AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **29** (2004), Pages 35–47

A minimum degree result for disjoint cycles and forests in bipartite graphs

Danhong Zhang   Hong Wang

Department of Mathematics
The University of Idaho
Moscow
Idaho 83844
USA

Abstract

Let $F = (U_1, U_2; W)$ be a forest with $|U_1| = |U_2| = s$, where $s \geq 2$, and let $G = (V_1, V_2, E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq 2k + s$, where k is a nonnegative integer. Suppose that the minimum degree of G is at least $k + s$. We show that if $n > 2k + s$ then G contains the disjoint union of the forest F and k disjoint cycles. Moreover, if $n = 2k + s$, then G contains the disjoint union of the forest F, $k - 1$ disjoint cycles and a path of order 4.

1 Introduction

A set of graphs is called disjoint if no two of them have any vertex in common. Schuster [5] investigated the disjoint cycles and a forest in a graph. He proved the following result:

Theorem A. ([5], Theorem) *Let F be a forest on s edges without isolated vertices and let G be a graph of order at least $3k + |V(F)|$ with minimum degree at least $2k + s$, where k and s are nonnegative integers. Then G contains the disjoint union of the forest F and k disjoint cycles.*

In this paper, we consider a similar problem in bipartite graphs. About the maximum number of disjoint cycles in a bipartite graph, H. Wang proved the following theorems:

Theorem B. ([7], Theorem 1) *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n > 2k$, where k is a positive integer. Suppose that the minimum degree of G is at least $k + 1$. Then G contains k disjoint cycles.*

Theorem C. ([7], Theorem 2) *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n = 2k$, where k is a positive integer. Suppose that the minimum degree* *of G is at least $k+1$. Then G contains $k-1$ disjoint 4-cycles and a path of order 4 such that the path is disjoint from all the $k-1$ 4-cycles.*

This paper proves two theorems as follows:

Theorem 1. *Let $F=(U_1,U_2;W)$ be a forest with $|U_1|=|U_2|=s$, where $s\geq 2$. Let $G=(V_1,V_2,E)$ be a bipartite graph with $|V_1|=|V_2|=n>2k+s$, where k is a nonnegative integer. Suppose that the minimum degree of G is at least $k+s$. Then G contains the disjoint union of the forest F and k disjoint cycles.*

Theorem 2. *Let $F=(U_1,U_2;W)$ be a forest with $|U_1|=|U_2|=s$, where $s\geq 2$. Let $G=(V_1,V_2;E)$ be a bipartite graph with $|V_1|=|V_2|=n=2k+s$, where k is a nonnegative integer. Suppose that the minimum degree of G is at least $k+s$. Then G contains the disjoint union of the forest F, $k-1$ disjoint cycles and a path of order 4.*

All graphs considered in this paper are finite simple graphs in standard terminology and notation from [1] except as indicated. Let $G=(V,E)$ be a graph. For any $u\in V$, if G' is a subgraph of G, we define $N(u,G')$ to be $N_G(u)\cap V(G')$ and let $d(u,G')=|N(u,G')|$. If $d(u,G)=0$ or 1 we say that u is an isolated vertex or an endvertex of G, respectively. The minimum degree of G is denoted by $\delta(G)$. For a subset U of V, $G[U]$ is the subgraph of G induced by U. For two disjoint subgraphs G_1 and G_2 of G, $E(G_1,G_2)$ is the set of all edges of G between G_1 and G_2. Let $e(G_1,G_2)=|E(G_1,G_2)|$, i.e. $e(G_1,G_2)=\sum_{x\in V(G_1)} d(x,G_2)$. A set of pairwise disjoint edges of G is called a matching in G. If M is a matching with the property that every vertex of G is incident with an edge of M, then M is called a perfect matching in G. The disjoint union of two graphs S and T is denoted by $S\,\dot\cup\,T$. We use the symbol $\bigcirc^k$ to denote the disjoint union of k cycles; for $k=1$ we simply write $\bigcirc$ instead of $\bigcirc^1$. An embedding of a graph H into a graph G is an injective mapping $\sigma: V(H)\to V(G)$ so that for every edge $xy\in E(G)$, the edge $\sigma(x)\sigma(y)$ is contained in $E(G)$. We write $H\subseteq G$ or $G\supseteq H$ if there is an embedding of H into G. For an embedding σ of H into G and a subgraph M of H, let $\sigma(M)$ denote the image of M in G, i.e., $\sigma(M)$ is the subgraph of G with vertex set $\{\sigma(x): x\in V(M)\}$ and edge set $\{\sigma(x)\sigma(y): xy\in E(M)\}$. We use $(X,Y;E)$ to denote a bipartite graph with (X,Y) as its bipartition and E as its edge set. The length of a cycle C is denoted by $l(C)$, and a 4-cycle is a cycle of length 4. An acyclic graph is a graph without cycles.

2 Lemmas

For all lemmas listed below, $G=(V_1,V_2;E)$ is a given bipartite graph.

Lemma 2.1 ([7], Lemma 2.1) *Let C be a cycle of G and x a vertex of G not on C. Suppose $d(x,C)\geq 2$. Then either C is a 4-cycle or $C+x$ contains a cycle C' such that $l(C')<l(C)$.*

Lemma 2.2 ([7], Lemma 2.2) *Let C be a 4-cycle of G. Let $x \in V_1$ and $y \in V_2$ be two vertices not on C. Suppose $d(x, C) + d(y, C) \geq 3$. Then there exists $z \in V(C)$ such that either $C - z + x$ is a 4-cycle and $yz \in E$, or $C - z + y$ is a 4-cycle and $xz \in E$.*

Lemma 2.3 ([7], Lemma 2.3) *Let T be a tree of order at least 2 with a bipartition (X, Y) such that $|Y| \geq |X|$. Let $p = |Y| - |X|$. Then Y contains at least $p+1$ endvertices of T.*

Lemma 2.4 ([7], Lemma 2.4) *Let $P = x_1x_2x_3$ and $Q = y_1y_2y_3$ be two disjoint paths of G with $x_1 \in V_1$ and $y_1 \in V_2$. Let C be a 4-cycle of G such that C is disjoint from both P and Q. Suppose $d(x_1, C) + d(x_3, C) + d(y_1, C) + d(y_3, C) \geq 5$. Then $G[V(C \cup P \cup Q)]$ contains a 4-cycle C' and a path P' of order 6 such that P' is disjoint from C'.*

Lemma 2.5 ([7], Lemma 2.5) *Let C be a 4-cycle of G. Let uv and xy be two disjoint edges of G such that they are disjoint from C. Suppose $d(u, C) + d(v, C) + d(x, C) + d(y, C) \geq 5$. Then $G[V(C) \cup \{u, v, x, y\}]$ contains a 4-cycle C' and a path P' of order 4 such that P' is disjoint from C'.*

Lemma 2.6 ([7], Lemma 2.6) *Let C be a 4-cycle and P a path of order 4 in G such that P is disjoint from C and $\sum_{x \in V(P)} d(x, C) \geq 6$. Then either $G[V(C \cup P)]$ contains two disjoint quadrilaterals, or P has an endvertex, say z, such that $d(z, C) = 0$.*

Lemma 2.7 ([7], Lemma 2.7) *Let C be a 4-cycle and P a path of order $s \geq 6$ in G such that C is disjoint from P. If $\sum_{x \in V(P)} d(x, C) \geq s + 1$, then $G[V(C \cup P)]$ contains two disjoint cycles.*

Lemma 2.8 ([7], Lemma 2.8) *Let s and t be two integers such that $t \geq s \geq 2$ and $t \geq 3$. Let C_1 and C_2 be two disjoint cycles of G with lengths $2s$ and $2t$, respectively. Suppose that $\sum_{x \in V(C_2)} d(x, C_1) \geq 2t + 1$. Then $G[V(C_1 \cup C_2)]$ contains two disjoint cycles C' and C'' such that $l(C') + l(C'') < 2s + 2t$.*

Lemma 2.9 *Let $F = (U_1, U_2; W)$ be a forest with $|U_1| = |U_2| = s$, where $s \geq 1$. Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq s$ and $\delta(G) \geq s$. Then $G \supseteq F$.*

Proof. Without loss of generality, assume F is a tree. The lemma is trivial for $s = 1$. By Lemma 2.3, each of U_1 and U_2 contains an endvertex of F, say x and y, respectively. Let $F' = F - \{x, y\}$. By induction on s, there exists an embedding σ of F' in G. Suppose $x_1x, y_1y \in W$ with $\{x_1, y_1\} \subseteq V(F')$. Since $\delta(G) \geq s$, $N(\sigma(x_1), G - V(\sigma(F'))) \neq \emptyset$ and $N(\sigma(y_1), G - V(\sigma(F'))) \neq \emptyset$, and it follows $G \supseteq F$.

Lemma 2.10 *Let $F = (U_1, U_2; W)$ be a forest in G with $|U_1| = |U_2| = s$, where $s \geq 3$. Let $C = (A_1, A_2; B)$ be a cycle in G with $|A_1| = |A_2| = t \geq 3$, and C is disjoint from F. Suppose $e(C, F) \geq 2ts - 4$, then $G[V(C \cup F)] \supseteq C' \dot\cup F$, where C' is a 4-cycle.*

Proof. Since $e(C,F) \geq 2ts-4$, $t \geq 3$ and $s \geq 3$, there exist $\{x,y\} \subseteq V(C)$ with $x \neq y$ and $d(x,F) = d(y,F) = s$. We may choose x and y such that $x \in A_1$ and $y \in A_2$. Suppose this is not the case, say, for any $z \in A_2$, $d(z,F) \leq s-1$. Let $C = z_1z_2...z_{2t}z_1$ with $z_1 \in A_1$. As $e(C,F) \geq 2ts-4$, either $d(z_1,F) = s$ or $d(z_5,F) = s$. If $w \in N(z_2,F) \cap N(z_4,F)$, then $G[V(C \cup F)] \supseteq C' \dot\cup F$, where C' is the 4-cycle $wz_2z_3z_4w$ and $F \subseteq F - w + z_i$ for some $i \in \{1,5\}$ with $d(z_i,F) = s$. So we may assume $N(z_2,F) \cap N(z_4,F) = \emptyset$. Therefore $d(z_2,F) + d(z_4,F) \leq s$. Then $e(C,F) \leq t(s-1) + s(t-1) = 2ts - t - s < 2ts - 4$, a contradiction, hence the claim is true. Then we see that for any $i \in \{1,...,t-1\}$ with $z_{2i+1} \neq x$, $N(z_{2i},F) \cap N(z_{2i+2},F) = \emptyset$, and $N(z_2,F) \cap N(z_{2t},F) = \emptyset$ if $x \neq z_1$, for otherwise $G[V(C \cup F)] \supseteq C' \dot\cup F$, where C' is a 4-cycle. When t is even, it's easy to deduce that $\sum_{i=1}^{t} d(z_{2i},F) \leq s(t/2)$ and $\sum_{i=1}^{t} d(z_{2i-1},F) \leq s(t/2)$. So $2ts-4 \leq e(C,F) \leq ts$, implying $st \leq 4$, a contradiction. Similarly, when t is odd, we obtain $e(C,F) \leq 2((t-1)s/2 + s) < 2ts - 4$, a contradiction.

Lemma 2.11 *Let $F = (U_1,U_2;W)$ be a forest in $G = (V_1,V_2;E)$ with $|U_1| = |U_2| = s$, where $s \geq 3$. Let uv and xy be two disjoint edges of G such that they are disjoint from F. Suppose $d(u,F) + d(v,F) + d(x,F) + d(y,F) \geq 4s-3$ and $G[V(F)] = K_{s,s}$. Then $G[V(F) \cup \{u,v,x,y\}] \supseteq F \dot\cup P$, where P is a path of order 4.*

Proof. As $\sum_{t \in T} d(t,F) \geq 4s-3$ where $T = \{u,v,x,y\}$, either $N(u,F) \cap N(x,F) \neq \emptyset$ or $N(v,F) \cap N(y,F) \neq \emptyset$. Say the former holds, and let $w \in N(u,F) \cap N(x,F)$. For the same reason, either $d(v,F) > 0$ or $d(y,F) > 0$. Say $d(y,F) > 0$. Clearly, $G[V(F)] - w + y$ contains F since $G[V(F)] = K_{s,s}$. As $vuwx$ is a path of G, the lemma follows.

Lemma 2.12 *Let $F = (U_1,U_2;W)$ be a forest in G with $|U_1| = |U_2| = s$, where $s \geq 3$. Let $P = x_1x_2...x_{2t}$ be a path in G, where $t \geq 3$. Suppose P is disjoint from F, $G[V(F)] = K_{s,s}$ and $e(P,F) \geq 2t(s-1)+1$. Then $G[V(F \cup P))] \supseteq F \dot\cup \bigcirc$.*

Proof. Without loss of generality, suppose $U_1 \subseteq V_1$. Suppose that there exists $v \in U_1$ such that $v \in N(x_i,F) \cap N(x_{i+2},F)$ for some $i \in \{1,...,2t-2\}$. Then $vx_ix_{i+1}x_{i+2}v$ is a 4-cycle in G. If $d(x_j,F) \geq 1$ for some $x_j \in V(P) \cap V_1 - \{x_{i+1}\}$, then $G[V(F) - \{v\}] + x_j$ contains F and so the lemma holds. So we may assume $d(x_j,F) = 0$ for all $x_j \in V(P) \cap V_1 - \{x_{i+1}\}$. It follows that $2t(s-1)+1 \leq e(P,F) \leq ts+s$, which implies $(t-1)(s-2)-1 \leq 0$, a contradiction. So we may assume $N(x_i,F) \cap N(x_{i+2},F) = \emptyset$ and therefore $d(x_i,F) + d(x_{i+2},F) \leq s$ for all $i \in \{1,...,2t-2\}$. If t is odd, then $2t(s-1)+1 \leq e(P,F) \leq s(t-1)+2s$, implying $(t-1)(s-2)-1 \leq 0$, a contradiction. If t is even, Then $2t(s-1)+1 \leq e(P,F) \leq ts$, which implies $t(s-2)+1 \leq 0$, a contradiction again.

Lemma 2.13 *Let $P = x_1x_2x_3$ and $Q = y_1y_2y_3$ be two disjoint paths of G with $x_1 \in V_1$ and $y_1 \in V_2$. Let $F = (U_1,U_2;W)$ be a forest in G with $|U_1| = |U_2| = s$, where $s \geq 3$. suppose F is disjoint from both P and Q, and $d(x_1,F) + d(x_3,F) + d(y_1,F) + d(y_3,F) \geq 4s-2$. Then $G[V(F \cup P \cup Q)] \supseteq F \dot\cup C$, where C is a 4-cycle.*

Proof. First we claim that $N(x_1, F) \cap N(x_3, F) \neq \emptyset$ and $N(y_1, F) \cap N(y_3, F) \neq \emptyset$. Suppose not, without loss of generality, say $N(x_1, F) \cap N(x_3, F) = \emptyset$, then $d(x_1, F) + d(x_3, F) \leq s$. It follows that $2s \geq d(y_1, F) + d(y_3, F) \geq 4s - 2 - s = 3s - 2$, implying $s \leq 2$, a contradiction. Clearly there exists one of $\{x_1, x_3, y_1, y_3\}$, say x_1, such that $d(x_1, F) = s$. Let $u \in N(y_1, F) \cap N(y_3, F)$. Then we see that $F - u + x_1 \supseteq F$ and uQu is a 4-cycle disjoint from $F - u + x_1$, where $u \in N(y_1, F) \cap N(y_3, F)$.

3 Proofs of the Theorems

To prove the theorems, we introduce the following terminology: For a graph H and a path $P = x_1 x_2 ... x_t$ of H, we define $\sigma(P, H) = \max\{d(x_2, H), d(x_{t-1}, H)\}$ if $t \geq 2$ and $\sigma(P, H) = d(x_1, H)$ if $t = 1$.

Let G and F be given as stated in the two theorems. We may assume that F is connected. If $s = 2$, F is a path of order 4. Since $|V_1| = |V_2| = n \geq 2k + 2 = 2(k + 1)$ and $\delta(G) \geq k + 2 > k + 1$, we see that if $s = 2$ then $G \supseteq F \dot{\cup} \bigcirc^k$ by Theorem B and Theorem C. Therefore we suppose $s \geq 3$ and need to show the following:

$$G \supseteq \bigcirc^k \dot{\cup} F \text{ if } n > 2k + s \text{ and}$$
$$G \supseteq \bigcirc^{k-1} \dot{\cup} F \dot{\cup} P \text{ if } n = 2k + s, \text{ where } P \text{ is a path of order 4.} \qquad (1)$$

We use induction on k to prove (1). If $k = 0$, (1) follows from Lemma 2.9. Since $n \geq 2k + s = 2(k - 1) + (s + 2)$ and $\delta(G) \geq k + s = (k - 1) + (s + 1)$, by induction on k, $G \supseteq \bigcirc^{k-1} \dot{\cup} F \dot{\cup} K_2$. Let $C_1, C_2, ..., C_{k-1}$ be $k - 1$ disjoint cycles of G. Let σ be an embedding of F in $G - V(\bigcup_{i=1}^{k-1} C_i)$. We choose $C_1, C_2, ..., C_{k-1}$ and $\sigma(F)$ such that

$$\sum_{i=1}^{k-1} l(C_i) \qquad \text{is minimum.} \qquad (2)$$

Subject to (2), we choose $C_1, C_2, ..., C_{k-1}$ and $\sigma(F)$ such that

$$e(G[\sigma(F)]) \qquad \text{is maximum.} \qquad (3)$$

Let $D = G - V(\bigcup_{i=1}^{k-1} C_i) - V(\sigma(F))$. Subject to (2) and (3), we choose $C_1, C_2, ...,$ C_{k-1} and $\sigma(F)$ such that

$$\text{the length of a longest path in } D \text{ is maximal.} \qquad (4)$$

Let $P = x_1 x_2 ... x_p$ be a fixed longest path of D. Without loss of generality, assume $x_1 \in V_1$. Subject to (2), (3) and (4), we choose $C_1, C_2, ..., C_{k-1}$ and $\sigma(F)$ such that

$$\sigma(P, D) \qquad \text{is minimum.} \qquad (5)$$

Let $D_0 = D - V(P)$. Subject to (2) to (5), we choose $C_1, C_2, ..., C_{k-1}$ and $\sigma(F)$ such that

$$\text{the length of a longest path in } D_0 \text{ is maximal.} \qquad (6)$$

Let $Q = y_1y_2...y_q$ be a fixed longest path of D_0. Without loss of generality, assume $y_1 \in V_1$ if q is even. Subject to (2) to (6), we finally choose $C_1, C_2, ..., C_{k-1}$ and $\sigma(F)$ such that

$$\text{if } q \text{ is odd, then } \sigma(Q, D_0) \text{ is minimum;} \tag{7}$$

$$\text{if } q \text{ is even, then } d(y_2, D_0) \text{ is minimum.} \tag{8}$$

Clearly, $p \geq q$. Let $H = \bigcup_{i=1}^{k-1} C_i$ and $|V(D)| = 2d$. We will prove a number of claims. First, we claim

$$d \geq 2. \tag{9}$$

Proof of (9). Suppose $d \leq 1$. Without loss of generality, assume that $l(C_1) \leq l(C_2) \leq ... \leq l(C_{k-1})$ and $l(C_{k-1}) = 2t$. Then $t \geq 3$, for otherwise $n = 2(k-1) + s + 1 < 2k + s$. By Lemma 2.8 and (2), $e(C_{k-1}, C_i) \leq 2t$ for all $i \in \{1, ..., k-2\}$. By Lemma 2.1 and (2), $d(x, C_{k-1}) \leq 1$ for all $x \in V(D)$. Therefore $e(C_{k-1}, \sigma(F)) \geq 2t(k+s) - 2t(k-2) - 4t - 2 = 2ts - 2$. Then $G[V(C_{k-1} \cup \sigma(F))] \supseteq C' \dot\cup F$ by Lemma 2.10, where C' is a 4-cycle, contradicting (2).

We claim

$$p \geq 3 \text{ and if } |V(D_0)| \geq 4 \text{ then } q \geq 3. \tag{10}$$

Proof of (10). First we show $p \geq 3$. To the contrary, suppose $p \leq 2$. If $p < 2$, then for any $x \in V(D) \cap V_1$ and $y \in V(D) \cap V_2$, $d(x, D) = d(y, D) = 0$. It follows that $d(x, H) + d(y, H) \geq 2(k+s) - 2s = 2k$. Then there exists a C_i in H such that $d(x, C_i) + d(y, C_i) \geq 3$. By Lemma 2.1 and (2), C_i is a 4-cycle. By Lemma 2.2, $G[V(C_i) \cup \{x, y\}] \supseteq C'_i \dot\cup K_2$, where C'_i is a 4-cycle. This is a contradiction to $p < 2$. So $p = 2$. Let $P = x_1x_2$. We may choose $C_1, C_2, ..., C_{k-1}$ and $\sigma(F)$ such that $D_0 \supseteq K_2$ while (2), (3) and (4) are maintained. If this is not the case, then by (4), $d(x, D) = 0$ for all $x \in D_0$. For any $x \in V(D_0) \cap V_1$ and $y \in V(D_0) \cap V_2$, if there exists a cycle, say C_1, such that $d(x, C_1) + d(y, C_1) \geq 3$, then by Lemma 2.1 and (2), C_1 must be a 4-cycle. By Lemma 2.2, $G[V(C_1) \cup \{x, y\}]$ contains a 4-cycle C' and an edge e' disjoint from C'. So we may assume $d(x, C_i) + d(y, C_i) \leq 2$ for all $C_i \in H$. It follows that $d(x, \sigma(F)) + d(y, \sigma(F)) \geq 2(k+s) - 2(k-1) = 2s + 2$, a contradiction. Hence $D_0 \supseteq K_2$. This argument allows us to choose $C_1, C_2, ..., C_{k-1}$ and $\sigma(F)$ such that D has a perfect matching. Let $uv \in E(D_0)$ and $R = \{x_1, x_2, u, v\}$. If there exists a cycle C_i in H such that $\sum_{x \in R} d(x, C_i) \geq 5$, then by Lemma 2.5, $G[V(C_i) \cup R]$ contains the disjoint union of a 4-cycle and a path of order 4, contradicting $p = 2$. So $\sum_{x \in R} d(x, C_i) \leq 4$ for all $C_i \in H$. Therefore $\sum_{x \in R} d(x, \sigma(F)) \geq 4(k+s) - 4(k-1) - 4 = 4s$, i.e. $d(x, \sigma(F)) = s$ for all $x \in R$. Clearly $G[V(\sigma(F)) \cup R] \supseteq F \dot\cup C$, where C is a 4-cycle, implying (1). Hence $p \geq 3$.

Suppose $q \leq 2$ when $|V(D_0)| \geq 4$. By a similar argument, we may choose $C_1, C_2, ..., C_{k-1}, \sigma(F)$ and P such that $D_0 \supseteq 2K_2$. Let u_1v_1 and u_2v_2 be two independent edges in D_0, and $T = \{u_1, v_1, u_2, v_2\}$. Since D is acyclic, $\sum_{x \in T} d(x, D) \leq 6$. By Lemmas 2.1 and 2.5, $\sum_{x \in T} d(x, C_i) \leq 4$ for all $C_i \in H$. So $\sum_{x \in T} d(x, \sigma(F)) \geq 4(k+$

$s) - 4(k-1) - 6 = 4s - 2$. Clearly there exists $x \in T$ such that $d(x, \sigma(F)) = s$. Then $G[V(\sigma(F))] = K_{s,s}$ follows from (3). By Lemma 2.11, $G[V(\sigma(F)) \cup T] \supseteq F \dot{\cup} Q'$, where Q' is a path of order 4 while (2), (3), (4), (5) are maintained, contradicting $q \leq 2$. Hence (10) holds.

The argument in the above paragraph shows that if $|V(D_0)| \geq 2$, then $q \geq 2$. We claim

$$\sigma(P, D) = 2,\ \sigma(Q, D_0) \leq 2 \text{ if } q \text{ is odd and } d(y_2, D_0) \leq 2 \text{ if } q \text{ is even.} \quad (11)$$

Proof of (11). First we suppose that $\sigma(Q, D_0) \geq 3$ if q is odd and $d(y_2, D_0) \geq 3$ if q is even. In the former case, we may assume $d(y_2, D_0) \geq 3$ and $q \geq 3$. Let $\{a, b\} = \{1, 2\}$ such that $y_1 \in V_a$. Let u be an endvertex of D_0 such that $uy_2 \in E$ and $u \notin \{y_1, y_q\}$. Clearly, either $d(u, P) = 0$ or $d(y_1, P) = 0$ as D is acyclic. Without loss of generality, assume that $d(u, P) = 0$. Let (A, B) be the bipartition of $D_0 - V(Q) \cup \{u\}$ with $A \subseteq V_a$ and $B \subseteq V_b$. Clearly $|B| > |A|$, so $D_0 - V(Q) \cup \{u\}$ has a component T such that$|V(T) \cap B| > |V(T) \cap A|$. As there is at most one edge between Q and T and by Lemma 2.3, we can choose a vertex $v \in V(T) \cap B$ such that $d(v, D_0) \leq 1$. We deduce that $d(u, D) + d(v, D) \leq 3$ as D is acyclic.

If there exists C_i in H such that $d(u, C_i) + d(v, C_i) \geq 3$, then by Lemma 2.1 and (2), C_i must be a 4-cycle. By Lemma 2.2, $G[V(C_i) \cup \{u, v\}] \supseteq C' \dot{\cup} e'$, where C' is a 4-cycle and e' is an edge, and exactly one of u and v is an endvertex of e'. Let $D' = G - (V(\bigcup_{j \neq i} C_j) \cup V(C')) - V(\sigma(F))$ and $D'_0 = D' - V(P)$. By (4), P is still a longest path of D'. So neither of the two endvertices of e' is adjacent to x_2 or x_{p-1} and therefore $\sigma(P, D') \leq \sigma(P, D)$. Subsequently, Q is still a longest path of D'_0 by (6). So neither of the two endvertices of e' is adjacent to y_2 or y_{q-1}. Thus $u \in V(C')$, $d(y_2, D'_0) = d(y_2, D_0) - 1$ and $d(y_{q-1}, D'_0) \leq d(y_{q-1}, D_0)$. Repeating this argument for y_{q-1} if q is odd and $d(y_{q-1}, D'_0) \geq 3$, we obtain a contradiction with (7) or (8) while (2) to (6) are maintained.

So we may assume $d(u, C_i) + d(v, C_i) \leq 2$ for all $C_i \in H$. It follows that $d(u, \sigma(F)) + d(v, \sigma(F)) \geq 2(k + s) - 2(k - 1) - 3 = 2s - 1$. By (3), it is easy to see that $G[V(\sigma(F))] = K_{s,s}$. If $d(v, \sigma(F)) = s$, then $d(u, \sigma(F)) \geq s - 1$. Clearly $G[V(\sigma(F) \cup D_0)] \supseteq K_{s,s} \dot{\cup} Q'$, where Q' is a path with $l(Q') > l(Q)$ without violating (2) to (5). Therefore $d(u, \sigma(F)) = s$ and $d(v, \sigma(F)) = s - 1$. Let $F' = \sigma(F) - w + u$ and $D'_0 = D_0 - u + w$, where $w \in N(v, \sigma(F))$. Then $d(y_2, D'_0) = d(y_2, D_0) - 1$ and $d(y_{q-1}, D'_0) \leq d(y_{q-1}, D_0)$. If q is even, we obtain a contradiction to (8) while (2) to (6) are maintained. If q is odd, we can obtain a contradiction to (7) by applying the same argument to y_{q-1}. A similar but simpler argument shows that $\sigma(P, D) = 2$ as we have no concerns for the priorities (6) to (8). So (11) holds.

We claim

$$p \geq 2d - 1 \quad (12)$$

Proof of (12). Suppose $p \leq 2d - 2$. We distinguish two cases: p is even or odd.

Case 1. p is even.

By (10), $p \geq 4$. Let $R = \{x_1, x_p, y_1, y_2\}$. By (11), $d(y_1, D_0) + d(y_2, D_0) \leq 3$. Since $e(P, Q) \leq 1$ and $d(x_1, D) + d(x_p, D) = 2$, $\sum_{x \in R} d(x, D) \leq 6$.

If there exists C_i in H such that $\sum_{x \in R} d(x, C_i) \geq 5$, then by Lemma 2.1 and (2), C_i must be a 4-cycle. Let $C_i = u_1u_2u_3u_4u_1$. Without loss of generality, assume $\{u_1, x_1, y_1\} \subseteq V_1$. Clearly, either $d(x_1, C_i) + d(y_2, C_i) \geq 3$ or $d(x_p, C_i) + d(y_1, C_i) \geq 3$. Without loss of generality, say the former holds. By Lemma 2.2, $G[V(C_i) \cup \{x_1, y_2\}]$ contains a 4-cycle C' and an edge e' disjoint from C' such that exactly one of x_1 and y_2 is an endvertex of e'. By (4), x_1 is not an endvertex of e'. So $d(x_1, C_i) = 2$ and $d(y_2, C_i) = 1$. As $d(y_1, C_i) + d(x_p, C_i) \geq 2$, we have either $d(y_1, C_i) > 0$ or $N(x_p, C_i) \cap N(y_2, C_i) \neq \emptyset$. In either case, it is easy to see that $G[V(C_i \cup P) \cup \{y_1, y_2\}] \supseteq C'' \dot\cup P'$, where C'' is a 4-cycle and P' is a path of order $p + 2$, contradicting (4).

So we may assume $\sum_{x \in R} d(x, C_i) \leq 4$ for all $C_i \in H$. It follows that

$$\sum_{x \in R} d(x, \sigma(F)) \geq 4(k + s) - 4(k - 1) - 6 = 4s - 2.$$

Clearly there exists $z \in R$ such that $d(z, \sigma(F)) = s$, so $G[V(\sigma(F))] = K_{s,s}$ by (3). we have either $d(x_1, \sigma(F)) + d(y_2, \sigma(F)) \geq 2s - 1$ or $d(x_p, \sigma(F)) + d(y_1, \sigma(F)) \geq 2s - 1$. Without loss of generality, say the former holds. If $d(y_2, \sigma(F)) = s$, then we readily see that $G[V(\sigma(F) \cup P) \cup \{y_1, y_2\}]$ contains $K_{s,s}$ and a path of order $p+1$ which is disjoint from $K_{s,s}$, contradicting (4). So $d(y_2, \sigma(F)) = s - 1$ and $d(x_1, \sigma(F)) = s$. And moreover, $N(y_2, \sigma(F)) \cap N(x_p, \sigma(F)) = \emptyset$, for otherwise $G[V(\sigma(F) \cup D)] \supseteq K_{s,s} \dot\cup P'$, where P' is a path of order $p + 2$, contradicting (4). Therefore $d(y_2, \sigma(F)) + d(x_p, \sigma(F)) \leq s$. It follows that $2s \geq d(y_1, \sigma(F)) + d(x_1, \sigma(F)) \geq 4s - 2 - s = 3s - 2$, implying $s \leq 2$, a contradiction.

Case 2. p is odd.

Notice that $|V(D_0)|$ is odd. We claim that if $q = 3$, then we may choose Q such that $y_1 \in V_2$. Suppose that this is not true, i.e. $y_1 \in V_1$. Let (A, B) be the bipartition of $D_0 - V(Q)$ such that $A \subseteq V_1$ and $B \subseteq V_2$. Then $|B| = |A| + 2$. As D is acyclic and by Lemma 2.3, we can choose a vertex $y_0 \in B$ such that $d(y_0, D_0) \leq 1$. Clearly, $d(y_0, P) \leq 1$ and $d(y_1, P) + d(y_3, P) \leq 1$. We may assume $d(y_1, P) = 0$. So $d(y_0, D) + d(y_1, D) \leq 3$.

If there exists a C_i in H such that $d(y_0, C_i) + d(y_1, C_i) \geq 3$, then by Lemma 2.1, (2) and Lemma 2.2, C_i must be a 4-cycle, and moreover, $G[V(C_i) \cup \{y_0, y_1\}]$ contains a 4-cycle C' and an edge e' disjoint from C' such that exactly one of y_0 and y_1 is an endvertex of e'. Replacing C_i with C' and by (4), we see that neither of the two endvertices of e' is adjacent to a vertex in $\{x_1, x_2, x_{p-1}, x_p\}$. Therefore (2) to (5) are maintained. By (6), y_1 is not an endvertex of e'. So $e' = y_0z_0$ for some $z_0 \in V(C_i)$. Let $H' = (H - V(C_i)) \cup C'$, $D' = D - y_1 + z_0$ and $D'_0 = D' - V(P)$. Then D'_0 does not contain a path of order 3 with its two endvertices in V_2. It follows from (11) that $d(y_2, D'_0) = 1$. Furthermore, $\sum_{z \in S} d(z, D'_0) \leq 5$, where $S = \{y_2, y_3, y_0, z_0\}$. As D' is acyclic, $\sum_{z \in S} d(z, D') \leq 7$. We distinguish two subcases:

Subcase 1.1. There exists a cycle C'' in H' such that $\sum_{z \in S} d(z, C'') \geq 5$.

By Lemma 2.1 and (2), C'' must be a 4-cycle. By Lemma 2.5, $G[V(C'') \cup S]$ contains a 4-cycle C''' and a path Q' of order 4 such that Q' is disjoint from C'''. By (4), no vertex of Q' is adjacent to a vertex in $\{x_1, x_2, x_{p-1}, x_p\}$. Thus we obtain a contradiction to (6) while (2) to (5) are maintained.

Subcase 1.2. $\sum_{z \in S} d(z, C'_i) \leq 4$ for all $C'_i \in H'$.

Clearly $\sum_{z \in S} d(z, \sigma(F)) \geq 4(k+s) - 7 - 4(k-1) = 4s - 3$. Then there exists $z \in S$ such that $d(z, \sigma(F)) = s$. It follows from (3) that $G[V(\sigma(F))] = K_{s,s}$. By Lemma 2.11, $G[V(\sigma(F) \cup Q) \cup \{y_0, z_0\}] \supseteq F \dot\cup Q'$, where Q' is a path of order 4, contradicting $q = 3$.

So we may assume $d(y_0, C_i) + d(y_1, C_i) \leq 2$ for all $C_i \in H$. Consequently

$$d(y_0, \sigma(F)) + d(y_1, \sigma(F)) \geq 2(k+s) - 2(k-1) - 3 = 2s - 1.$$

If $d(y_0, \sigma(F)) = s$, it's easy to see that $G[V(\sigma(F)) \cup \{y_1, y_2, y_3, y_0\}]$ contains F and a disjoint path of order 4, contradicting $q = 3$. So $d(y_0, \sigma(F)) = s - 1$ and $d(y_1, \sigma(F)) = s$. Let $y_0 z_0 \in E$ for some $z_0 \in V(\sigma(F))$. By (6), $y_2 z_0 \notin E$. Let $\sigma'(F) = \sigma(F) - z_0 + y_1$, $D'_0 = D_0 - y_1 + z_0$ and $D' = D'_0 \cup P$. Then $d(y_2, D'_0) = 1$, and moreover, $d(z_0, {D_0}') \leq 1$ for otherwise we have a path of order 3 with both endvertices in V_2. Let $T = \{y_2, y_3, y_0, z_0\}$. Then $\sum_{z \in T} d(z, D) \leq 7$ as $\sum_{z \in T} d(z, P) \leq 2$. Therefore $\sum_{z \in T} d(z, \sigma'(F)) \geq 4(k+s) - 4(k-1) - 7 = 4s - 3$. Again $G[V(\sigma'(F))] = K_{s,s}$ follows from (3). By Lemma 2.11, $G[V(\sigma'(F)) \cup T] \supseteq F \dot\cup Q'$, where Q' is a path of order 4, contradicting $q = 3$.

Now $y_1 \in V_2$ for $q = 3$, so we can choose three distinct vertices z_1, z_2, z_3 from D_0 with $z_1 \in V_1$ and $\{z_2, z_3\} \subseteq V_2$ such that $\{z_1, z_2\} = \{y_1, y_2\}$, and if $q \geq 3$ then $z_3 \in \{y_{q-1}, y_q\}$. If $q = 2$, then $|V(D_0)| = 3$ by (10) and therefore z_3 is an isolated vertex of D_0. Let $T = \{x_1, x_{p-1}, x_p, z_1, z_2, z_3\}$. As D is acyclic and $d(z_3, P) \leq 1$, we deduce from (11) that $\sum_{u \in T} d(u, D) \leq 10$.

If there exists a C_i in H such that $\sum_{u \in T} d(u, C_i) \geq 7$, then by Lemma 2.1 and (2), C_i must be a 4-cycle. Let $C_i = v_1 v_2 v_3 v_4 v_1$ with $v_1 \in V_1$. If $d(z_2, C_i) = 2$ or $d(z_3, C_i) = 2$, it is easy to see, by observing two situations that either $d(x_1, C_i) + d(x_p, C_i) \geq 1$ or $d(x_1, C_i) + d(x_p, C_i) = 0$, that $G[V(C_i \cup P) \cup \{z_1, z_2, z_3\}]$ contains a 4-cycle C' and a path P' disjoint from C' but longer than P, contradicting (4). Hence $d(z_2, C_i) \leq 1$ and $d(z_3, C_i) \leq 1$. We distinguish two subcases. Note that $z_1 z_2 \in E$.

Subcase 2.1. $q \geq 3$.

We first suppose that $d(z_1, C_i) \geq 1$ and $d(z_2, C_i) = 1$. Without loss of generality, say $\{v_1 z_2, v_2 z_1\} \subseteq E$. Then $C' = v_1 v_2 z_1 z_2 v_1$ is a 4-cycle, and $e(\{x_1, x_{p-1}, x_p\}, \{v_3, v_4\}) = 0$ By (4). As $\sum_{u \in T} d(u, C_i) \geq 7$, we deduce that $d(u, C_i) = 1$ for all $u \in T - \{z_1\}$ and $d(z_1, C_i) = 2$. Then $z_1 z_2 v_1 v_4 z_1$ and $v_2 P v_2$ are two disjoint cycles in $G[V(C_i \cup P) \cup \{z_1, z_2\}]$. So either $d(z_1, C_i) = 0$ or $d(z_2, C_i) = 0$. Suppose the former holds. We have $d(x_1, C_i) + d(x_{p-1}, C_i) + d(x_p, C_i) \geq 5$ and therefore $N(x_1, C_i) \cap N(x_p, C_i) \neq \emptyset$. For $v_2 \in N(x_1, C_i) \cap N(x_p, C_i)$, clearly $G[V(C_i \cup Q)] - v_2$ is disjoint from $v_2 P v_2$ and therefore is acyclic. So $d(z_2, C_i) + d(z_3, C_i) \leq 1$. Consequently, $d(x_1, C_i) =$

$d(x_{p-1}, C_i) = d(x_p, C_i) = 2$ and $d(z_j, C_i) = 1$ for some $j \in \{2, 3\}$. Without loss of generality, say $z_j v_1 \in E$. Then the 4-cycle $x_{p-1}x_p v_4 v_3 x_{p-1}$ is disjoint from the path $z_j v_1 v_2 x_1 x_2 ... x_{p-2}$ which is longer than P, contradicting (4). Therefore $d(z_1, C_i) > 0$ and $d(z_2, C_i) = 0$.

If $d(z_3, C_i) = 0$, then there exists $u' \in \{x_1, x_{p-1}, x_p, z_1\}$ such that $d(u, C_i) = 2$ for all $u \in \{x_1, x_{p-1}, x_p, z_1\} - \{u'\}$ and $d(u', C_i) \geq 1$. This implies that $\{v_i z_1, v_i x_1, v_j x_p\} \subseteq E$ for some $\{i, j\} = \{2, 4\}$ and $x_{p-1} v_h \in E$ for some $h \in \{1, 3\}$. Then the 4-cycle $x_{p-1} x_p v_j v_h x_{p-1}$ is disjoint from the path $z_2 z_1 v_i x_1 x_2 ... x_{p-2}$ which is longer than P, contradicting (4). Therefore $d(z_3, C_i) = 1$. Say $\{v_1 z_3, v_2 z_1\} \subseteq E$. Then $G[V(Q) \cup \{v_1, v_2\}]$ contains a cycle and therefore $G[V(P) \cup \{v_3, v_4\}]$ is acyclic. Hence

$$e(\{x_1, x_{p-1}, x_p\}, \{v_3, v_4\}) \leq 1.$$

This implies that $d(x_1, C_i) + d(x_{p-1}, C_i) + d(x_p, C_i) = 4$ as $d(z_1, C_i) + d(z_3, C_i) \leq 3$. Thus $d(z_1, C_i) = 2$ and $x_{p-1} v_1 \in E$. Then the 4-cycle $z_1 v_2 v_3 v_4 z_1$ is disjoint from the path $x_1 x_2 ... x_{p-1} v_1 z_3$ which is longer than P, contradicting (4) again.

Subcase 2.2. $q = 2$. Notice that $d(z_3, D) \leq 1$.

First suppose that there exists C_i in H such that $d(x_p, C_i) + d(z_3, C_i) \geq 3$, then by Lemma 2.1, Lemma 2.2, (2) and (3) as before, we see that C_i is a 4-cycle, $d(x_p, C_i) = 2$ and $d(z_3, C_i) = 1$. Let $L_1 = C_i - z_4 + x_p$ where $z_4 \in V(C_i)$ such that $z_3 z_4 \in E$. Let $H_1 = (H - V(C_i)) \cup L_1$ and $D_1 = G - V(H_1) - V(\sigma(F))$. As D_1 is acyclic, $\sum_{i=1}^4 d(z_i, D_1) \leq 7$. If there exists a cycle C' in H_1 such that $\sum_{i=1}^4 d(z_i, C') \geq 5$, then by Lemma 2.1 and (2), C' must be a 4-cycle. By Lemma 2.5, $G[V(C') \cup \{z_1, z_2, z_3, z_4\}] \supseteq C'' \dot\cup Q'$, where C'' is a 4-cycle and Q' is a path of order 4. If $\sum_{i=1}^4 d(z_i, C'_i) \leq 4$ for all $C'_i \in H_1$, then $\sum_{i=1}^4 d(z_i, \sigma(F)) \geq 4(k+s) - 7 - 4(k-1) = 4s - 3$. Again $G[V(\sigma(F))] = K_{s,s}$ by (3). It follows from Lemma 2.11 that $G[V(\sigma(F)) \cup \{z_1, z_2, z_3, z_4\}] \supseteq F \dot\cup Q'$, where Q' is a path of order 4. So in both cases we obtain a path Q' of order 4. Without loss of generality, say the former case holds. As p is odd and by (4), $p \geq 5$. Let $H_2 = (H_1 - V(C')) \cup C''$, $D_2 = G - V(H_2) - V(\sigma(F))$, $P' = P - x_p$ and $Q' = u_1 u_2 u_3 u_4$ with $u_1 \in V_1$. Then D_2 is acyclic and $e(P', Q') \leq 1$.

When $p \geq 7$, if there exists a cycle C''' in H_2 such that $\sum_{i=1}^{p-1} d(x_i, C''') \geq p$, then by Lemma 2.1 and (2), C''' must be a 4-cycle. It follows from Lemma 2.7 that $G[V(C''' \cup P')] \supseteq \bigcirc^2$, implying (1). So we may assume $\sum_{i=1}^{p-1} d(x_i, C_i'') \leq p - 1$ for all $C''_i \in H_2$. Therefore $\sum_{i=1}^{p-1} d(x_i, \sigma(F)) \geq (p-1)(k+s) - 2(p-2) - 1 - (p-1)(k-1) = (s-1)(p-1) + 1$. By Lemma 2.12, $G[V(\sigma(F) \cup P)] \supseteq F \dot\cup \bigcirc$, which implies (1).

When $p = 5$, we have $e(\{x_1, x_3\}, \{u_2, u_4\}) = 0$. Let $W = \{x_1, x_3, u_2, u_4\}$. Then $\sum_{w \in W} d(w, D_2) = 6$ as D_2 is acyclic. If there exists a cycle L' in H_2 such that $\sum_{w \in W} d(w, L') \geq 5$, then by Lemma 2.1 and (2), L' must be a 4-cycle. By Lemma 2.4, $G[V(L') \cup \{x_1, x_2, x_3, u_2, u_3, u_4\}] \supseteq L'' \dot\cup P''$, where L'' is a 4-cycle and P'' is a path of order 6, contradicting $p = 5$. So $\sum_{w \in W} d(w, L_i) \leq 4$ for all $L_i \in H_2$. Therefore $\sum_{w \in W} d(w, \sigma(F)) \geq 4(k+s) - 6 - 4(k-1) = 4s - 2$. Evidently (1) follows from Lemma 2.13.

So we can assume $d(x_p, C_i) + d(z_3, C_i) \leq 2$ for all $C_i \in H$, then $d(x_p, \sigma(F)) + d(z_3, \sigma(F)) \geq 2(k+s) - 2 - 2(k-1) = 2s$. Clearly $G[V(\sigma(F) \cup P)] \supseteq F \dot\cup P'$, where P' is a path of order $p+1$, a contradiction to (4). This proves the subcase 2.2.

Now we may assume that $\sum_{u \in T} d(u, C_i) \leq 6$ for all $C_i \in H$. Then

$$\sum_{u \in T} d(u, \sigma(F)) \geq 6(k+s) - 10 - 6(k-1) = 6s - 4.$$

Again $G[V(\sigma(F))] = K_{s,s}$ by (3). We claim that there exists $x \in \{x_1, x_p\}$, say x_1, such that $d(x_1, \sigma(F)) \geq 1$. Suppose that this is not the case, then $d(x_1, \sigma(F)) = d(x_p, \sigma(F)) = 0$. It follows that $4s \geq d(x_{p-1}, \sigma(F)) + d(z_1, \sigma(F)) + d(z_2, \sigma(F)) + d(z_3, \sigma(F)) \geq 6s - 4$, implying $s \leq 2$, a contradiction. Similarly there exists $z \in \{z_2, z_3\}$ say z_2 such that $d(z_2, \sigma(F)) \geq 1$. Let $\{ux_1, vz_2\} \subseteq E$, where $\{u, v\} \subseteq V(\sigma(F))$. Then $\sigma(F) - u + z_2 \supseteq F$ and $P + u$ is a path disjoint from F, a contradiction to (4). So (12) holds.

We are now in the position to complete the proofs. By (9) and (12), $p \geq 2d - 1 \geq 3$. As D is acyclic, $e(P, D) \leq 2(p-1) + 1$. We distinguish two cases:

Case 1. There exists a C_i in H such that $e(P, C_i) \geq p + 1$.

By Lemma 2.1 and (2), C_i must be a 4-cycle. If $p \geq 6$, then by Lemma 2.7, $G[V(C_i \cup P)] \supseteq \bigcirc^2$, implying (1). So assume $p \leq 5$ and therefore $d = 2$ or $d = 3$.

If $d = 2$, we will prove Theorem 2. First we prove $p = 4$. If $p \neq 4$, then by (10), $p = 3$. Without loss of generality, assume $\{x_1, x_3\} \subseteq V_1$. Let $x_0 \in D - V(P)$. Clearly $d(x_0, D) + d(x_3, D) = 1$. If there exists a cycle C_i in H such that $d(x_3, C_i) + d(x_0, C_i) \geq 3$, then by Lemma 2.1 and (2), C_i must be a 4-cycle and $G[V(C_i) \cup \{x_0, x_3\}]$ contains a 4-cycle C' and an edge e' disjoint from C', and moreover, we must have $e' = x_0 z$ for some $z \in V(C_i)$, for otherwise $G[V(C_i \cup D)] \supseteq C_i' \dot\cup L$, where C_i' is a 4-cycle and L is a path of order 4, a contradiction. Let $D' = D - x_3 + z$ and $H' = (H - V(C_i)) \cup C'$. If there exists a cycle, say C_1' in H' such that $e(D', C_1') \geq 5$, then by Lemma 2.5, $G[V(C_1' \cup D')]$ contains a 4-cycle and a disjoint path of order 4, contradicting $p = 3$. So we may assume $e(D', C_i') \leq 4$ for all $C_i' \in H'$. It follows that $e(D', \sigma(F)) \geq 4(k+s) - 4(k-1) - 4 = 4s$, which implies $G[V(\sigma(F) \cup D')] \supseteq F \dot\cup M$, where M is a path of order 4, a contradiction. Thus $d(x_3, C_i) + d(x_0, C_i) \leq 2$ for all $C_i \in H$, implying $d(x_3, \sigma(F)) + d(x_0, \sigma(F)) \geq 2(k+s) - 1 - 2(k-1) = 2s + 1$, a contradiction again. Hence $p = 4$.

Now we prove $n = 2k + s$. Suppose $l(C_1) \leq l(C_2) \leq ... \leq l(C_{k-1}) = 2t$. It's enough to show $t = 2$. If $t \geq 3$, then by Lemma 2.8 and (2), $e(C_{k-1}, C_i) \leq 2t$ for all $i \in \{1, ..., k-2\}$, and moreover, $e(C_{k-1}, P) \leq 4$ by Lemma 2.1 and (2). Therefore $e(C_{k-1}, \sigma(F)) \geq 2t(k+s) - 2t(k-2) - 4t - 4 = 2ts - 4$. By Lemma 2.10, $G[V(C_{k-1} \cup \sigma(F))] \supseteq C' \dot\cup F$, where C' is a 4-cycle, contradicting $t \geq 3$. Hence Theorem 2 holds.

If $d = 3$, then $p = 5$. Let $z_0 \in V(D) - V(P)$. If $d(x_1, C_i) + d(z_0, C_i) \leq 2$ for all $C_i \in H$, then $d(x_1, \sigma(F)) + d(z_0, \sigma(F)) \geq 2(k+s) - 2(k-1) - 2 = 2s$. Clearly $G[V(\sigma(F) \cup D)] \supseteq F \dot\cup L$, where L is a path of order 6, a contradiction to (4). So we may assume that there exists $C_i \in H$, say C_1 such that $d(x_1, C_1) + d(z_0, C_1) \geq 3$.

As before, by Lemma2.1, Lemma 2.2, (2) and (3), we see that C_1 is a 4-cycle, $d(x_1, C_1) = 2$ and $d(z_0, C_1) = 1$. Let $H_1 = H - V(C_1)$ and $z_1 \in V(C_1)$ be such that $z_1 z_0 \in E$. Consider $\{x_5, z_0\}$.

If there exists $C_j \in H_1$, say C_2 such that $d(x_5, C_2) + d(z_0, C_2) \geq 3$. Then C_2 is a 4-cycle, $d(x_5, C_2) = 2$ and $d(z_0, C_2) = 1$. Let $z_2 \in V(C_2)$ be such that $z_0 z_2 \in E$. Let $H' = (H - V(C_1 \cup C_2)) \cup (C_1 - z_1 + x_1) \cup (C_2 - z_2 + x_5)$, $D' = G - V(H') - V(\sigma(F))$ and $U = \{x_2, x_4, z_1, z_2\}$. Clearly H' consists of $k-1$ disjoint cycles satisfying (2). Then $d(u, D') = 1$ for all $u \in U$, for otherwise D' contains a path of order 6, contradicting (4). If there exists $C' \in H'$ such that $\sum_{u \in U} d(u, C') \geq 5$, then by Lemma 2.1 and (2), C' is a 4-cycle. By Lemma 2.4, $G[V(C' \cup D')] \supseteq C'' \,\dot{\cup}\, P'$, where C'' is a 4-cycle and P' is a path of order 6, a contradiction. So we may assume $\sum_{u \in U} d(u, C'_i) \leq 4$ for all $C'_i \in H'$. Therefore $\sum_{u \in U} d(u, \sigma(F)) \geq 4(k+s) - 4(k-1) - 4 = 4s$. It follows that $G[V(\sigma(F) \cup D')] \supseteq F \,\dot{\cup}\, C'''$, where C''' is a 4-cycle, implying (1).

So we may suppose that $d(x_5, C_i) + d(z_0, C_i) \leq 2$ for all $C_i \in H_1$. It follows that $d(x_5, \sigma(F)) + d(z_0, \sigma(F)) \geq 2(k+s) - 2(k-2) - 5 = 2s - 1$. If $d(z_0, \sigma(F)) = s$, clearly $G[V(\sigma(F) \cup D)] \supseteq F \,\dot{\cup}\, L$, where L is a path of order 6, contradicting $p = 5$. So we may assume $d(z_0, \sigma(F)) = s - 1$ and $d(x_5, \sigma(F)) = s$. Let $w \in N(z_0, \sigma(F))$ and $W = \{x_2, x_4, z_1, w\}$. It's easy to see that $G[V(C_1 \cup D \cup \sigma(F))] \supseteq {C_1}' \,\dot{\cup}\, D' \,\dot{\cup}\, F$, where C'_1 is a 4-cycle and $D' = G[\{x_2, x_3, x_4, z_1, z_0, w\}]$. If $\sum_{u \in W} d(u, \sigma(F)) = 4s$, then evidently $G[V(\sigma(F) \cup D')] \supseteq F \,\dot{\cup}\, \bigcirc$, implying (1). So we may assume $e(W, \sigma(F)) \leq 4s - 1$. Furthermore, we have $e(W, D') = 4$, thus $e(W, H') \geq 4(k+s) - 4 - (4s - 1) = 4(k-1) + 1$, where $H' = H_1 \cup C'_1$. This implies that there exists a cycle C' in H' such that $e(W, C') \geq 5$. Again by Lemma 2.1 and (2), C' is a 4-cycle. By Lemma 2.4, $G[V(C' \cup D')] \supseteq F \,\dot{\cup}\, P'$, where P' is a path of order 6, a contradiction.

Case 2. $e(P, C_i) \leq p$ for all $C_i \in H$.

We have $e(P, \sigma(F)) \geq p(k+s) - p(k-1) - (2(p-1)+1) = p(s-1)+1$. If p is even, let $p = 2t$. If $t = 2$ then $d = 2$. So assume $t \geq 3$. It follows from Lemma 2.12 that $G[V(P \cup \sigma(F))] \supseteq F \,\dot{\cup}\, \bigcirc$, implying (1). If p is odd, let $p = 2t+1$. If $t = 2$ then $p = 5$. So assume $t \geq 3$. If $d(x_1, \sigma(F)) \leq s - 1$ or $d(x_p, \sigma(F)) \leq s - 1$, then let $P' = P - x_1$ or $P - x_p$. We have $e(P', \sigma(F)) \geq (2t+1)(s-1) + 1 - (s-1) = 2t(s-1) + 1$. By Lemma 2.12, $G[V(P' \cup \sigma(F))] \supseteq F \,\dot{\cup}\, \bigcirc$. So $d(x_1, \sigma(F)) = d(x_p, \sigma(F)) = s$. Let $T = \{x_{2i} : i = 1, ..., (p-1)/2\}$. If there exists $\{x, y\} \subseteq T$ such that $N(x, \sigma(F)) \cap N(y, \sigma(F)) \neq \emptyset$, then clearly $G[V(P \cup \sigma(F))] \supseteq F \,\dot{\cup}\, \bigcirc$. Therefore $\sum_{x \in T} d(x, \sigma(F)) \leq s$. Let $U = \{x_{2i+1} : i = 1, ..., (p-3)/2\}$. We have $e(U, \sigma(F)) = 0$, for otherwise $G[V(P \cup \sigma(F))] \supseteq F \,\dot{\cup}\, \bigcirc$. It follows that $3s \geq e(P, \sigma(F)) \geq (2t+1)(s-1) + 1$, implying $(s-1)(t-1) \leq 1$, a contradiction. This completes the proofs of the theorems.

4 References

[1] B. Bollobás, Extremal Graph Theory, Academic Press, London (1978).

[2] S. Brandt, Subtrees and subforests of graphs, *J. Combin. Theory, Ser. B* **61** (1994), 63–70.

[3] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, *Acta Math. Acad. Sci. Hunger.* **14** (1963), 423–439.

[4] L. Lesniak, Independent cycles in graphs, *JCMCC* **17** (1995), 55–63.

[5] G. W. Schuster, A minimum degree result for disjoint cycles and forests in graphs. *Combinatorica* **18**(3)(1998) 425–436.

[6] H. Wang, Independent cycles with limited size in a graph. *Graphs and Combinatorics* **10** (1994), 271–281.

[7] H. Wang, On the maximum number of independent cycles in a bipartite graph. *J. Combin. Theory, Ser. B* **67** (1996), 152–164.

[8] H. Wang, On the maximum number of independent cycles in a graph. *Discrete Mathematics* **205** (1999), 183–190.

(Received 30 Jan 2002)