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On potentially $(K_4 - e)$-graphic sequences

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Abstract

In this paper, we characterize the potentially $(K_4 - e)$-graphic sequences where $K_4 - e$ is the graph obtained from K_4 by removing one edge. This characterization implies a theorem due to C. H. Lai (*Australas. J. Combin.* 24 (2001), 123–127) and a characterization of potentially C_4-graphic sequences due to R. Luo (*Ars Combin.* 64 (2002), 301–318).

1 Introduction

An n-term nonincreasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be *graphic* if it is the degree sequence of a simple graph G of order n; such a graph G is referred to as a *realization of* π. We denote by $\sigma(\pi)$ the sum of all the terms of π. K_n is the complete graph on n vertices. $K_n - e$ is the graph obtained from K_n by removing one edge. C_n is the cycle of length n. Let H be a graph. A graphic sequence π is said to be *potentially H-graphic* if it has a realization G containing H as its subgraph.

In [1], Erdös, Jacobson and Lehel considered the following problem about potentially K_k-graphic sequences: determine the smallest positive even number $\sigma(k, n)$ such that every n-term graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ without zero terms and with degree sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$ at least $\sigma(k, n)$ is potentially K_k-graphic. They gave a lower bound of $\sigma(k, n)$ by the example $\pi_0 = ((n-1)^{k-2}, (k-2)^{n-k+2})$, i.e., $\sigma(k, n) \geq (k-2)(2n-k+1)+2$, and they further conjectured that this lower bound is the exact value of $\sigma(k, n)$. They proved the conjecture is true for $k = 3$ and $n \geq 6$, i.e., $\sigma(3, n) = 2n$ for $n \geq 6$. The conjecture is confirmed in [2, 6, 7, 8] for any $k \geq 4$ and for n sufficiently large. Recently, J. S. Li and J. H. Yin in [9] determined the values $\sigma(k, n)$ for any $k \geq 4$ and $n \geq k$.

In [2], Gould, Jacobson and Lehel generalized the above problem: for a given simple graph H, determine the smallest positive even number $\sigma(H, n)$ such that every n-term graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ without zero terms and with degree sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$ at least $\sigma(H, n)$ is potentially H-graphic. They determined the values $\sigma(C_4, n)$ and $\sigma(pK_2, n)$, where pK_2 is the matching consisted of p edges. They also pointed out: "*It would be nice to see where in the range for* $3n-2$ *to* $4n-4$*, the value* $\sigma(K_4-e,n)$ *lies*". In [5], C. H. Lai determined the values $\sigma(K_4-e,n)$.

Motivated by the above problems, we considered a stronger problem: given a graph H, characterize the potentially H-graphic sequences. Luo [10] has characterized the potentially C_k-graphic sequences for $k=3,4,5$.

In this paper, we characterize the potentially (K_4-e)-graphic sequences without zero terms. This characterization implies a theorem due to C. H. Lai [5] and a characterization of potentially C_4-graphic sequences due to R. Luo [10].

2 Preliminaries

Let $\pi=(d_1,d_2,\cdots,d_n)$ be a nonincreasing positive integer sequence. Then $\pi'=(d_1-1,d_2-1,\cdots,d_{d_n}-1,d_{d_n+1},\cdots,d_{n-1})$ is the *residual sequence* obtained by laying off d_n from π. We will denote the nonincreasing sequence π' by $(d_1',d_2',\cdots,d_{n-1}')$. From here on, all the graphic sequences have no zero terms. In order to prove our main result, we need the following results.

Lemma 2.1 (Kleitman and Wang [4], Hakimi[3]) *π is graphic if and only if π' is graphic.*

The following corollary is obvious.

Corollary 2.2 *Let H be a simple graph. If π' is potentially H-graphic, then π is potentially H-graphic.*

As Corollary 2.2 is a basic tool, which we will use frequently, sometimes it will be used without reference.

We will also use the following simple fact:

Fact 2.3 *Let π be an n-term graphic sequence with $n\geq 4$. If $d_1=n-1$ and $d_2\geq 3$, then π is potentially (K_4-e)-graphic.*

3 Potentially (K_4-e)-graphic sequences

Our main result is as follows:

Theorem 3.1 *Let $\pi=(d_1,d_2,\cdots,d_n)$ be a graphic sequence with $n\geq 4$. The sequence π is potentially (K_4-e)-graphic if and only if the following conditions hold:*

(1) $d_2\geq 3$.

(2) $d_4\geq 2$.

(3) If $n=5,6$, then $\pi\neq(3^2,2^{n-2})$ and $\pi\neq(3^6)$.

Proof: First we assume that π is potentially $(K_4 - e)$-graphic. In this case the necessary conditions (1), (2) and (3) are obvious.

Now we prove the sufficient conditions. Suppose the graphic sequence π satisfies the conditions (1), (2), and (3). Our proof is by induction on n. We consider the base cases $n = 4, 5, 6$, and 7. For many of the subcases below we show that π' (with $n-1$ terms) satisfies conditions (1), (2), and (3), and thus, by a previously established case, is potentially $(K_4 - e)$-graphic. Then, by Corollary 2.2, π is potentially $(K_4 - e)$-graphic. In other subcases, it is easy to verify directly that π is potentially $(K_4 - e)$-graphic.

Case 1. $n = 4$.

In this case all d_is are less than or equal to 3. Thus, condition (1) implies $d_1 = d_2 = 3$. Then, by Fact 2.3, π is potentially $(K_4 - e)$-graphic.

Case 2. $n = 5$.

Suppose $d_5 \geq 3$. Note $\pi \neq (3^5)$, since this sequence is not graphic. For all other possible π, π' satisfies the assumption. Thus, by Case 1, π' is potentially $(K_4 - e)$-graphic. Therefore, π is potentially $(K_4 - e)$-graphic by Corollary 2.2.

Now we assume that $d_5 \leq 2$. By condition (1), $d_2 \geq 3$. If $d_1 = 4$, π is potentially $(K_4 - e)$-graphic by Fact 2.3. So we may assume that $d_1 = d_2 = 3$.

If $d_5 = 2$, then $\pi = (3^k, 2^{5-k})$ where $2 \leq k \leq 4$. Since $\sigma(\pi)$ is even, k must be even, that is $k = 2$ or $k = 4$. If $k = 2$, then $\pi = (3^2, 2^3)$, which is impossible since π satisfies condition (3). Hence, $k = 4$, that is, $\pi = (3^4, 2)$ and in this case π is potentially $(K_4 - e)$-graphic.

Suppose $d_5 = 1$. By condition (2), $d_4 \geq 2$, and thus, d_3 is 2 or 3. Thus, $\pi = (3^k, 2^{5-k-1}, 1)$ where $2 \leq k \leq 4$. Since $\sigma(\pi)$ is even, k must be odd, that is $k = 3$. Hence, $\pi = (3^3, 2, 1)$, and it is easy to see that π is potentially $(K_4 - e)$-graphic.

Case 3. $n = 6$.

Suppose $d_6 \geq 3$. Note that $\pi' \neq (3^2, 2^3)$, otherwise $\pi = (3^6)$ which is contrary to the assumption that π satisfies condition (3). For all other possible π, π' satisfies the assumption. Thus, by Case 2, π' is potentially $(K_4 - e)$-graphic.

If $d_6 = 2$, then $\pi' = (d_1 - 1, d_2 - 1, d_3, d_4, d_5)$. Since $d_2 \geq 3$ by condition (1), we have $d'_i \geq 2$ for $1 \leq i \leq 5$. If $d'_2 \geq 3$ and $\pi' \neq (3^2, 2^3)$, then π' satisfies the assumption. Thus, π' is potentially $(K_4 - e)$-graphic. If $\pi' = (3^2, 2^3)$, then π is $(4^2, 2^4)$, $(4, 3^2, 2^3)$ or $(3^4, 2^2)$, and it is easy to see that these sequences are potentially $(K_4 - e)$-graphic. Hence, we may assume $d'_2 = 2$, that is $\pi' = (d'_1, 2^4)$. Since $\sigma(\pi')$ is even, $d'_1 = 4$ or $d'_1 = 2$. But $\pi' = (2^5)$ is impossible, otherwise $\pi = (3^2, 2^4)$ which contradicts condition (3). Therefore, $d'_1 = 4$, that is $\pi' = (4, 2^4)$. Hence, $\pi = (5, 3, 2^4)$, which is potentially $(K_4 - e)$-graphic by Fact 2.3.

If $d_6 = 1$, then $\pi' = (d_1 - 1, d_2, d_3, d_4, d_5)$. Since $d_2 \geq 3$ by condition (1) and $d_4 \geq 2$ by condition (2), we have $d'_2 \geq 2$ and $d'_4 \geq 2$. If $d'_2 = 4$, then π' satisfies the assumption, and thus, π' is potentially $(K_4 - e)$-graphic. If $d'_2 = 3$ and $\pi' \neq (3^2, 2^3)$, then π' satisfies the assumption. Thus, π' is potentially $(K_4 - e)$-graphic. If $\pi' = (3^2, 2^3)$, then $\pi = (3^3, 2^2, 1)$ or $\pi = (4, 3, 2^3, 1)$, and it is easy to see both of these sequences are potentially $(K_4 - e)$-graphic. If $d'_2 = 2$, then we have $\pi = (3^2, 2^2, 1^2)$, which is obviously potentially $(K_4 - e)$-graphic.

Case 4. $n = 7$.

If $d_7 \geq 3$ and $\pi' \neq (3^6)$, then π' satisfies the assumption, and thus by Case 3, is potentially $(K_4 - e)$-graphic. If $\pi' = (3^6)$, then $\pi = (4^3, 3^4)$. It is easy to see that $\pi = (4^3, 3^4)$ is potentially $(K_4 - e)$-graphic.

If $d_7 = 2$, then $\pi' = (d_1 - 1, d_2 - 1, d_3, d_4, d_5, d_6)$. Since $d_2 \geq 3$, we have $d'_i \geq 2$ for $1 \leq i \leq 6$. If $d'_2 > 3$, then π' satisfies the assumption, and thus π' is potentially $(K_4 - e)$-graphic. If $d'_2 = 3$, $\pi' \neq (3^2, 2^4)$, and $\pi' \neq (3^6)$, then π' satisfies the assumption. Thus, π' is potentially $(K_4 - e)$-graphic. If $\pi' = (3^2, 2^4)$, then π is $(4^2, 2^5)$, $(4, 3^2, 2^4)$ or $(3^4, 2^3)$, and it is easy to see that all of these sequences are potentially $(K_4 - e)$-graphic. If $\pi' = (3^6)$, then $\pi = (4^2, 3^4, 2)$, which is potentially $(K_4 - e)$-graphic. Hence, we may assume $d'_2 = 2$. Then, $\pi' = (d'_1, 2^5)$, and since $\sigma(\pi')$ is even, $d'_1 = 2$ or $d'_1 = 4$. If $d'_1 = 2$, then $\pi = (3^2, 2^5)$, which is potentially $(K_4 - e)$-graphic. If $d'_1 = 4$, then $\pi = (5, 3, 2^5)$, which is also potentially $(K_4 - e)$-graphic.

If $d_7 = 1$, then $\pi' = (d_1 - 1, d_2, d_3, d_4, d_5, d_6)$. Since $d_2 \geq 3$ by condition (1) and $d_4 \geq 2$ by condition (2), we have $d'_1 \geq 3$, $d'_2 \geq 2$, and $d'_4 \geq 2$. If $d'_2 \geq 3$, $\pi' \neq (3^2, 2^4)$ and $\pi' \neq (3^6)$, then π' satisfies the assumption, and thus π' is potentially $(K_4 - e)$-graphic. If $\pi' = (3^2, 2^4)$, then $\pi = (3^3, 2^3, 1)$ or $\pi = (4, 3, 2^4, 1)$, and it is easy to see both of these are potentially $(K_4 - e)$-graphic. If $\pi' = (3^6)$, then $\pi = (4, 3^5, 1)$, which is potentially $(K_4 - e)$-graphic. Hence, we may assume $d'_2 = 2$. Then $\pi' = (d'_1, 2^k, 1^{5-k})$ where $3 \leq k \leq 5$ and $d'_1 \geq 3$. If $d'_1 > 3$, then $\pi = (d'_1 + 1, 2^k, 1^{6-k})$, which is impossible by condition (1). If $d'_1 = 3$, since $\sigma(\pi')$ is even, $k = 4$, that is $\pi' = (3, 2^4, 1)$. Hence, $\pi = (3^2, 2^3, 1^2)$, which is potentially $(K_4 - e)$-graphic.

Case 5. $n > 7$.

We now prove that the sufficient condition is true for all $n > 7$. Assume the sufficient condition is true for $n - 1$. Let $\pi = (d_1, \cdots, d_n)$ be a graphic sequence with n terms that satisfies the conditions (1) and (2). We only need to show that π is potentially $(K_4 - e)$-graphic.

If $d_n \geq 3$, then π' satisfies the assumption, and thus, by the induction hypothesis, π' is potentially $(K_4 - e)$-graphic. Therefore, π is potentially $(K_4 - e)$-graphic by Corollary 2.2.

If $d_n = 2$, then $\pi' = (d_1 - 1, d_2 - 1, d_3, \cdots, d_{n-1})$. Since $d_2 \geq 3$ by condition (1) and $d_4 \geq 2$ by condition (2), we have $d'_2 \geq 2$ and $d'_4 \geq 2$. If $d'_2 \geq 3$, then π' satisfies the assumption, and thus, by the induction hypothesis, π' is potentially $(K_4 - e)$-graphic. Hence, we may assume $d'_2 = 2$, and thus, $\pi' = (d'_1, 2^{n-2})$ and $\pi = (d'_1 + 1, 3, 2^{n-2})$. Clearly, π is potentially $(K_4 - e)$-graphic.

If $d_n = 1$, then $\pi' = (d_1 - 1, d_2, d_3, \cdots, d_{n-1})$ is such that $d'_4 \geq 2$ by conditions (1) and (2). If $d_1 > 3$ or $d_3 \geq 3$, then π' satisfies the assumption, and thus, by the induction hypothesis, π' is potentially $(K_4 - e)$-graphic. Hence, we may assume $d_1 = d_2 = 3$ and $d_3 = d_4 = 2$. Thus, $\pi = (3^2, 2^k, 1^{n-k-2})$ where $2 \leq k \leq n - 4$ and

$n - k - 2$ is even, and π is potentially $(K_4 - e)$-graphic. □

4 Applications

Using Theorem 3.1, we give a simple proof of the following theorem due to C. H. Lai [5]:

Theorem 4.1 (Lai [5]) *For $n = 4, 5$ and $n \geq 7$,*

$$\sigma(K_4 - e, n) = \begin{cases} 3n - 1, & \text{if } n \text{ is odd,} \\ 3n - 2, & \text{if } n \text{ is even.} \end{cases}$$

If π is a 6-term graphical sequence with $\sigma(\pi) \geq 16$, then either there is a realization of π containing $(K_4 - e)$ or $\pi = (3^6)$. (Thus, $\sigma(K_4 - e, 6) = 20$.)

Proof: First we claim that for $n \geq 4$

$$\sigma(K_4 - e, n) \geq \begin{cases} 3n - 1, & \text{if } n \text{ is odd,} \\ 3n - 2, & \text{if } n \text{ is even.} \end{cases}$$

It is enough to show that there exist π_1 with $\sigma(\pi_1) = 3n - 3$, for odd n, and π_2 with $\sigma(\pi_2) = 3n - 4$, for even n, such that both π_1 and π_2 are not $(K_4 - e)$-graphic. If $n = 2k + 1$, take $\pi_1 = (2k, 2^{2k})$. Then $\sigma(\pi_1) = 2k + 4k = 6k = 3n - 3$, and it is easy to see that π_1 is not potentially $(K_4 - e)$-graphic. If $n = 2k + 2$, take $\pi_2 = (2k + 1, 2^{2k}, 1)$. Then $\sigma(\pi_2) = 2k + 1 + 4k + 1 = 6k + 2 = 3n - 4$, and it is easy to see that π_2 is not potentially $(K_4 - e)$-graphic.

Now we show that if π is an n-term ($n \geq 4$) graphical sequence with $\sigma(\pi) \geq 3n-1$, when n is odd, or $\sigma(\pi) \geq 3n - 2$, when n is even, then there is a realization of π containing a $K_4 - e$ (unless $\pi = (3^6)$). Hence, it suffices to show that π is potentially $(K_4 - e)$-graphic (unless $\pi = (3^6)$).

If $d_4 = 1$, then $\sigma(\pi) = d_1 + d_2 + d_3 + (n-3)$ and $d_1 + d_2 + d_3 \leq 6 + (n-3) = n+3$. Therefore, $\sigma(\pi) \leq 2n < 3n - 2$, which is a contradiction. Thus, $d_4 \geq 2$.

If $d_2 \leq 2$, then $\sigma(\pi) \leq d_1 + 2(n - 1) \leq n - 1 + 2(n - 1) = 3n - 3$, which is a contradiction. Thus, $d_2 \geq 3$.

Clearly, if $n = 5, 6$, then $\pi \neq (3^2, 2^{n-2})$. Moreover, if $\pi \neq (3^6)$, then π satisfies the conditions (1)-(3) in Theorem 3.1. Thus, π is potentially $(K_4 - e)$-graphic. □

Theorem 3.1 also implies the following theorem due to R. Luo [10].

Theorem 4.2 (Luo [10]) *Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq 4$. Then π is potentially C_4-graphic if and only if the following conditions hold:*

(1) $d_4 \geq 2$.

(2) $d_1 = n - 1$ implies $d_2 \geq 3$.

(3) If $n = 5, 6$, then $\pi \neq (2^n)$.

Proof: The necessary conditions are obvious. We now prove the sufficient conditions.

Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence satisfying conditions (1)-(3). If $\pi = (3^2, 2^{n-2})$ where $n \in \{5, 6\}$ or $\pi = (3^6)$, π is obviously potentially C_4-graphic. Thus, we may assume $\pi \neq (3^2, 2^{n-2})$ where $n \in \{5, 6\}$ and $\pi \neq (3^6)$. In this case we show that π is potentially C_4-graphic.

If $d_2 \geq 3$, then by Theorem 3.1, we are done.

Now assume that $d_2 = 2$. Then by condition (1), $\pi = (d_1, 2^k, 1^{n-k-1})$ where $3 \leq k \leq n-1$. Also note, by condition (2), $d_1 \leq n-2$, and by condition (3), if $n = 5$ or 6, $\pi \neq (2^n)$. It is easy to see that π is potentially C_4-graphic if $n = 4, 5, 6$. Thus, we may assume $n \geq 7$. We use induction to prove this case.

First we consider $n = 7$. If $d_7 = 2$, then $d'_4 \geq 2$ and $\pi' \neq (2^6)$. Hence, π' satisfies the assumption, and π' is potentially C_4-graphic. Then by Corollary 2.2, π is potentially C_4-graphic. If $d_7 = 1$, then $d'_4 \geq 2$, otherwise $\pi = (2^4, 1^3)$ which is not graphic since $\sigma(\pi)$ is odd. Thus, if $\pi' \neq (2^6)$, then π' is potentially C_4-graphic. Then by Corollary 2.2, π is potentially C_4-graphic. If $\pi' = (2^6)$, then $\pi = (3, 2^5, 1)$, and clearly π is potentially C_4-graphic.

We now prove that the sufficient condition is true for all $n > 7$. Assume the sufficient condition is true for $n-1$.

If $d_n = 2$, then $\pi = (d_1, 2^{n-1})$ where d_1 is even and $2 \leq d_1 \leq n-2$. In this case, it is easy to see that π is potentially C_4-graphic.

Suppose $d_n = 1$. If $3 \leq d_1 \leq n-2$, then π' satisfies the assumption, and by the induction hypothesis, π' is potentially C_4-graphic. If $d_1 = 2$ and $d_5 = 2$, then again the induction hypothesis implies that π' is potentially C_4-graphic. Hence, we may assume $\pi = (2^4, 1^{n-4})$. Since $\sigma(\pi)$ must be even, $n-4$ is even. Hence, π is potentially C_4-graphic. □

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