fixed k. To see this, let $G = (V, E)$ be a graph with $|V| = n$. If $k \geq n$, then V is a TDS of G of cardinality at most k. On the other hand, if $k < n$, then consider all the r-subsets of V, where $r = 1, \ldots, k$. There are $\sum_{r=1}^{k} \binom{n}{r}$ of these subsets, which is bounded above by the polynomial $\sum_{r=1}^{k} n^r$. It takes a polynomial amount of time to verify that a set is or is not a TDS. These remarks show that it takes a polynomial amount of time to verify whether G has a TDS of cardinality at most k when k is fixed. Hence for fixed k, **TD** $\in P$.

In contrast, we now show that for a fixed k, **TMD** may be NP-complete. To see this, we will demonstrate a polynomial time reduction of **TMD** to the following decision problem.

Zero Total Minus Domination (ZTMD)

Instance: A graph $G = (V, E)$.

Question: Does G have a TMDF of weight at most 0?

Theorem 3 **ZTMD** *is NP-complete, even when restricted to bipartite or chordal graphs.*

Proof. It is obvious that **ZTMD** is a member of NP since we can, in polynomial time, guess at a function $f : V(G) \to \{-1, 0, 1\}$ and verify that f has weight at most 0 and is a TMDF.

We next show how a polynomial time algorithm for **ZTMD** could be used to solve **TMD** in polynomial time. Before proceeding further, we prove the following helpful result.

Lemma 4 $\gamma_t^-(G_i) = \gamma_t^s(G_i) = -1$ *for* $i = 1, 2$ *(see the above figure).*

Proof. Suppose f is a $\gamma_t^-(G)$-function ($\gamma_t^s(G)$-function, respectively). Every vertex adjacent to an endvertex must receive 1 under f, since otherwise that endvertex would not have an open neighborhood sum of at least 1 under f. If any endvertex has a value other than -1 assigned to it by f, we may reassign -1 to it and the resulting function will still be a TMDF (TSDF, respectively) of G_i, which is a contradiction. Thus, each endvertex of G_i is assigned -1 by f. It now follows that $\gamma_t^-(G_i) = \gamma_t^s(G_i) = -1$. □

Note that G_1 is bipartite, while G_2 is chordal.

Given a graph $H = (V, E)$ and a positive integer ℓ, let $J_1 = H \cup \bigcup_{j=1}^{\ell} H_{1,j}$, where $H_{1,j} \cong G_1$ for $j = 1, \ldots, \ell$ ($J_2 = H \cup \bigcup_{j=1}^{\ell} H_{2,j}$, where $H_{2,j} \cong G_2$ for $j = 1, \ldots, \ell$, respectively). It is clear that J_1 (J_2, respectively) can be constructed in polynomial time. Note that if H is bipartite (chordal, respectively), then so too is J_1 (J_2, respectively).

We now show that $\gamma_t^-(H) \leq \ell$ if and only if $\gamma_t^-(J_i) \leq 0$ for $i = 1, 2$. Let $1 \leq i \leq 2$. Suppose first $\gamma_t^-(H) \leq \ell$ and f is a $\gamma_t^-(H)$-function. Let f_j be any TDMF of weight -1 for $H_{i,j}$ for $j = 1, \ldots, \ell$. Define $g : V(G) \to \{-1, 0, 1\}$ by $g(x) = f_j(x)$ if $x \in V(H_{i,j})$, ($j = 1, \ldots, \ell$), while $g(x) = f(x)$ for $x \in V(H)$. Then g is a TMDF of G of weight $\gamma_t^-(H) + \ell(-1) \leq \ell - \ell = 0$. Conversely, suppose $\gamma_t^-(J_i) \leq 0$ and g is a $\gamma_t^-(J_i)$-function. Let f be the restriction of g on $V(H)$ and let f_j be the restriction of g on $V(H_{i,j})$ for $j = 1, \ldots, \ell$. Then $\gamma_t^-(H) + \ell(-1) = \gamma_t^-(H) + \sum_{j=1}^{\ell} \gamma_t^-(H_{i,j}) \leq f(V(H)) + \sum_{j=1}^{\ell} f_j(V(H_{i,j})) = g(V(J_i)) = \gamma_t^-(J_i) \leq 0$, so that $\gamma_t^-(H) \leq \ell$. □

Henning [23] showed that the following decision problem is NP-complete.

Total Signed Domination (TSD)

Instance: A graph $H = (V, E)$ and a positive integer $\ell \leq |V|$.

Question: Does H have a TSDF of weight ℓ or less?

Theorem 5 **TSD** *is NP-complete, even when restricted to bipartite or chordal graphs.*

As before, by using Lemma 4, one may show that the following decision problem is NP-complete, even for bipartite and chordal graphs.

Zero Total Signed Domination (ZTSD)

Instance: A graph $G = (V, E)$.

Question: Does G have a TSDF of weight at most 0?

3 A Linear Algorithm for Trees for Computing the Total Minus Domination Number

Next we present a linear algorithm for finding a $\gamma_t^-(T)$-function in a nontrivial tree T. The variable *OpenSum* denotes the sum of the values assigned to the open neighborhood of v.

Algorithm: Total Minus Domination(TMD). *Given a nontrivial tree T on n vertices, root the tree T and label the vertices of T from 1 to n so that label(w) > label(y) if the level of vertex w is less than the level of vertex y. Note the root of T will be labeled n.*

```
for i := 1 to n do
    f(i) ← −1;

for i := 1 to n do
begin

    1.  if vertex i is a leaf and i < n
        then begin
                    OpenSum ← 1;
                    f(parent(i)) ← 1;
             end
             else OpenSum ← f(N(i));

    2.  if i < n
        then while (OpenSum < 1) and (f(parent(i)) < 1) do
             begin
                    f(parent(i)) ← f(parent(i)) + 1;
                    OpenSum ← OpenSum + 1;
             end;
    3.  while OpenSum < 1 do
        begin
              Choose a child of i, say v, for which f(v) < 1;
              while (OpenSum < 1) and (f(v) < 1) do
              begin
                    f(v) ← f(v) + 1;
                    OpenSum ← OpenSum + 1;
              end
end;
```

Theorem 6 *Algorithm* **TMD** *produces a* $\gamma_t^-(T)$*-function in a nontrivial tree* T.

Proof. Let $T = (V, E)$ be a nontrivial tree of order n and let f be the function produced by the Algorithm **TMD**. Then $f : V \to \{-1, 0, 1\}$. For convenience, the variable *OpenSum* which was used by Algorithm **TMD** when it considered the vertex v, will be denoted by $OpenSum(v)$.

Lemma 7 *The function* f *produced by Algorithm* **TMD** *is a TMDF.*

Proof. First consider the case when v is a leaf. The algorithm assigns, in Step 1, the value 1 to the parent of v, and since values are never decreased by the algorithm, the open neighborhood sum of v is at least one.

Next consider the case when v is not a leaf. If $OpenSum(v) \geq 1$, we are done. If not, then Steps 2 and 3 of the algorithm increase the value of vertices in the open neighborhood of v such that $OpenSum(v) \geq 1$, as required. □

To show that the function f obtained by Algorithm **TMD** is a $\gamma_t^-(T)$-function, let g be any $\gamma_t^-(T)$-function for the rooted tree T. If $f \neq g$, then we will show that g can be transformed into a new $\gamma_t^-(T)$-function g' that will differ from f in fewer values than g did. This process will continue until $f = g'$. Suppose, then, that $f \neq g$. Let v be the lowest labeled vertex for which $f(v) \neq g(v)$. Then *all* descendants of v are assigned the same value under g as under f.

Lemma 8 *If $g(v) < f(v)$, then the initial value assigned to the vertex v was increased in Step 3 of Algorithm* **TMD**.

Proof. Suppose the value of v was increased in Step 1. Then v is the parent of some leaf, say u. Since $g(v) < f(v)$, we have $g(v) \leq 0$. But then $g(N(u)) = g(v) \leq 0$, contradicting the fact that g is a TMDF of T.

Suppose the value of v was increased in Step 2. This occurred when the algorithm was processing a vertex, say u, whose parent is v. Then $f(N(u)) \leq 1$ and $g(N(u)) = g(N(u)-\{v\})+g(v) = f(N(u)-\{v\})+g(v) = f(N(u))-f(v)+g(v) < f(N(u)) \leq 1$, which contradicts the fact that g is a TMDF for T. □

Lemma 9 *If $g(v) < f(v)$, then the function g' defined by $g'(u) = f(u)$ if $u \in N(parent(v))$ and $g'(u) = g(u)$ if $u \notin N(parent(v))$ is a $\gamma_t^-(T)$-function that differs from f in fewer values than does g.*

Proof. By Lemma 8, the initial value of v is increased in Step 3 of Algorithm **TMD**, which occurs when the parent of v was being processed. Let w be the parent of v. So g' is defined by $g'(u) = f(u)$ if $u \in N(w)$ and $g'(u) = g(u)$ for all remaining vertices in V.

The algorithm ensures that $f(N(w)) = 1$. Also, since g is a TMDF of T, $f(N(w)) = 1 \leq g(N(w))$. Furthermore, $g'(V) = g'(V - N(w)) + g'(N(w)) = g(V - N(w)) + f(N(w)) \leq g(V - N(w)) + g(N(w)) = g(V)$. Thus, $g'(V) \leq g(V)$.

Since all the descendants of w, other than its children, have the same values under g as under f, $g'(N(u)) = f(N(u))$ if $u = w$ or if u is a descendant of w, other than a child of w. Moreover, since the value of v was increased in Step 3, then, if w had a parent, its value was either already 1 or otherwise it was increased to 1 in Step 2. Thus, $g'(N(u)) \geq g(N(u))$ for all vertices u different from w or a descendant of w, other than a child of w. Thus, since f and g are TMDFs of T, so too is g'. Since $g'(V) \leq g(V)$, g' is a $\gamma_t^-(T)$-function of T that differs from f in fewer values than does g. □

We now consider the case where $f(v) < g(v)$. We will need the following result.

Lemma 10 *A TMDF on a graph $G = (V, E)$ is minimal if and only if for every vertex $v \in V$ with $f(v) \in \{0, 1\}$, there exists a vertex $u \in N(v)$ with $f(N(u)) = 1$.*

Proof. Let f be a minimal TMDF of G. Suppose there is a vertex $v \in V$ with $f(v) \in \{0, 1\}$ and $f(N(u)) \geq 2$ for every vertex $u \in N(v)$. Define a function $g : V \to \{-1, 0, 1\}$ by $g(v) = f(v) - 1$ and $g(w) = f(w)$ for all $w \neq v$. Thus $g(N(w)) = f(N(w)) \geq 1$ for all $w \notin N(v)$ and $g(N(w)) = f(N(w)) - 1 \geq 1$ for all $w \in N(v)$. So g is a TMDF with $g < f$, contradicting the minimality of f.

Conversely, let f be a TMDF such that for every vertex $v \in V$ with $f(v) \in \{0, 1\}$, there exists a vertex $u \in N(v)$ with $f(N(u)) = 1$. Suppose f is not minimal. Then there exists a TMDF g with $g < f$. Thus, $g(w) \leq f(w)$ for all $w \in V$ and there exists a vertex $v \in V$ such that $g(v) < f(v)$. Therefore $f(v) \in \{0, 1\}$ and by the assumption there is a vertex $u \in N(v)$ with $f(N(u)) = 1$. So $g(N(u)) \leq f(N(u)) - 1 = 0$, which contradicts the fact that g is a TMDF. □

If the vertex v is the root then $f(V) < g(V) = \gamma_t^-(T)$ which is a contradiction. Thus, we may assume that v is not the root of T.

Since the labeling at each level is arbitrary, if any vertex x at the same level as v has $g(x) < f(x)$, we can proceed as before to find a TMDF g' that agrees with f in more values than g does. Thus we may assume that every vertex x at the same level as v has $f(x) \leq g(x)$.

Since $f(v) < g(v)$, we know that $g(x) \in \{0, 1\}$. By Lemma 10, there must be a vertex $x \in N(v)$ such that $g(N(x)) = 1$. Let w be the parent of v and u be the parent of w. If $f(u) \leq g(u)$, then $f(N(x)) = f(N(x) - \{v\}) + f(v) \leq g(N(x) - \{v\}) + g(v) - 1 = g(N(x)) - 1 = 0$, which contradicts the fact that f is a TMDF.

Thus $f(u) > g(u)$. Suppose $f(u) = g(u) + r$ and $f(v) = g(v) - s$ where $r, s \in \{1, 2\}$. Define $g' : V \to \{-1, 0, 1\}$ as follows: $g'(y) = g(y)$ for all vertices $y \in V - \{u, v\}$,

$$g'(u) = \begin{cases} f(u) - 1 & \text{if } r = 2 \text{ and } s = 1 \\ f(u) & \text{otherwise} \end{cases}$$

and

$$g'(v) = \begin{cases} f(v) + 1 & \text{if } r = 1 \text{ and } s = 2 \\ f(v) & \text{otherwise} \end{cases}.$$

Then

$$\begin{aligned} g'(u) &= \begin{cases} f(u) - 1 & \text{if } r = 2 \text{ and } s = 1 \\ f(u) & \text{otherwise} \end{cases} \\ &= \begin{cases} g(u) + r - 1 & \text{if } r = 2 \text{ and } s = 1 \\ g(u) + r & \text{otherwise} \end{cases} \\ &\geq g(u) + 1. \end{aligned}$$

It follows that the only vertex with possibly a smaller value under g' than under g is v. For each child x of v, we have $g'(N(x)) = g'(N(x) - \{v\}) + g'(v) \geq f(N(x) - \{v\}) + f(v) = f(N(x)) \geq 1$.

Furthermore,

$$\begin{aligned} g'(u) + g'(v) &= \begin{cases} f(u) + f(v) + 1 & \text{if } r = 1 \text{ and } s = 2 \\ f(u) - 1 + f(v) & \text{if } r = 2 \text{ and } s = 1 \\ f(u) + f(v) & \text{otherwise} \end{cases} \\ &= \begin{cases} (g(u) + 1) + (g(v) - 2) + 1 & \text{if } r = 1 \text{ and } s = 2 \\ (g(u) + 2) - 1 + (g(v) - 1) & \text{if } r = 2 \text{ and } s = 1 \\ g(u) + g(v) & \text{otherwise} \end{cases} \\ &= g(u) + g(v). \end{aligned}$$

Thus, $g'(N(w)) = g'(N(w) - \{u, v\}) + g'(u) + g'(v) = g(N(w) - \{u, v\}) + g(u) + g(v) = g(N(w)) \geq 1$ and $g'(V) = g'(V - \{u, v\}) + g'(u) + g'(v) = g(V - \{u, v\}) + g(u) + g(v) = g(V)$. This shows that g' is a $\gamma_t^-(T)$-function which differs from f in fewer values than does g. $\square$

4 A Linear Algorithm for Trees for Computing the Total Signed Domination Number

In our final section, we present a linear algorithm for finding a minimum total signed dominating function in a nontrivial tree T. The algorithm roots the tree T and associates various variables with the vertices of T as it proceeds. For any vertex v, the variable $MinSum$ denotes the miminum possible sum of values that may be assigned to the open neighborhood of v. So $MinSum = 1$ or 2 depending on whether v has odd or even degree, respectively. The variable $OpenSum$ denotes the sum of the values assigned to the open neighborhood of v.

Algorithm: Total Signed Domination (TSD). *Given a nontrivial tree T on n vertices, root the tree T and relabel the vertices of T from 1 to n so that $label(w) > label(y)$ if the level of vertex w is less than the level of vertex y. Note the root of T will be labeled n.*

```
for i := 1 to n do
      f(i) ← −1;

for i := 1 to n do
begin

      1.   deg i ← degree of the vertex i in T;

      2.   if deg i is odd
           then MinSum ← 1
           else MinSum ← 2;

      3.   if vertex i is a leaf and i < n
           then begin
                         OpenSum ← 1;
      3.1.               f(parent(i)) ← 1;
                end
           else OpenSum ← f(N(i));

      4.   if OpenSum < MinSum
           then begin
                         if i < n and f(parent(i)) = −1
                         then begin
      4.1.                             f(parent(i) = 1;
                                       OpenSum ← OpenSum + 2;
                              end;

                         while OpenSum < MinSum do
                         begin
      4.2.                     increase the value of one of the children of i;
                               OpenSum ← OpenSum + 2;
                         end;
                end;
end;
```

We now verify the validity of Algorithm **TSD**.

Theorem 11 *Algorithm* **TSD** *produces a* $\gamma_t^s(T)$*-function in a nontrivial tree* T*.*

Proof. Let $T = (V, E)$ be a nontrivial tree of order n, and let f be the function produced by Algorithm **TSD**. Then $f : V \to \{-1, 1\}$. For convenience, the variables *MinSum* and *OpenSum*, which were used by Algorithm **TSD** when it considered the vertex v, will be denoted by $MinSum(v)$ and $OpenSum(v)$, respectively.

Lemma 12 *The function* f *produced by Algorithm* **TSD** *is a TSDF for* T*.*

Proof. First consider the case when v is a leaf. The algorithm assigns, in Step 3, the value 1 to the parent of v, and since values are never decreased by the algorithm, the open neighborhood sum of v is at least one.

Next consider the case when v is not a leaf. If $OpenSum(v) \geq MinSum(v) \geq 1$, we are done. If not, then Step 4 of the algorithm increases the value of vertices in the open neighborhood of v such that $OpenSum(v) \geq MinSum(v) \geq 1$, as required. □

To show that the TSDF f obtained by Algorithm **TSD** is minimum, let g be any $\gamma_t^s(T)$-function for the rooted tree T. If $f \neq g$, then we will show that g can be transformed into a new $\gamma_t^s(T)$-function g' that will differ from f in fewer values than g did. This process will continue until $f = g'$. Suppose, then, that $f \neq g$. Let v be the lowest labeled vertex for which $f(v) \neq g(v)$. Then *all* descendants of v are assigned the same value under g as under f.

Lemma 13 *If* $g(v) < f(v)$*, then the initial value assigned to the vertex* v *was increased in Step* 4.2 *of Algorithm* **TSD***.*

Proof. Suppose the value of v was increased in Step 3.1. Then v is the parent of some leaf, say u. But then $g(N(u)) = g(v) = -1$, contradicting the fact that g is a TSDF of T.

Suppose the value of v was increased in Step 4.1. This occurred when the algorithm was processing a vertex, say u, whose parent is v. Then $f(N(u)) = MinSum(u) \leq 2$ and $g(N(u)) = g(N(u) - \{v\}) + g(v) = f(N(u) - \{v\}) - 1 = f(N(u)) - f(v) - 1 = f(N(u)) - 2 \leq 0$, which is a contradiction.

Thus, the value of v was increased in Step 4.2. of Algorithm **TSD**. □

Lemma 14 *If* $g(v) < f(v)$*, then the function* g' *defined by* $g'(u) = f(u)$ *if* $u \in N(parent(v))$ *and* $g'(u) = g(u)$ *if* $u \notin N(parent(v))$ *is a* $\gamma_t^s(T)$*-function of* T *that differs from* f *in fewer values than does* g*.*

Proof. By Lemma 13, the initial value assigned to the vertex v was increased in Step 4.2 of Algorithm **TSD** and this occurs when the parent of v was being processed. Let w be the parent of v. Thus g' is defined by $g'(u) = f(u)$ if $u \in N(w)$ and $g'(u) = g(u)$ for all remaining vertices u in V.

Then $f(N(w)) = MinSum(w)$. If $\deg w$ is even, then $MinSum(w) = 2$, so $g(N(w)) \geq 2 = MinSum(w) = f(N(w))$. If $\deg w$ is odd, then $g(N(w)) \geq 1 = MinSum(w) = f(N(w))$. Hence, $f(N(w)) \leq g(N(w))$. Furthermore, $g'(V) = g'(V-N(w))+g'(N(w)) = g(V-N(w))+f(N(w)) \leq g(V-N(w))+g(N(w)) = g(V)$. Since all the descendants of w, other than its children, have the same values under g as under f, $g'(N(u)) = f(N(u))$ if $u = w$ or if u is a descendant of w, other than a child of w. Moreover, since the value of v was increased in Step 4.2, then, if w had a parent, its value was either already 1 or otherwise it was increased to 1 in Step 4.1. Thus, $f(parent(w)) = 1$, so that $g'(N(u)) \geq g(N(u))$ for all vertices u different from w or a descendant of w, other than a child of w. Thus, since f and g are TSDFs of T, so too is g'. Since $g'(V) \leq g(V)$, g' is a $\gamma_t^s(T)$-function of T that differs from f in fewer values than does g. □

It remains for us to consider the case where $f(v) < g(v)$. We will need the following result from [23].

Lemma 15 *A TSDF f on a graph $G = (V, E)$ is minimal if and only if for every vertex $v \in V$ with $f(v) = 1$, there exists a vertex $u \in N(v)$ with $f(N(u)) \in \{1, 2\}$.*

Here the vertex v is not the root of T, for otherwise $f(V) < g(V) = \gamma_t^s(T)$, which is impossible. Since the labeling of the vertices was arbitrary at each level, if any vertex x at the same level as v has $g(x) < f(x)$, we can proceed as before to find a TSDF g' that agrees with f in more values than under g. So we may assume in what follows that every vertex x at the same level as v has $f(x) \leq g(x)$.

Since $f(v) < g(v)$, it follows that $f(v) = -1$ and $g(v) = 1$. By the minimality of g (cf. Lemma 15), there exists a vertex $x \in N(v)$ such that $g(N(x)) \in \{1, 2\}$. Let w be the parent of v and let u be the parent of w. If $f(u) \leq g(u)$, then $f(N(x)) = f(N(x) - \{v\}) + f(v) \leq g(N(x) - \{v\}) + g(v) - 2 = g(N(x)) - 2 \leq 0$, which is a contradiction.

Hence $f(u) > g(u)$, i.e., $f(u) = 1$ and $g(u) = -1$. Define a function $g' : V \to \{-1, 1\}$ by $g'(y) = g(y)$ if $y \in V - \{v, u\}$, $g'(v) = -1$ and $g'(u) = 1$. Note that $f(v) = g'(v) = -1$ and $f(u) = g'(u) = 1$. The only vertices whose neighborhood sums are decremented under g' are the children of v. However, these open neighborhood sums under g' are at least as large as under f. Thus, since g are f are TSDFs, so too is g'. Furthermore, $g'(V) = g(V)$, so that g' is a $\gamma_t^s(T)$-function which differs from f in fewer values than does g. □

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