

# A game based on vertex-magic total labelings

E. BOUDREAU, B. HARTNELL, K. SCHMEISSER AND J. WHITELEY

*Saint Mary's University*  
*Halifax, Nova Scotia*  
CANADA

## Abstract

For a given graph  $G$ , let  $V$  be the number of vertices and  $E$  be the number of edges. Consider a labeling of the vertices and edges which is a one-to-one mapping of the set of integers  $\{1, 2, \dots, V + E\}$  onto the vertices and edges of the graph, with the property that for every vertex the sum of the labels assigned to that vertex and all edges incident with it is some constant  $k$ . Although work to date has been on deciding which graphs admit such a labeling (called a *vertex-magic total labeling*), we shall consider a game based on this concept. For certain families of graphs a winning strategy will be described.

## 1 Introduction

We shall examine the following two person game played on a graph  $G$ , where  $V$  is the number of vertices and  $E$  is the number of edges. The set  $\{1, 2, \dots, V + E\}$  will be the set of labels, each of which can be used at most once. Each vertex and edge can be assigned at most one label. The players alternate assigning an unused label from  $\{1, 2, \dots, V + E\}$  to an unlabeled vertex or edge. For a given vertex  $x$ , let  $S(x)$  be the set consisting of  $x$  and all the edges meeting  $x$ . If not all elements of  $S(x)$  have been assigned a label yet, we call the *partial weight* of  $x$  the sum of the labels of those elements of  $S(x)$  that have a label. Once all of  $S(x)$  has a label we call the *weight* of the vertex  $x$  the sum of the labels assigned to itself and all of its incident edges. The first time some vertex and all incident edges are labeled, the sum of those labels is called the *magic constant*,  $k$ . If the first time this occurs it involves two vertices (by labeling the edge joining them), both weights must be the same. Once the magic constant has been set, all other weights must be equal to this magic constant. If this cannot be accomplished at some vertex, then either that vertex or one of its incident edges must remain unlabeled. Note that as long as either the vertex or an incident edge remains unlabeled, the partial weight may actually exceed  $k$ .

The last player able to make a legal move wins. Thus, in general, the game will be over before all the vertices and edges have been labeled. In fact, for many graphs it is impossible to find a one-to-one mapping of the set of integers  $\{1, 2, \dots, V + E\}$  onto the vertices and edges of the graph, with the property that for every vertex the weight is a constant,  $k$ . If such a labeling does exist, it is called a *vertex-magic total labeling* [3]. To our knowledge, work to date has been on determining which graphs admit such a labeling. The reader is referred to [2,4] for further references to the extensive literature on labelings in general.

Several definitions will be useful in what follows. A vertex of degree one is called a *leaf* while a vertex with at least one leaf as a neighbour is called a *stem*. The subgraph consisting of a stem and all the neighbours that are leaves along with the edges joining the stem to those leaves is called a *star*. If the stem has at least three leaves it will be called a *big star*, otherwise it is called a *small star*. We shall call a leaf and the edge joining it to the adjacent stem a *one-arm*. The *core* of a graph is the subgraph that results when all the one-arms of the graph are deleted. Let us consider an illustration of the game. If  $G$  is the path on 3 vertices, then the set of possible labels is  $\{1, 2, 3, 4, 5\}$ . If Player 1 assigned the label 1 to an edge and Player 2 responded by giving the label 5 to the leaf incident with that edge, then the magic constant would be set to 6. If Player 1 then gave the label 2 to the vertex of degree two, Player 2 could move by playing the number 3 on the last edge (the weight of the middle vertex is 6 which is legal). But now Player 1 cannot play, as placing a 4 on the last leaf would give a weight of 7 which is not legal. Hence Player 2 would win in this case. On the other hand, if after three moves the label 4 was assigned to the vertex of degree two and the edges had the labels 2 and 3, then no further move is possible (the magic constant is 9 but this cannot be achieved at either leaf).

As another example, if  $G$  were the graph consisting of the path on two vertices and Player 1 and Player 2 have assigned labels to the vertices, then it is now not legal to assign a number to the edge since so doing would simultaneously result in two vertices having different weights (thus the magic constant cannot be set). Hence Player 2 would win.

## 2 Some Results

We first show that if the game is in progress then, under suitable conditions, the second player to move can win.

If the magic constant, say  $k$ , has been set, and two labels, say  $x$  and  $y$ , add to  $k$ , then we say that  $x$  is the *match* for  $y$ .

**LEMMA A** Let  $G$  be a graph in which every vertex is either a leaf or a stem and every stem has at least two leaves as neighbours. Furthermore,  $G$  has been partially labeled and satisfies the following conditions:

- (i) the magic constant has been set;

- (ii) the labels that have not been used yet can be partitioned into pairs, each of which adds to the magic constant;
- (iii) either (a) every stem has at least one unlabeled edge to a leaf and the number of unlabeled vertices and edges in the core is even  
or (b) there is exactly one stem with all its edges to leaves labeled. Furthermore, the partial weight of this stem exceeds the magic constant and the number of unlabeled vertices and edges in the core is odd;
- (iv) each stem either has at least two one-arms that have no label assigned or else no labels can be assigned to any of its one-arms (these one-arms already have two labels or in the case a one-arm has just one label, then the match has already been used elsewhere).

Then the second player to move on the graph  $G$  can win.

**Proof.** Let  $G$  be a graph on which the game is being played and which satisfies the hypothesis of the lemma. Let  $k$  be the magic constant. Partition the labels that have not yet been used into pairs adding to  $k$  and call each of the labels in these pairs the match for the other.

The next player to play on the graph must make a move either on the core or on a one-arm. If they play on the core, the second player responds by playing on the core as well with the match. If they play on a one-arm and there are still two other one-arms in the same star that have no labels yet, the second player responds by playing the match on the same one-arm. If there is only one other one-arm in the same star without a label, the second player responds by assigning the match to the leaf of that one-arm. This strategy guarantees that if we have a graph  $G$  satisfying the hypothesis of the lemma that the remaining number of moves that will be made on the core is even. If (iii) (a) holds, all of the core is played on and if (iii)(b) holds all but one of the vertices and edges of the core is used. In addition, two labels will be used for every one-arm of each star except for the last two one-arms of that star played on and each of these will be assigned one label. Thus the total number of moves remaining is even and the second player will win.  $\square$

Consider the family of graphs  $\mathcal{G}$  where for a graph  $G$  belonging to  $\mathcal{G}$  every vertex is either a leaf or a stem. Furthermore, one stem has at least three leaves attached and the other stems have at least two leaves attached. Let the graph  $H$  formed by deleting all the one-arms (every leaf and incident edge) be called the *core*. If the total number of vertices and edges in  $H$  is even we call the core even, otherwise it is called odd. Given such a graph,  $G$ , consider the core,  $H$ . If the core is odd, then consider a spanning tree of the core. Let  $v$  be any vertex of that spanning tree and form a directed tree rooted at  $v$  and directed outwards from  $v$ . Associate each vertex (other than the root) with the unique edge directed towards it. Call a vertex and the edge associated with it *companions*. Arbitrarily form pairs of the remaining edges (those not in the spanning tree) and call the two edges in each pair companions. If the core is even, then form a spanning unicyclic subgraph of the core. Let  $v$  be a vertex in

the cycle and form a directed cycle, ending back at  $v$ . The rest of the subgraph is directed away from this cycle. Associate each vertex with the unique edge directed towards it and call them companions. Arbitrarily form pairs of the remaining edges in the graph  $H$  and call the two edges in each pair companions.

We now show that for any graph  $G$  in the family  $\mathcal{G}$  with an odd core that Player 1 has a winning strategy while if  $G$  has an even core that Player 2 has a winning strategy.

**THEOREM 1** Let  $G$  be a graph in which every vertex is either a stem or a leaf where at least one stem has at least three leaves as neighbours and all other stems have at least two adjacent leaves. If  $G$  has an odd core, then Player 1 can win. Otherwise, Player 2 can win.

**Proof.** Let  $G$  be a graph satisfying the hypothesis of the theorem and let  $V$  and  $E$  be the sets of vertices and edges of  $G$ , respectively. Let  $k = V + E$ . Then the set of numbers or labels that are allowed is  $\{1, 2, \dots, k\}$ . First let  $k$  be odd. Consider the core of  $G$  and any spanning tree of that core. Let  $v$  be a leaf of the spanning tree and consider the pairs of companions based on the tree and the leaf  $v$ . Player 1 begins by assigning the label  $k$  (the eventual magic constant) to the vertex  $v$ .

Observe that the remaining labels can be considered as pairs  $(1, k - 1), (2, k - 2), \dots, ([k - 1]/2, [k - 1]/2 + 1)$ . Thus, every label has a potential match.

Player 1 responds to any label Player 2 chooses by selecting its potential match. If Player 2 plays on the core, Player 1 responds by assigning the potential match to the companion of the vertex or edge that Player 2 labeled. Because the total number of moves in the core is odd, at some stage Player 2 must play on a leaf, say  $x$ , or an edge, say  $y$ , not in the core. In the former case, Player 1 responds by playing the potential match on the unique edge incident with  $x$  and, in the latter situation, on the leaf that  $y$  meets. At this point the magic constant is set and equals  $k$ .

If these moves were made on a big star, then Player 1 can now win by Lemma A.

If they were made on a small star, say  $B^*$ , then let the stem of that star be  $s^*$ , the leaf not yet labeled be  $v^*$  and the edge joining  $s^*$  and  $v^*$  be  $e^*$ . Player 2 has several choices for their next move.

If Player 2 moves on the core, Player 1 follows by playing the match on the companion to the edge or vertex that Player 2 labeled. If Player 2 plays on a one-arm of a small star other than  $B^*$ , then Player 1 responds by playing the match on the leaf of the other one-arm of that star. Observe that now the edge of that one-arm cannot be labeled as the sum of the labels on the leaf and the incident edge cannot be made to equal  $k$ . Also note that the leaf or the edge meeting the leaf (whichever is not yet labeled) of the other one-arm can never be labeled either. While Player 2 moves on either the core or a small star other than  $B^*$ , Player 1 responds in this fashion.

Eventually Player 2 must move on either the star  $B^*$  or a big star.

If Player 2 moves on a one-arm of a big star, then Player 1 responds by playing the match on the leaf  $v^*$  of the star  $B^*$  which now makes the edge  $e^*$  ineligible for a label and the unlabeled vertex or edge of the one-arm in the big star that Player 2 assigned a label also ineligible. Hence Player 1 can now win by Lemma A.

If Player 2 moves on the star  $B^*$  by assigning a label to the leaf  $v^*$ , then Player 1 responds by assigning the match to a leaf of a big star. This makes the edge  $e^*$  incident with  $v^*$  as well as the edge in the big star incident with the leaf labeled ineligible. Again Player 1 can win by Lemma A.

If Player 2 moves on the star  $B^*$  by assigning a label to the edge  $e^*$ , and either there are at least two of the edges meeting  $s^*$  that are unlabeled or  $s^*$  and at least one edge meeting it remain unlabeled, then Player 1 assigns the match to one of these choices. Observe that this means that the sum of the labels assigned so far to  $s^*$  and edges incident with  $s^*$  now exceeds  $k$  (thus its partial weight exceeds  $k$ ). Hence it will not be possible to assign labels to  $s^*$  and every edge incident with it. We note that in this case the number of unlabeled vertices and edges in the core is odd. Also observe that the leaf  $v^*$  in  $B^*$  is ineligible for any remaining label. Hence Player 1 can win by Lemma A.

It will never occur that Player 2 will play on  $e^*$  and  $s^*$  remains unlabeled while all its other incident edges are labeled. The vertex  $s^*$  is the companion of one of those edges (since  $s^*$  is unlabeled it is not the special vertex  $v$ ), so when Player 2 played on that edge, Player 1 would respond on the vertex  $s^*$ .

If Player 2 moves on the star  $B^*$  by assigning a label to the edge  $e^*$ , and  $s^*$  is labeled, and there is only one edge meeting it in the core that has not yet been labeled, this implies that the sum of labels at  $s^*$  and on incident edges is already more than  $k$  since  $s^*$  and its companion must be already labeled (or  $s^*$  was the special vertex  $v$  and has the label  $k$ ). In this case, Player 1 assigns the match to the companion of the unique edge meeting  $s^*$  that is not yet labeled. Again this results in an odd number of unlabeled vertices and edges in the core and Player 1 can win by Lemma A.

This completes the argument when the core is odd.

If the core of  $G$  is even, then consider the labels as pairs  $(1, k), (2, k-1), \dots, (k/2, [k+2]/2)$  and for each label consider the other label in its pair as its potential match. Now the roles of the players are reversed and regardless of how Player 1 makes their first move, Player 2 (responding as Player 1 did above) can win.  $\square$

The game appears more difficult when there are vertices that are neither stems nor leaves. We address this situation in the special circumstances when there is only one such vertex.

Consider the family of graphs,  $\mathcal{J}$ , where for a graph  $J$ . belonging to  $\mathcal{J}$  all but one vertex is a leaf or a stem. Furthermore, all stems have at least three adjacent leaves. We will show that if the core is odd and the total number of leaves is even, then there is a winning strategy for Player 1.

**THEOREM 2** Let  $J$  be a graph where every vertex but one, say  $v$ , is either a leaf or a stem, where all stems have at least three adjacent leaves. Furthermore, let the total number of leaves be even. If  $J$  has an odd core, then Player 1 can win.

**Proof.** Let  $J$  be a graph satisfying the hypothesis of the theorem and let  $V$  and  $E$  be the sets of vertices and edges of  $J$ , respectively. Let  $m = V + E$ . Then the set of labels that are allowed is  $\{1, 2, \dots, m\}$ . Player 1 begins by labeling the vertex  $v$  with the label  $m$ . Then the remaining labels can be divided into pairs  $(1, m-1), (2, m-2), \dots, ([m-1]/2, [m-1]/2+1)$ . Thus, every label has a potential match and Player 1 can respond to any label that Player 2 chooses by selecting the other label in that pair.

If Player 2 plays on the core, then Player 1 responds by assigning the potential match to any other part of the core as long as it is not the second last unlabeled edge in the core incident with  $v$ . If Player 2 labels the second last unlabeled edge in the core incident with  $v$  (Player 1 will never have to do this because whenever Player 2 plays in the core, there is an odd number of moves remaining in the core), then Player 1 responds on the last unlabeled edge incident with  $v$ . This sets  $k$  to a value of at least  $2m$ . This implies that a leaf and its incident edge can never both be labeled since there are no two available labels whose sum is at least  $2m$ . Hence the core can be completely labeled if Player 1 ensures that every stem has at least one adjacent leaf labelled (since this ensures the edge between that leaf and the stem cannot be labelled). This means there is an even number of moves in the core and an even number of moves on the one-arms, and therefore Player 1 can win.

If Player 2 does not label the second last unlabeled edge in the core incident with  $v$ , then Player 2 must play on a one-arm.

If Player 2 labels any one-arm attached to a stem  $s$ , then Player 1 immediately sets  $k = m$  by playing the match on that same one-arm. This results in the partial weight of  $v$  being at least  $k$  (so now not all edges incident with vertex  $v$  can be labeled). Player 2 can play either on the core, on a different one-arm not attached to  $s$ , or on a one-arm attached to  $s$ .

If Player 2 plays on the core, Player 1 responds by playing the match on a one-arm not attached to  $s$ . Player 1 can now win by following the strategy given in Lemma A.

If Player 2 plays on a one-arm not attached to  $s$ , Player 1 responds with the match somewhere on the core. Player 1 can again win by using the strategy of Lemma A.

The last remaining option would be for Player 2 to play on a one-arm attached to  $s$ . If  $s$  has more than three leaves, then Player 1 responds with the match on the core and can win again by the strategy used in Lemma A.

If  $s$  has only three leaves, then Player 1 responds on the unlabeled leaf of the other one-arm attached to  $s$ . This leaves Player 2 with two options once again. Player 2 can play on the core, or on a one-arm not attached to  $s$ . For each case, Player 1 will respond as described above. Once again Player 1 has the winning strategy.  $\square$

We remark that in the situation that all vertices are either stems or leaves and there are exactly two stems each with a single leaf where these two stems are adjacent and all other stems have three or more, then there is a similar winning strategy for Player 1 when the total number of leaves is even and the core is odd. We did not feel it was sufficiently different to warrant inclusion but refer the interested reader to [1].

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## References

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