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Exterior derivatives and Laplacians on digraphs

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Abstract

Given a digraph $G = (V, E)$ with the set of vertices V and the set of edges E, let $d : \mathfrak{F} \to \Omega^1$ be the exterior derivative map from the space of complex-valued functions on V to the complex vector space spanned by E. We introduce the Laplacian $\Delta : \mathfrak{F} \to \mathfrak{F}$ and the label difference map $\widehat{d} : \mathfrak{F} \to (\Omega^1)^*$ of $\mathfrak{F}$ into the dual space $(\Omega^1)^*$ of Ω^1 and establish their connections with d. In particular, we prove that, given elements ϕ and ψ of $\mathfrak{F}$, the image of the conjugate $\overline{d\psi}$ of $d\psi$ under $\widehat{d}\phi$ is equal to the value of the Hermitian product between $\Delta\phi$ and ψ and that $\widehat{d}\phi$ is a flow in G associated to $\Delta\phi$.

1 Introduction

It is well-known that graph theory is related to many areas of pure and applied mathematics (see e.g. [1]). One such area is differential geometry (see e.g. [3]), and the purpose of this paper is to investigate some of the aspects of differential geometric structures on digraphs.

Let $G = (V, E)$ be a digraph having $V = \{1, \ldots, n\}$ as the set of its vertices and E the set of its edges. Thus E is a subset of $V \times V$, and an element (i, j) represents the directed edge from the vertex i to the vertex j. Throughout this paper we assume that the digraph G does not have loops or multiple edges. We denote by $\mathfrak{F}$ the space of complex-valued functions on V and by Ω^1 the complex vector space spanned by E. The spaces $\mathfrak{F}$ and Ω^1 are the analogues of the spaces of smooth functions and differential 1-forms, respectively, on a differentiable manifold. The exterior derivative map $d : \mathfrak{F} \to \Omega^1$ on G can then be defined in such a way that it has the properties analogous to the ones satisfied by the usual exterior differentiation on the space of smooth functions on a manifold. The map d is related to a map from $\mathcal{F}$ to the dual space of Ω constructed as follows. Let $\mathcal{X} = \mathrm{Hom}_{\mathbb{C}}(\Omega^1, \mathbb{C})$ be the complex vector space dual to Ω^1, which can be regarded as the space of vector fields on G. Then the label difference map $\widehat{d} : \mathfrak{F} \to \mathcal{X}$ is the complex linear map satisfying

$$\widehat{d}\psi(i, j) = \psi(j) - \psi(i)$$

for $\psi \in \mathfrak{F}$ and $(i, j) \in E$, and it can be considered as the dual of d in some sense.

Spectra of matrices representing graphs play an important role in graph theory. For example, the eigenvalues of the adjacency matrix of a graph are closely linked to the geometric and combinatorial properties of the graph (see e.g. [2], [6]). Other important matrices associated to graphs include Laplacian matrices for undirected graphs, which may be regarded as the finite analogue of the Laplacian operator in Riemannian geometry. Spectral properties of such Laplacian matrices are related to structures of the associated graphs. Numerous papers have been devoted to the study of various aspects of Laplacian matrices and their eigenvalues over the years (see e.g. [5]). Another map of interest for us in this paper is a linear map $\Delta : \mathfrak{F} \to \mathfrak{F}$ called the Laplacian, which is analogous to the notion of Laplacian matrices for undirected graphs.

Each vector field $\eta \in \mathcal{X}$ on the digraph G determines a function on the set of edges, and therefore the edges are weighted by the values of this function. Thus we can consider the flows on G associated to elements of $\mathfrak{F}$. In this paper, we establish various connections among the maps d, $\widehat{d}$ and Δ defined on $\mathfrak{F}$. In particular, we prove that, given elements ϕ and ψ of $\mathfrak{F}$, the image of the conjugate $\overline{d\psi}$ of $d\psi$ under $\widehat{d\phi}$ is equal to the value of the Hermitian product between $\Delta\phi$ and ψ and that $\widehat{d\phi}$ is a flow in G associated to $\Delta\phi$.

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2 Differential calculi on digraphs

In this section we describe a differential calculus on a digraph studied by Dimakis and Müller-Hoissen (see e.g. [3]). Throughout this paper we denote by V the set of the positive integers $\{1, 2, \ldots, n\}$ and fix a subset E of the Cartesian product $V \times V$ of V. We denote by $G = (V, E)$ the associated digraph which has V as the set of its vertices and E the set of its edges, where $(i, j) \in E$ represents the directed edge from i to j.

Let $\mathfrak{F}$ be the set of all complex-valued functions on V. Thus each element of $\mathfrak{F}$ may be regarded as a vertex labeling of G, and $\mathfrak{F}$ has the structure of an algebra over $\mathbb{C}$ with respect to the usual addition, multiplication, and scalar multiplication operations on complex-valued functions. For each $i \in V$, we denote by $\chi^i : V \to \mathbb{C}$ the characteristic function of the vertex $i \in V$, that is, the function defined by $\chi^i(j) = \delta_{ij}$ for all $i, j \in V$, where δ_{ij} is the Kronecker delta that is equal to 1 for $i = j$ and 0 otherwise. Then the functions $\chi^1, \ldots, \chi^n$ form a basis for the algebra $\mathfrak{F}$, and they satisfy

$$\sum_{i \in V} \chi^i = 1, \qquad \chi^i \cdot \chi^j = \delta_{ij} \chi^j \tag{2.1}$$

for all $i, j \in V$; here 1 denotes the function with constant value $1 \in \mathbb{C}$.

We now define the space of 1-forms Ω^1 on the digraph G by

$$\Omega^1 = \bigoplus_{(i,j) \in E} \mathbb{C}(i, j).$$

Thus Ω^1 is the complex vector space spanned by the set E of edges (i,j) of G, and it has the structure of an $\mathfrak{F}$-bimodule determined by

$$\chi^i \cdot (j,k) = \delta_{ij}(j,k), \quad (j,k)\cdot \chi^i = \delta_{ki}(j,k) \tag{2.2}$$

for all $i \in V$ and $(j,k) \in E$. We denote by ρ the 1-form on G given by

$$\rho = \sum_{(i,j)\in E} (i,j) \in \Omega^1, \tag{2.3}$$

and define the *exterior derivative map* $d : \mathfrak{F} \to \Omega^1$ to be the $\mathbb{C}$-linear map sending an element $\psi \in \mathfrak{F}$ to its *differential* $d\psi$ given by

$$d\psi = \rho\cdot\psi - \psi\cdot\rho. \tag{2.4}$$

Thus we see that $dK = 0$ for each constant function K on V and

$$\begin{aligned} d(\phi\psi) &= \rho\cdot\phi\cdot\psi - \phi\cdot\psi\cdot\rho \\ &= \rho\cdot\phi\cdot\psi - \phi\cdot\rho\cdot\psi + \phi\cdot\rho\cdot\psi - \phi\cdot\psi\cdot\rho \\ &= (d\phi)\cdot\psi + \phi\cdot(d\psi) \end{aligned}$$

for all $\phi, \psi \in \mathfrak{F}$. In particular, by (2.1) we obtain

$$\sum_{i\in V} d\chi^i = 0, \quad \chi^i\cdot d\chi^j = -(d\chi^i)\cdot\chi^j + \delta_{ij}(d\chi^j) \tag{2.5}$$

for all $(i,j) \in E$.

Lemma 2.1 *Let $\psi : V \to \mathbb{C}$ be an element of $\mathfrak{F}$. Then the differential $d\psi$ of ψ is given by*

$$d\psi = \sum_{(i,j)\in E} (\psi(j) - \psi(i))(i,j), \tag{2.6}$$

and we have

$$(k,\ell) = \chi^k \cdot d\chi^\ell = \delta_{k\ell}(d\chi^\ell) - (d\chi^k)\cdot\chi^\ell \tag{2.7}$$

for all $(k,\ell) \in E$.

Proof. Since $\chi^i(j) = \delta_{ij}$ for $i,j \in V$, we see that $\psi = \sum_{k\in V}\psi(k)\chi^k$. Hence, using (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned} d\psi &= \Big(\sum_{(i,j)\in E}(i,j)\Big)\cdot\Big(\sum_{k\in V}\psi(k)\chi^k\Big) - \Big(\sum_{k\in V}\psi(k)\chi^k\Big)\cdot\Big(\sum_{(i,j)\in E}(i,j)\Big) \\ &= \sum_{(i,j)\in E}\psi(j)(i,j) - \sum_{(i,j)\in E}\psi(i)(i,j) \\ &= \sum_{(i,j)\in E}(\psi(j)-\psi(i))(i,j); \end{aligned}$$

hence (2.6) follows. In particular, for $(k,\ell) \in E$, we have

$$\begin{aligned} \chi^k \cdot d\chi^\ell &= \sum_{(i,j)\in E} (\chi^\ell(j) - \chi^\ell(i))\chi^k \cdot (i,j) \\ &= \sum_{(i,j)\in E} (\chi^\ell(j) - \chi^\ell(i))\delta_{ki}(i,j) \\ &= \sum_j (\chi^\ell(j) - \chi^\ell(k))(k,j) = (k,\ell). \end{aligned}$$

Thus (2.7) follows from this and (2.5), and therefore the proof of the lemma is complete. □

Remark 2.2 Higher order differential forms on G can also be obtained by concatenation of 1-forms as follows. Given a positive integer $r > 1$, suppose that there are vertices $i_0, i_1, \dots, i_r \in V$ of $G = (V, E)$ such that

$$(i_0, i_1), (i_1, i_2), \dots, (i_{r-1}, i_r) \in E.$$

Then we set

$$(i_0, i_1, \cdots, i_r) = (i_0, i_1)(i_1, i_2) \cdots (i_{r-1}, i_r),$$

and define the space Ω^r of r-forms on G to be the complex vector space spanned by elements of the form $(i_0, i_1, \cdots, i_r)$. Using (2.2), we see that Ω^r has the structure of an $\mathfrak{F}$-bimodule. Since we have

$$(i_0, \cdots, i_r)(j_0, \cdots, j_s) = \delta_{i_r j_0} \cdot (i_0, \cdots, i_{r-1}, j_0, \cdots, j_s),$$

we also see that $\Omega^r \Omega^s \subset \Omega^{r+s}$ for positive integers r and s. The exterior derivative map $d : \Omega^r \to \Omega^{r+1}$ on r-forms are given by

$$d(i_0, \cdots, i_r) = \sum_j \sum_{k=0}^{r+1} (-1)^k (i_0, \cdots, i_{k-1}, j, i_k, \cdots, i_r)$$

for all $(i_0, \cdots, i_r) \in \Omega^r$ (see e.g. [3] for details).

3 Label difference maps

Let Ω^1 be the complex vector space of 1-forms on the digraph $G = (V, E)$ considered in Section 2, and let $\mathcal{X} = \mathrm{Hom}_{\mathbb{C}}(\Omega^1, \mathbb{C})$ be its dual space. Thus $\mathcal{X}$ is the analogue of the tangent bundle in the theory of manifolds, and therefore elements of $\mathcal{X}$ may be called vector fields on G. In this section we introduce the label difference map and prove that it is dual to the exterior map in some sense.

Given a vertex labeling $\psi \in \mathfrak{F}$ of G, we consider the associated map $\delta\psi : E \to \mathbb{C}$ which assigns to each edge the difference in labels of its incident vertices, that is, the map given by

$$(\delta\psi)(i,j) = \psi(j) - \psi(i) \tag{3.1}$$

for all $(i,j) \in E$. We denote by

$$\widetilde{\delta\psi} : \Omega^1 \to \mathbb{C}$$

the complex linear map obtained by extending $\delta\psi$ linearly.

Definition 3.1 The *label difference map* for the digraph $G = (V, E)$ is the complex linear map

$$\widehat{d} : \mathfrak{F} \to \mathcal{X} = \mathrm{Hom}_{\mathbb{C}}(\Omega^1, \mathbb{C})$$

sending $\psi \in \mathfrak{F}$ to $\widetilde{\delta\psi} \in \mathcal{X}$.

If $\omega = \sum_{(i,j)\in E} K_{ij}(i,j) \in \Omega^1$ with $K_{ij} \in \mathbb{C}$, then by Definition 3.1 we see that

$$\widehat{d}\psi(\omega) = \sum_{(i,j)\in E} (\psi(j) - \psi(i))K_{ij} = \sum_{(i,j)\in E} K_{ij}(\delta\psi)(i,j).$$

The label difference map $\widehat{d}$ may be regarded as the dual of the exterior derivative map $d : \mathfrak{F} \to \Omega^1$ in some sense according to the next theorem.

Theorem 3.2 *Let $\{\varepsilon_{ij} \mid (i,j) \in E\}$ be the basis of $\mathcal{X}$ that is dual to the basis E of Ω^1. Then the label difference map $\widehat{d}$ and the exterior derivative map d satisfy the relations*

$$d\psi = \sum_{(i,j)\in E} (\widehat{d}\psi(i,j))(i,j), \tag{3.2}$$

$$\widehat{d}\psi = \sum_{(i,j)\in E} (\varepsilon_{ij}(d\psi))\varepsilon_{ij} \tag{3.3}$$

for all $\psi \in \mathfrak{F}$.

Proof. Given an element $\psi \in \mathfrak{F}$, by (2.6) we have

$$d\psi = \sum_{(i,j)\in E} (\psi(j) - \psi(i))(i,j).$$

Using (3.1) and the relation $\varepsilon_{ij}(k,\ell) = \delta_{ik}\delta_{j\ell}$ for $(i,j),(k,\ell) \in E$, we see that

$$\varepsilon_{ij}(d\psi) = \psi(j) - \psi(i) = \delta\psi(i,j) = \widehat{d}\psi(i,j) \tag{3.4}$$

for all $(i,j) \in E$; hence (3.2) follows immediately. On the other hand, if $\omega = \sum_{(i,j)\in E} K_{ij}(i,j) \in \Omega^1$, then $\varepsilon_{ij}(\omega) = K_{ij}$ for each $(i,j) \in E$. Hence, using this and (3.4), we have

$$\begin{aligned}
(\widehat{d}\psi)(\omega) &= \sum_{(i,j)\in E} \widehat{d}\psi(i,j)K_{ij} \\
&= \sum_{(i,j)\in E} \widehat{d}\psi(i,j)\varepsilon_{ij}(\omega) \\
&= \sum_{(i,j)\in E} \varepsilon_{ij}(d\psi)\varepsilon_{ij}(\omega).
\end{aligned}$$

Thus we obtain (3.3), and therefore the proof of the theorem is complete. □

4 Laplacians

In this section we introduce the Laplacian for our digraph G, which is analogous to the Laplacian matrix for an undirected graph. We discuss relations of this Laplacian with the exterior derivative map $d : \mathfrak{F} \to \Omega^1$ and the label difference map $\widehat{d} : \mathfrak{F} \to \mathcal{X}$.

For each vertex $i \in V$ of the digraph $G = (V, E)$ we consider the subsets V_i^O and V_i^I of V given by

$$V_i^O = \{j \in V \mid (i,j) \in E\}, \qquad V_i^I = \{j \in V \mid (j,i) \in E\}. \tag{4.1}$$

Thus the vertex j belongs to V_i^O (resp. V_i^I) if and only if (i,j) (resp. (j,i)) is an edge of G, and the number of elements of V_i^O (resp. V_i^I) is the outdegree (resp. indegree) of the vertex i. We now set

$$V_i = V_i^O \cup V_i^I \tag{4.2}$$

for each $i \in V$, and define Laplacians on digraphs as follows.

Definition 4.1 The *Laplacian* on the digraph $G = (V, E)$ is the complex linear map $\Delta : \mathfrak{F} \to \mathfrak{F}$ given by

$$(\Delta\phi)(i) = \sum_{j \in V_i} \widehat{d}\phi(j, i) \tag{4.3}$$

for all $\phi \in \mathfrak{F}$ and $i \in V$.

Let A be the adjacency matrix of the digraph G. Thus the ij-entry of A is 1 if there is a directed edge from i to j, and is 0 otherwise. Therefore, if A_{ij} denotes the ij-entry of A, we see that

$$A_{ij} = \sum_{k \in V_i^O} \delta_{kj} = \sum_{k \in V_j^I} \delta_{ki}, \tag{4.4}$$

where δ is the Kronecker delta.

Proposition 4.2 *Let D be the diagonal matrix whose ii-entry is the degree of the vertex i of G for $1 \leq i \leq n$. Then the matrix M_Δ representing the Laplacian $\Delta : \mathfrak{F} \to \mathfrak{F}$ with respect to the basis $\{\chi^i \mid 1 \leq i \leq n\}$ of $\mathfrak{F}$ is given by*

$$M_\Delta = D - (A + A^t),$$

where A^t is the transpose of the adjacency matrix A of G.

Proof. For $1 \leq i, j \leq n$ let M_{ij}, D_{ij} and A_{ij} be the ij-entries of M_Δ, D and A, respectively. Then we have

$$D_{ij} = (\deg j)\delta_{ij}, \quad \Delta(\chi^i) = \sum_{j=1}^n M_{ij}\chi^j. \tag{4.5}$$

Since $\chi^i(j) = \delta_{ij}$ for all $i, j \in V$, we see that $M_{ij} = (\Delta\chi^i)(j)$ for $1 \leq i, j \leq n$. However, using (4.2), (4.3) and the fact that $\widehat{d}\chi^i(j,k) = \chi^i(j) - \chi^i(k)$ for $(j,k) \in E$, we see that

$$\begin{aligned}(\Delta\chi^i)(j) = \sum_{k\in V_j} \widehat{d}\chi^i(j,k) &= \sum_{k\in V_j^O} (\chi^i(j) - \chi^i(k)) + \sum_{k\in V_j^I} (\chi^i(j) - \chi^i(k)) \\ &= (\deg j)\chi^i(j) - \sum_{k\in V_j^O} \chi^i(k) - \sum_{k\in V_j^I} \chi^i(k) \\ &= (\deg j)\delta_{ij} - \sum_{k\in V_j^O} \delta_{ik} - \sum_{k\in V_j^I} \delta_{ik},\end{aligned}$$

where we used the relation $\deg j = \sum_{k\in V_j^O}(1) + \sum_{k\in V_j^I}(1)$ for each $j \in V$. Thus by (4.4) and (4.5) we obtain

$$M_{ij} = (\Delta\chi^i)(j) = D_{ij} - A_{ji} - A_{ij}$$

for $1 \leq i, j \leq n$, and therefore the proposition follows. □

Remark 4.3 If at most one of (i,j) and (j,i) is contained in E for $1 \leq i, j \leq n$, then $A + A^t$ is the usual adjacency matrix of the undirected graph with entries in $\{0, 1\}$ obtained from G by ignoring the direction of each edge, and M_Δ coincides with the usual Laplacian matrix for an undirected graph (cf. [5]). Note that, as is mentioned in the introduction, the digraph G is assumed to have no loops or multiple edges.

Let $\langle\ ,\ \rangle_{\Omega^1}$ be the standard Hermitian product on Ω^1 given by

$$\langle \xi, \eta\rangle_{\Omega^1} = \sum_{(i,j)\in E} C_{ij} \cdot \overline{C'_{ij}}$$

for $\xi = \sum_{(i,j)\in E} C_{ij}(i,j)$ and $\eta = \sum_{(i,j)\in E} C'_{ij}(i,j)$. We also denote by $\langle\ ,\ \rangle_{\mathfrak{F}}$ and $\langle\ ,\ \rangle_{\mathcal{X}}$ the Hermitian products on the complex vector spaces $\mathfrak{F}$ and $\mathcal{X}$, respectively, given by

$$\langle \phi, \psi\rangle_{\mathfrak{F}} = \sum_{i\in V} \phi(i) \cdot \overline{\psi(i)}, \qquad \langle f, g\rangle_{\mathcal{X}} = \sum_{(i,j)\in E} f(i,j) \cdot \overline{g(i,j)}$$

for all $\phi, \psi \in \mathfrak{F}$ and $f, g \in \mathcal{X}$.

Theorem 4.4 *The linear maps $d : \mathfrak{F} \to \Omega^1$, $\widehat{d} : \mathfrak{F} \to \mathcal{X}$, and $\Delta : \mathfrak{F} \to \mathfrak{F}$ satisfy the relation*

$$\widehat{d}\phi(\overline{d\psi}) = \langle d\phi, d\psi\rangle_{\Omega^1} = \langle \widehat{d}\phi, \widehat{d}\psi\rangle_{\mathcal{X}} = \langle \Delta\phi, \psi\rangle_{\mathfrak{F}}.$$

for all $\phi, \psi \in \mathfrak{F}$

Proof. Let the differentials $d\phi, d\psi \in \Omega^1$ of $\phi, \psi \in \mathfrak{F}$ be given by

$$d\phi = \sum_{(i,j)\in E} C_{ij}(i,j), \qquad d\psi = \sum_{(i,j)\in E} C'_{ij}(i,j)$$

with $C_{ij}, C'_{ij} \in \mathbb{C}$. Then by (3.3) we have

$$\widehat{d}\phi = \sum_{(i,j)\in E} C_{ij}\varepsilon_{ij}, \qquad \widehat{d}\psi = \sum_{(i,j)\in E} C'_{ij}\varepsilon_{ij},$$

where $\{\varepsilon_{ij} \mid (i,j) \in E\}$ is the basis of $\mathcal{X}$ dual to the basis $\{(i,j) \mid (i,j) \in E\}$ of Ω^1 as in Theorem 3.2. Using the relations

$$C_{ij} = \widehat{d}\phi(i,j), \quad C'_{ij} = \widehat{d}\psi(i,j)$$

for $(i,j) \in E$, we obtain

$$\begin{aligned}\langle d\phi, d\psi\rangle_{\Omega^1} &= \sum_{(i,j)\in E} C_{ij} \cdot \overline{C'_{ij}} \\ &= \sum_{(i,j)\in E} \widehat{d}\phi(i,j) \cdot \overline{\widehat{d}\psi(i,j)} \\ &= \langle \widehat{d}\phi, \widehat{d}\psi\rangle_{\mathcal{X}}.\end{aligned}$$

Furthermore, since $d\psi = \sum_{(i,j)\in E} \widehat{d}\psi(i,j)(i,j)$, we also have

$$\widehat{d}\phi(\overline{d\psi}) = \widehat{d}\phi\Big(\sum_{(i,j)\in E} \overline{\widehat{d}\psi(i,j)}(i,j)\Big) = \sum_{(i,j)\in E} \overline{\widehat{d}\psi(i,j)}\widehat{d}\phi(i,j) = \langle \widehat{d}\phi, \widehat{d}\psi\rangle_{\mathcal{X}}.$$

On the other hand, we have

$$\begin{aligned}\langle \Delta\phi, \psi\rangle_{\mathfrak{F}} &= \sum_{i\in V} \Delta\phi(i) \cdot \overline{\psi(i)} \\ &= \sum_{i\in V}\Big[\sum_{j\in V_i^O} (\phi(i) - \phi(j))\overline{\psi(i)} + \sum_{j\in V_i^I} (\phi(i) - \phi(j))\overline{\psi(i)}\Big].\end{aligned}$$

Using (4.1), we see that

$$\sum_{i\in V}\sum_{j\in V_i^O} = \sum_{(i,j)\in E}, \qquad \sum_{i\in V}\sum_{j\in V_i^I} = \sum_{(j,i)\in E}. \tag{4.6}$$

Hence we obtain

$$\begin{aligned}\langle \Delta\phi, \psi\rangle_{\mathfrak{F}} &= \sum_{(i,j)\in E} (\phi(i) - \phi(j))\overline{\psi(i)} + \sum_{(j,i)\in E} (\phi(i) - \phi(j))\overline{\psi(i)} \\ &= \sum_{(i,j)\in E} \Big[(\phi(i) - \phi(j))\overline{\psi(i)} + (\phi(j) - \phi(i))\overline{\psi(j)}\Big] \\ &= \sum_{(i,j)\in E} (\phi(j) - \phi(j))(\overline{\psi(j)} - \overline{\psi(i)}) \\ &= \langle \widehat{d}\phi, \widehat{d}\psi\rangle_{\mathcal{X}},\end{aligned}$$

and therefore the theorem follows. □

Corollary 4.5 *For each* $\phi \in \mathfrak{F}$ *we have*

$$\langle \Delta\phi, \phi \rangle_{\mathfrak{F}} = \widehat{d}\phi(\overline{d\phi}) = \sum_{(i,j)\in E} |\widehat{d}\phi(i,j)|^2 \geq 0.$$

Proof. This follows immediately from Theorem 4.4 by letting $\psi = \phi$. □

5 Flows in digraphs

A flow on a digraph is usually defined to be a real-valued function on the set of edges satisfying certain conditions. In this section we generalize such a notion of flows by introducing a flow on our digraph $G = (V, E)$ as a certain complex-valued function on Ω^1 associated to an element of $\mathfrak{F}$ (cf. [4]) and discuss connections between such a flow and the maps d, $\widehat{d}$ and Δ.

Given a vector field $\eta \in \mathcal{X} = \mathrm{Hom}_{\mathbb{C}}(\Omega^1, \mathbb{C})$ on $G = (V, E)$, we define the associated function $\mathcal{D}\eta : V \to \mathbb{C}$ by

$$(\mathcal{D}\eta)(i) = \sum_{j\in V_i^I} \eta(j,i) - \sum_{j\in V_i^O} \eta(i,j) \tag{5.1}$$

for all $i \in V$. Then we see that $\eta \mapsto \mathcal{D}\eta$ determines a $\mathbb{C}$-linear map $\mathcal{D} : \mathcal{X} \to \mathfrak{F}$.

Definition 5.1 Let $\phi : V \to \mathbb{C}$ be an element of $\mathfrak{F}$. A *flow* in the digraph $G = (V, E)$ associated to ϕ is a vector field $\eta \in \mathcal{X}$ on G such that $\phi = \mathcal{D}\eta$.

Remark 5.2 In the usual definition of a flow, η is a real-valued function on the set $E = \{(i,j) \mid (i,j) \in E\}$ of edges satisfying $0 \leq \eta(i,j) \leq c(i,j)$ for all $(i,j) \in E$, where c is a fixed real-valued function on E called the capacity. Furthermore, η is often required to satisfy $\mathcal{D}\eta(i) = 0$ for all vertices i except two, namely, the source and the sink.

Theorem 5.3 *Let* $\phi : V \to \mathbb{C}$ *be an element of* $\mathfrak{F}$. *Then a vector field* $\eta \in \mathcal{X}$ *on* G *is a flow associated to* ϕ *if and only if*

$$\langle \phi, \psi \rangle_{\mathfrak{F}} = \eta(\overline{d\psi})$$

for all $\psi \in \mathfrak{F}$.

Proof. As was noted in Section 2, $\mathfrak{F}$ is a complex vector space of dimension n with basis $\{\chi^j \mid 1 \leq j \leq n\}$. Since $\langle\ ,\ \rangle_{\mathfrak{F}}$ is the standard Hermitian form on $\mathfrak{F}$, we see that $\phi_1 = \phi_2 \in \mathfrak{F}$ if and only if $\langle \phi_1, \psi \rangle_{\mathfrak{F}} = \langle \phi_2, \psi \rangle_{\mathfrak{F}}$ for all $\psi \in \mathfrak{F}$. Thus, by Definition 5.1 it suffices to show that

$$\langle \mathcal{D}\eta, \psi \rangle_{\mathfrak{F}} = \eta(\overline{d\psi})$$

for all $\psi \in \mathfrak{F}$. Given $\psi \in \mathfrak{F}$, using (5.1), we have

$$\begin{aligned}\langle \mathcal{D}\eta, \psi \rangle_{\mathfrak{F}} &= \sum_{i\in V} \mathcal{D}\eta(i) \cdot \overline{\psi(i)} \\ &= \sum_{i\in V} \Big[\sum_{j\in V_i^I} \eta(j,i) - \sum_{j\in V_i^O} \eta(i,j) \Big] \cdot \overline{\psi(i)}.\end{aligned}$$

Thus, using (4.6), we see that

$$\begin{aligned}\langle \mathcal{D}\eta, \psi\rangle_{\mathfrak{F}} &= \sum_{(j,i)\in E} \eta(j,i)\overline{\psi(i)} - \sum_{(i,j)\in E} \eta(i,j)\overline{\psi(i)} \\ &= \sum_{(i,j)\in E} \eta(i,j)(\overline{\psi(j)} - \overline{\psi(i)}) \\ &= \sum_{(i,j)\in E} \overline{\widehat{d}\psi(i,j)}\eta(i,j) \\ &= \eta\Big(\sum_{(i,j)\in E} \overline{\widehat{d}\psi(i,j)}(i,j)\Big).\end{aligned}$$

Hence the theorem follows from this and (3.2). □

Corollary 5.4 *Let ϕ be an element of $\mathfrak{F}$.*

(i) $\widehat{d}\phi$ is a flow in the digraph $G = (V, E)$ associated to $\Delta\phi$.

(ii) If $\eta \in \mathcal{X}$ is a flow in the digraph $G = (V, E)$ associated ϕ, then $\eta(\overline{d\phi})$ is a nonnegative real number.

Proof. By Theorem 4.4 we have $\langle \Delta\phi, \psi\rangle_{\mathfrak{F}} = \widehat{d}\phi(\overline{d\psi})$; hence (i) follows from this and Theorem 5.3. As for (ii), using Theorem 5.3, we see that $\eta(\overline{d\phi}) = \langle \phi, \phi\rangle_{\mathfrak{F}} \geq 0$. Therefore the corollary follows. □

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