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A note on max-leaves spanning tree problem in Halin graphs

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Abstract

A Halin graph H is a planar graph obtained by drawing a tree T in the plane, where T has no vertex of degree 2, then drawing a cycle C through all leaves in the plane. We write $H = T \cup C$, where T is called the characteristic tree and C is called the accompanying cycle. The problem is to find a spanning tree with the maximum number of leaves in a Halin graph. In this paper, we prove that the characteristic tree in a Halin graph H is one of the spanning trees with maximum number of leaves in H, and we design a linear time algorithm to find such a tree in the Halin graph.

All graphs in this paper are finite, undirected and simple. We follow the terminology and notation of [1] and [6].

A Halin graph H is a planar graph obtained by drawing a tree T in the plane, where T has no vertex of degree 2, then drawing a cycle through all leaves of T in the plane. We write $H = T \cup C$, where T is called the characteristic tree and C is called the accompanying cycle.

In this paper, we deal with the problem of finding a spanning tree with maximum number of leaves in a Halin graph. The problem of finding a spanning tree with maximum number of leaves for arbitrary graphs is NP-complete (see [3], Problem [ND 2]). When we confine an NP-complete problem to Halin graphs, we often get a polynomial time algorithm to solve it. (For example, see [2]). However, not every NP-complete problem confined to Halin graphs can be solved in polynomial time (see [5]). In this paper, we prove that the characteristic tree in a Halin graph H is one of the spanning trees with maximum number of leaves in H, and we design a linear time algorithm to find such a tree in the Halin graph.

Theorem 1. *The characteristic tree of a Halin graph H is a spanning tree with maximum number of leaves.*

Proof. Let T' be a spanning tree with maximum number of leaves in H, V_1' be the set of internal vertices of T' and V_2' be the set of leaves of T'. Assume that $|V_1'| = t'$ and $|V_2'| = s'$. Then we have

$$s' = \sum_{v \in V_1'} (d(v) - 2) + 2. \tag{1}$$

Let T be the characteristic tree of H, V_1 be the set of internal vertices of T and V_2 be the set of leaves of T. Assume that $|V_1| = t$ and $|V_2| = s$. By the same reason as above, we have

$$s = \sum_{v \in V_1} (d(v) - 2) + 2. \tag{2}$$

Let $V_{11} = V_1 \cap V_1'$, $V_{12} = V_1 \backslash V_1'$ and $V_{12}' = V_1' \backslash V_1 \subseteq V_2$.

Since T' is one of the spanning trees with maximum number of leaves of H, so $s' \geq s$. However, $s + t = s' + t' = \nu$. So $t \geq t'$. This implies $|V_{12}'| \leq |V_{12}|$.

Now, $\forall v \in V_{12}$ and $\forall x \in V_{12}'$, $d_H(v) \geq 3 = d_H(x)$. Therefore,

$$\begin{aligned} s &= \sum_{v \in V_1} (d(v) - 2) + 2 \\ &= \sum_{v \in V_{11}} (d(v) - 2) + \sum_{v \in V_{12}} (d(v) - 2) + 2 \\ &\geq \sum_{v \in V_{11}} (d(v) - 2) + \sum_{v \in V_{12}'} (d(v) - 2) + 2 \\ &= \sum_{v \in V_1'} (d(v) - 2) + 2 = s'. \end{aligned}$$

Since s$' \geq s$, we have $s' = s$. Hence the characteristic tree T is one of the spanning trees with maximum number of leaves. □

By the result of Theorem 1, we can find a spanning tree with maximum number of leaves using the following algorithm:

1. Use the algorithm of Hopcroft and Tarjan [4] to find a planar embedding $\widetilde{H}$ of the Halin graph H.

2. Traverse each face C in $\widetilde{H}$, if $T = H - E(C)$ is a tree, then T is the characteristic tree and C is the accompanying cycle.

Note that in Step 2, we decide whether $T = H - E(C)$ is a tree in the following way. For each face C, if $\varepsilon(H) - \varepsilon(C) = \nu(H) - 1$, then $T = H - E(C)$ is a tree as $H - E(C)$ is connected for each face in a Halin graph.

The correctness of the algorithm is shown in [2]. The time complexity of the above algorithm is bounded by $O(\nu)$, where $\nu = |V(H)|$. By [4], Step 1 takes $O(\nu)$ time. Step 2 takes $O(\varepsilon)$ time. However, in a Halin graph H, $\varepsilon(H) = \varepsilon(T) + \varepsilon(C) \leq 2(\nu - 1) = O(\nu)$. Hence Step 2 also takes $O(\nu)$ time.

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